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Northwestern University, Methods for Theroetical Physicists
Mathematical Methods for Physicists, Seventh Ed.- Arfken
October 2, 2015

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1 Problem #1.

Determine if each series converges conditionally or absolutely.

1.1 $\Sigma_{n=1}^{\infty} = \frac{(-5)^n}{n!}$

Starting with the definition of absolutely convergent, the absolute value of each term in this series is given by

$$|a_n| = \frac{5^n}{n!} \ . \tag{1}$$

If the series $\Sigma_{n=1}^{\infty} = |a_n|$ converges, then the series in question $\Sigma_{n=1}^{\infty} = a_n$ converges absolutely. In order to determine absolute convergence, the ratio

$$\frac{|a_{n+1}|}{|a_n|} = \frac{5^{n+1}}{(n+1)!}\frac{n!}{5^n} = \frac{5}{n+1} \ , \tag{2}$$

was formed. The limit of this ratio as $n \to \infty$ is zero (which is less than one). According to the ratio test, the series formed by the absolute values of the terms of the original series converges. Therefore the series in question converges absolutely.

1.2 $\Sigma_{n=0}^{\infty} = \frac{n!(2n)!}{(3n)!}$

Each term in this series is positive-definite, which means if it converges, it must converge absolutely. The ratio test can be used again, with

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!(2n+2)!}{(3n+3)!}\frac{(3n)!}{n!(2n)!} = \frac{(n+1)(2n+2)(2n+1)}{(3n+3)(3n+2)(3n+1)} \ , \tag{3}$$

which when the numerator and denominator are multiplied out, both show a n^3 dependence. Taking the limit as $n \to \infty$ and using L'Hôpital's Rule three times gives the result $\frac{4}{27}$, which again is less than one. The ratio test determined this series converges, and must converge absolutely because all terms in the series are positive.

1.3 $\Sigma_{n=3}^{\infty} = \frac{(n-\ln n)^2}{5n^5-3n^2+1}$

Again, each term in this series is positive-definite, and by the same reasoning as above, if the series converges, it does so absolutely. The terms in this series show roughly a n^{-3} dependence. Consider the function $f(x) = \frac{1}{x^3}$, which for any $n > 3$

$$a_n < f(n) \tag{4}$$

is true. This is true because they are both monotonically decreasing, and it is true for $n = 3$. This allows us to use the integral test, which says if $\int_3^{\infty} f(x)dx$ is finite, the series of a_n's converges. Evaluation of the integral yields

$$I = \int_3^{\infty} x^{-3}dx = -\frac{1}{2}x^{-2}\Big|_3^{\infty} = -\frac{1}{2}\left[0 - \frac{1}{9}\right] = \frac{1}{18} \ , \tag{5}$$

which is finite. This implies the series converges absolutely.

1.4 $\frac{1}{2} + \frac{1}{2^2} - \frac{1}{3} - \frac{1}{3^2} + \frac{1}{4} + \frac{1}{4^2} - \frac{1}{5} - \frac{1}{5^2} + \ldots$

This series can be written

$$\sum_{n=1}^{\infty} = (-1)^{n+1} a_n \ , \tag{6}$$

where

$$a_n = \frac{1}{n+1} + \frac{1}{(n+1)^2} = \frac{n+2}{n^2+2n+1} \ , \tag{7}$$

note that all $a_n > 0$. Furthermore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+2}{n^2+2n+1} = \lim_{n\to\infty} \frac{1}{2n+2} = 0 \ . \tag{8}$$

It can also be shown that the a_n's are monotonically decreasing, $a_{n+1} < a_n$, for all n,

$$a_{n+1} = \frac{n+3}{n^2+2n+1+2n+2+1} = \frac{n+3}{n^2+4n+4} < \frac{n+2}{n^2+2n+1} \ , \tag{9}$$

due to the $4n$ term in a_{n+1} dropping off slightly faster than the $2n$ term in the a_n. This series has been shown to completely satisfy the Leibniz Criterion, and therefore the alternating series converges. However, this series converges conditionally, which can be shown using a limit comparison test. Consider the harmonic series $\Sigma_{n=1}^{\infty} = b_n$, where $b_n = \frac{1}{n}$. The ratio

$$\frac{a_n}{b_n} = \frac{n^2+2n}{n^2+2n+1} \tag{10}$$

converges to a constant in the limit $n \to \infty$. Using L'Hôpital's Rule twice, $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, which demonstrates the fates of both series are bound. The harmonic series diverges and so must the series of interest, indicating the series of interest converges conditionally.

