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1 Arfken 11.2.3.

Determine the analytic function $w(z) = u(x, y)+iv(x, y)$, for a given $u(x, y)$ or $v(x, y)$. An analytic function is differentiable and single-valued in the complex plane. For a derivative to exist, the function w must satisfy the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \ . \tag{1}$$

1.1 $u(x, y) = x^3 - 3xy^2$

The function $u(x, y)$ is single-valued in the complex plane, so as long as the function $v(x, y)$ that satisfies the Cauchy-Riemann equations, and is single-valued, the function $w(z)$ will be analytic. The derivatives of u are

$$\frac{\partial u}{\partial x} = u_x = 3x^2 - 3y^2, \qquad \frac{\partial u}{\partial y} = u_y = -6xy \ , \tag{2}$$

which gives the derivatives of a function v that satisfies the Cauchy-Riemann equations,

$$\frac{\partial v}{\partial x} = v_x = -u_y = 6xy, \qquad \frac{\partial v}{\partial y} = v_y = u_x = 3x^2 - 3y^2 \ . \tag{3}$$

These partial derivatives are single valued, so the function v will be as well. Determining a $v(x, y)$ that satisfies Equation 3 will yield an analytic function. There must a a term $-y^3$ in v so that differentiating with respect to y yields a $-3y^2$ term. Looking at the equation for v_y, there must also be a term $3x^2y$ to get a $3x^2$ term when differentiating with respect to y. Luckily, the x derivative of this term is exactly what we expect from the equation for v_x. This gives an $v(x, y)$ of $3x^2y-3y^2$, yielding an analytic function,

$$w(z) = u(x, y) + iv(x, y) = [x^3 - 3xy^2] + i[3x^2y - y^3] \tag{4}$$

$$= x^3 + 3x(iy)^2 + 3x^2(iy) + (iy)^3 \tag{5}$$

$$= (x + iy)^3 = z^3 \ . \tag{6}$$

1.2 $v(x, y) = e^{-y} \sin x$

Following the same logic as in the previous section, the derivatives of v are

$$v_x = e^{-y} \cos x, \qquad v_y = -e^{-y} \sin x \ , \tag{7}$$

which gives derivatives of u,

$$u_x = v_y = -e^{-y} \sin x, \qquad u_y = -v_x = -e^{-y} \cos x. \tag{8}$$

It is easy to tell the integral with respect to x of the u_x equation is $e^{-y} \cos x$ because $\frac{d}{dx} \cos x = -\sin x$. Similarly, the integral with respect to y of u_y is also $e^{-y} \cos x$ because $\frac{d}{dy} e^{-y} = -e^{-y}$. This gives the analytic function,

$$w(z) = [e^{-y} \cos x] + i[e^{-y} sinx] = e^{-y}[\cos x + i \sin x] = e^{i^2y}e^{ix} = e^{i(x+iy)} = e^{iz} \ . \tag{9}$$

2 Arfken 11.2.7.

The function $f(z) = f(re^{i\theta}) = R(r,\theta)e^{i\Theta(r,\theta)}$, where $R(r,\theta)$ and $\Theta(r,\theta)$ are both differentiable real functions, is an arbitrary complex function in polar coordinates. Using this, the Cauchy-Riemann conditions can be found using a similar treatment of Arfken does in the beginning of Section 11.2. Start with the definition of the derivative

$$f'(z) = \lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z} = \frac{df}{dz} \ , \tag{10}$$

which must be equal at a point z_0 from all directions for the derivative to exist. Consider small increments in the polar coordinates r and θ, given by (δr) and $(\delta\theta)$, using the chain rule these give the small increment in z to be

$$\delta z = (\delta r)e^{i\theta} + ri(\delta\theta)e^{i\theta} \ . \tag{11}$$

A small increment of the function can be found the same way,

$$\delta f = (\delta R)e^{i\Theta} + Ri(\delta\Theta)e^{i\Theta} \ , \tag{12}$$

which gives the ratio of incremental movement of the function to the incremental movement of the coordinates. This is given by

$$\frac{\delta f(z)}{\delta z} = \frac{(\delta R)e^{i\Theta} + Ri(\delta\Theta)e^{i\Theta}}{(\delta r)e^{i\theta} + ri(\delta\theta)e^{i\theta}} \ , \tag{13}$$

which must be equal along every path approaching a point z_0.