2 Arfken 1.5.3.

Given the series

$$\sum_{n=1}^{\infty} u_n(p) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)\dots(n+p)} \ , \tag{11}$$

and applying a partial fraction decomposition to the first and last terms (as suggested in the Arfken Supplement),

$$\frac{1}{n(n+p)} = \frac{1}{p}\left[\frac{1}{n} - \frac{1}{n+p}\right] \ , \tag{12}$$

yields

$$\begin{aligned} u_n(p) &= \frac{1}{p}\left[\frac{1}{n} - \frac{1}{n+p}\right]\frac{1}{(n+1)\dots(n+p-1)} \\ &= \frac{1}{p}\left[\frac{1}{n(n+1)\dots(n+p-1)} - \frac{1}{(n+1)\dots(n+p)}\right] \ . \end{aligned} \tag{13}$$

Using the definition of $u_n(p)$, Equation 11, Equation 13 becomes

$$u_n(p) = \frac{1}{p}[u_n(p-1) - u_{n+1}(p-1)] \ . \tag{14}$$

Therefore the sum becomes

$$\sum_{n=1}^{\infty} u_n(p) = \frac{1}{p}\sum_{n=1}^{\infty}[u_n(p-1) - u_{n+1}(p-1)] \ . \tag{15}$$

Expanding the sum out shows that each term but the first cancels with the term following it,

$$\sum_{n=1}^{\infty}[u_n(p-1) - u_{n+1}(p-1)] = [u_1(p-1) - u_2(p-1)] + [u_2(p-1) - u_3(p-1)] + \dots \ , \tag{16}$$

and the term $u_1(p-1) = \frac{1}{(1)(2)(3)\dots[1+(p-1)]} = \frac{1}{p!}$, which gives the desired result,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)\dots(n+p)} = \frac{1}{p!p} \ . \tag{17}$$

3 Arfken 1.5.4.

Prove the Euler Transformation,

$$\sum_{n=0}^{\infty}(-1)^n c_n x^n \ , \tag{18}$$

can be written,

$$\frac{1}{1+x}\sum_{n=0}^{\infty}(-1)^n a_n \frac{x}{1+x}^{n} \ , \tag{19}$$

with $a_n = \sum_{j=0}^{n}(-1)^j \binom{n}{j} c_{n-j}$. Upon substituting the a_n into Equation 19 and writing out the first few terms of each sums gives,

$$\begin{aligned}\frac{1}{1+x}\Bigg\{\left(\frac{x}{1+x}\right)^0\left[c_0\binom{0}{0}\right] - \left(\frac{x}{1+x}\right)^1\left[c_1\binom{1}{0} - c_0\binom{0}{0}\right]& \\ + \left(\frac{x}{1+x}\right)^2\left[c_2\binom{2}{0} - c_1\binom{2}{1} + c_0\binom{2}{2}\right]& \\ - \left(\frac{x}{1+x}\right)^3\left[c_3\binom{3}{0} - c_2\binom{3}{1} + c_1\binom{3}{2} - c_0\binom{3}{3}\right] + \ldots\Bigg\}& \ .\end{aligned} \tag{20}$$

This expression can be rearranged by gathering terms with the same c_i coefficients,

$$\begin{aligned}\frac{1}{1+x}\Bigg\{& \\ c_0\Bigg[\left(\frac{x}{1+x}\right)^0\binom{0}{0} + \left(\frac{x}{1+x}\right)^1\binom{1}{1} + \left(\frac{x}{1+x}\right)^2\binom{2}{2} + \left(\frac{x}{1+x}\right)^3\binom{3}{3} + \ldots\Bigg]& \\ - c_1\Bigg[\left(\frac{x}{1+x}\right)^1\binom{1}{0} + \left(\frac{x}{1+x}\right)^2\binom{2}{1} + \left(\frac{x}{1+x}\right)^3\binom{3}{2} + \ldots\Bigg]& \\ + c_2\Bigg[\left(\frac{x}{1+x}\right)^2\binom{2}{0} + \left(\frac{x}{1+x}\right)^3\binom{3}{1} + \ldots\Bigg]& \\ - c_3\Bigg[\left(\frac{x}{1+x}\right)^3\binom{3}{0} + \ldots\Bigg]& \\ + \ldots\Bigg\}& \ .\end{aligned} \tag{21}$$

Equation 21 can be written as a double sum with both indexes running from zero to infinity,

$$\frac{1}{1+x}\sum_{s=0}^{\infty}(-1)^s c_s \sum_{p=0}^{\infty}\left(\frac{x}{1+x}\right)^{(s+p)}\binom{s+p}{p} \ , \tag{22}$$

which by splitting the $\frac{x}{1+x}^{(s+p)}$ term can be written,

$$\sum_{s=0}^{\infty}(-1)^s c_s x^s \left(\frac{1}{1+x}\right)^{(s+1)}\sum_{p=0}^{\infty}\left(\frac{x}{1+x}\right)^p\binom{s+p}{p} \ , \tag{23}$$

which has the same form as the Euler Transformation, Equation 18, under the condition,

$$\left(\frac{1}{1+x}\right)^{(s+1)}\sum_{p=0}^{\infty}\left(\frac{x}{1+x}\right)^p\binom{s+p}{p} = 1 \ , \tag{24}$$

or alternatively,

$$\left(\frac{1}{1+x}\right)^{-(s+1)} = \sum_{p=0}^{\infty}\left(\frac{x}{1+x}\right)^{p}\binom{s+p}{p} = (1+x)^{(s+1)} \ . \tag{25}$$

This can be shown to be true by using an alternative representation of the binomial series, as given by Wikipedia,

$$\frac{1}{(1-z)^{(\beta+1)}} = \sum_{k=0}^{\infty}\binom{\beta+k}{k} z^{k} \ , \tag{26}$$

and then taking $z = \frac{x}{1+x}$. This shows that,

$$\frac{1}{(1-\frac{x}{1+x})^{(s+1)}} = \sum_{p=0}^{\infty}\binom{s+p}{p}\left(\frac{x}{1+x}\right)^{p} \ , \tag{27}$$

where the left hand side simplifies to,

$$\frac{1}{(1-\frac{x}{1+x})^{(s+1)}} = \frac{1}{(\frac{x+1-x}{x+1})^{(s+1)}} = \left(\frac{x+1-x}{x+1}\right)^{-(s+1)} = \left(\frac{1}{x+1}\right)^{-(s+1)} = (x+1)^{(s+1)} \ , \tag{28}$$

which satisfies the condition set in Equation 25. This proves the Euler Transformation by showing Equation 23 is equivalent to Equation 18.

4 Problem #4.

The three roots of

$$0 = x^3 - 4.001x + 0.002 \tag{29}$$

can be found using perturbation theory, by introducing a small parameter ϵ,

$$0 = x^3 - (4 + \epsilon)x + 2\epsilon \ , \tag{30}$$

to retrieve the equation of interest, this equation is evaluated for $\epsilon = 0.001$.

4.1 Zeroth order solution.

To find the zeroth order solution, Equation 30 is solved for x when $\epsilon = 0$. When this is the case, the equation simplifies to

$$0 = x(x^2 - 4) \ , \tag{31}$$

which has roots $x = 2, 0, -2$.

4.2 Power Series in ϵ.

To find the roots of Equation 29 with a higher precision, the form of the answer is assumed to be a power series of ϵ,

$$x(\epsilon) = x_0 + a\epsilon + b\epsilon^2 + O(\epsilon^3) \ , \tag{32}$$

where x_0 is one of the thee roots found in Section 4.1. From this point forward any terms containing the third, or higher, power of ϵ will be neglected. If the solution to Equation 29 takes this form, the function $x(\epsilon)$ can be inserted into Equation 30,

$$0 = x(\epsilon)^3 - (4 + \epsilon)x(\epsilon) + 2\epsilon \ . \tag{33}$$

When everything is multiplied through and terms are collected by powers of ϵ, Equation 30 becomes

$$0 = \epsilon^0[-4x_0 + x_0^3] + \epsilon^1[2 - 4a - x_0 + 3ax_0] + \epsilon^2[-a - 4b + 3a^2x_0 + 3bx_0^2] + O(\epsilon^3) \ , \tag{34}$$

which results in a system of three equations, where each coefficient of an ϵ^n term is equal to zero. These equations can be solved to give values of a, b, and x_0. These x_0 values should be the same as the zeroth order roots calculated in Section 4.1, and will serve as a way to check the validity of the power series assumption. Solving the equations implied from Equation 34 gives

$$x_0 = 2, 0, -2 \tag{35}$$

$$a = \frac{x_0 - 2}{3x_0^2 - 4} \tag{36}$$

$$b = \frac{a - 3ax_0^2}{3x_0^2 - 4} \tag{37}$$

where each x_0 will result in a different a, b. Evaluating these expressions for each x_0 gives the power series, Equation 32, with explicit coefficients that is responsible for enhancing the precision of the zeroth order roots. The power series are

$$x_{(2)}(\epsilon) = 2 + 0\epsilon + 0\epsilon^2 + O(\epsilon^3) \tag{38}$$

$$x_{(-2)}(\epsilon) = -2 - \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 + O(\epsilon^3) \tag{39}$$

$$x_{(0)}(\epsilon) = 0 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + O(\epsilon^3) \ , \tag{40}$$

and by plugging in the correct value of $\epsilon = 0.001$, the second order roots of the equation of interest can be found to be

$$x_{(2)}^{(2)}(0.001) = 2 \; ; \qquad x_{(-2)}^{(2)}(0.001) = -2.0005 \; ; \quad x_{(0)}^{(2)}(0.001) = 0.000499875 \; . \tag{41}$$

4.3 Notes.

Numerically solving the equation of interest using Mathematica gives the roots to be $x = 2, -2.0005$, and 0.000499875 which shows the results of the second order power series are as exact as the precision of Mathematica's numerical solve algorithm.

5 Problem Review #1.

5.1 Calculate Vector Triple Product, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$.

In order to find the vector triple product $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$, the cross product of $\mathbf{vw}$ is calculated by finding the determinant of the matrix

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_i & v_j & v_k \\ w_i & w_j & w_k \end{pmatrix} = (v_j w_k - v_k w_j)\hat{i} + (v_k w_i - v_i w_k)\hat{j} + (v_i w_j - v_j w_i)\hat{k} \ . \tag{42}$$

To get the vector triple product, the determinant of the matrix,

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_i & u_j & u_k \\ (v_j w_k - v_k w_j) & (v_k w_i - v_i w_k) & (v_i w_j - v_j w_i) \end{pmatrix} \ , \tag{43}$$

must be calculated. After multiplying out all terms, the determinant becomes,

$$\begin{aligned} &[u_j v_i w_j - u_j v_j w_i - u_k v_k w_i + u_k v_i w_k]\hat{i} \\ &\qquad + [u_k v_j w_k - u_k v_k w_j - u_i v_i w_j + u_i v_j w_i]\hat{j} \\ &\qquad\qquad + [u_i v_k w_i - u_i v_i w_k - u_j v_j w_k + u_j v_k w_j]\hat{k} \ . \end{aligned} \tag{44}$$

This simplifies by factoring out the common coefficient from the first pair of terms and second pair of terms in each component of this vector,

$$\begin{aligned} &[v_i(u_j w_j + u_k w_k)]\hat{i} - [w_i(u_j v_j + u_k v_k)]\hat{i} \\ &\qquad [v_j(u_k w_k + u_i w_i)]\hat{j} - [w_j(u_k v_k + u_i v_i)]\hat{j} \\ &\qquad\qquad [v_k(u_i w_i + u_j w_j)]\hat{j} - [w_k(u_i v_i + u_j v_j)]\hat{k} \ . \end{aligned} \tag{45}$$

We are free to add any term to this vector without changing the result, so long as the value of the term is zero. Consider adding

$$0 = [v_i(u_i w_i)]\hat{i} - [w_i(u_i v_i)]\hat{i} + [v_j(u_j w_j)]\hat{j} - [w_j(u_j v_j)]\hat{j} + [v_k(u_k w_k)]\hat{k} - [w_k(u_k v_k)]\hat{k} \ , \tag{46}$$

to Equation 45 gives,

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = &[v_i(u_i w_i + u_j w_j + u_k w_k)]\hat{i} - [w_i(u_i v_i + u_j v_j + u_k v_k)]\hat{i} \\ &+ [v_j(u_j w_j + u_k w_k + u_i w_i)]\hat{j} - [w_j(u_j v_j + u_k v_k + u_i v_i)]\hat{j} \\ &+ [v_k(u_k w_k + u_i w_i + u_j w_j)]\hat{j} - [w_k(u_k v_k + u_i v_i + u_j v_j)]\hat{k} \ , \end{aligned} \tag{47}$$

such that,

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= [v_i(\mathbf{u} \cdot \mathbf{w})\hat{i} + v_j(\mathbf{u} \cdot \mathbf{w})\hat{j} + v_k(\mathbf{u} \cdot \mathbf{w})\hat{k}] - [w_i(\mathbf{u} \cdot \mathbf{v})\hat{i} + w_j(\mathbf{u} \cdot \mathbf{v})\hat{j} + w_k(\mathbf{u} \cdot \mathbf{v})\hat{k}] \\ &= \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v}) \ . \end{aligned} \tag{48}$$

5.2 Calculate $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v})$.

To show the sum of each permutation $\mathbf{z_a} \times (\mathbf{z_b} \times \mathbf{z_c})$, the three vector triple products are calculated,

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v}) \\ \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) &= \mathbf{w}(\mathbf{v} \cdot \mathbf{u}) - \mathbf{u}(\mathbf{v} \cdot \mathbf{w}) \\ \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) &= \mathbf{u}(\mathbf{w} \cdot \mathbf{v}) - \mathbf{v}(\mathbf{w} \cdot \mathbf{u}) \ , \end{aligned} \tag{49}$$

it is easy to see the sum of the three vector triple products is zero, for each term in one vector triple product cancels exactly with a term in another. Note that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.