Consider a path purely in the radial direction, such that $\delta\theta = 0$. On this path the derivative can be found by taking the limit in Equation 10,

$$\lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z} = \lim_{(\delta r) \to 0} \frac{(\delta R)e^{i\Theta} + Ri(\delta\Theta)e^{i\Theta}}{(\delta r)e^{i\theta}} = \lim_{(\delta r) \to 0} \frac{(\delta R)e^{i\Theta}}{(\delta r)e^{i\theta}} + i\frac{R(\delta\Theta)e^{i\Theta}}{(\delta r)e^{i\theta}} \tag{14}$$

$$\lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z} = \frac{\partial R}{\partial r}\frac{e^{i\Theta}}{e^{i\theta}} + i\frac{\partial \Theta}{\partial r}\frac{Re^{i\Theta}}{e^{i\theta}} \ . \tag{15}$$

Now consider a path in the tangental direction so that $(\delta r) = 0$. When taking the same limit, this becomes

$$\lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z} = \lim_{(\delta r) \to 0} \frac{(\delta R)e^{i\Theta} + Ri(\delta\Theta)e^{i\Theta}}{(ri\delta\theta)e^{i\theta}} = \lim_{(\delta r) \to 0} \frac{(\delta R)e^{i\Theta}}{ri(\delta\theta)e^{i\theta}} + i\frac{R(\delta\Theta)e^{i\Theta}}{ri(\delta\theta)e^{i\theta}} \tag{16}$$

$$\lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z} = -i\frac{\partial R}{\partial \theta}\frac{e^{i\Theta}}{re^{i\theta}} + \frac{\partial \Theta}{\partial \theta}\frac{Re^{i\Theta}}{re^{i\theta}} \ . \tag{17}$$

For the derivative to exist, these must give the same result. To ensure this, the real and imaginary parts of both are set equal to each other,

$$\frac{\partial R}{\partial r}\frac{e^{i\Theta}}{e^{i\theta}} = \frac{\partial \Theta}{\partial \theta}\frac{Re^{i\Theta}}{re^{i\theta}}, \qquad \frac{\partial \Theta}{\partial r}\frac{Re^{i\Theta}}{e^{i\theta}} = \frac{\partial R}{\partial \theta}\frac{e^{i\Theta}}{re^{i\theta}} \ , \tag{18}$$

which give the equivalent of the Cauchy-Riemann conditions in polar coordinates,

$$\frac{\partial R}{\partial r} = \frac{\partial \Theta}{\partial \theta}\frac{R}{r}, \qquad \frac{\partial \Theta}{\partial r}R = -\frac{\partial R}{\partial \theta}\frac{1}{r} \ . \tag{19}$$

3 Arfken 11.3.3.

Show the integral

$$\int_{3+4i}^{4-3i} (4z^2 - 3iz)dz \tag{20}$$

takes the same value along two paths: one along a straight line connecting the two points and one along an arc of the circle with radius 5, centered at the origin.

3.1 Straight Line

The integral goes from $3 + 4i$ to $4 - 3i$, so a straight line connecting these points has slope -7. The equation for this line is $y = -7x + b$, but when $x = 3$, $y = 4$, so $4 = -7(3) + b$ which implies $b = 25$. The equations for $y(x)$ and $x(y)$ are

$$y(x) = -7x + 25, \qquad x(y) = -\frac{y}{7} + \frac{25}{7} \ , \tag{21}$$

The function $f(z) = 4z^2 - 3iz$, plugging in for $z = x + iy$ becomes

$$f(x, y) = 4(x + iy)^2 - 3i(x + iy) = 4(x^2 - y^2 + 2ixy) + 3y - 3ix \tag{22}$$

$$= 4x^2 - 4y^2 + 3y - 3ix + 8ixy = 4x^2 - 4y^2 + 3y + i[8xy - 3x] \ . \tag{23}$$

This can be reduced to two functions of one variable each $f_x(x)$ and $f_y(y)$ by substituting in the expressions in Equation 21,

$$\begin{aligned} f_x(x) &= 4x^2 - 4[49x^2 + 25^2 - (2)(7)(25)x] + 3[-7x + 25] + i[8x(-7x + 25) - 3x] \\ &= [4 - (4)(49)]x^2 + [(-4)(-2)(7)(25) + (3)(-7)]x + [(-4)(25^2) + (3)(25)] \\ &\quad + i[(8)(-7)x^2 + \{(8)(25) - 3\}x] \\ &= -192x^2 + 1379x - 2425 + i[-56x^2 + 197x] \ , \end{aligned}$$

$$\begin{aligned} f_y(y) &= 4\left[\frac{y^2}{49} + \frac{25^2}{49} - \frac{2(25)y}{49}\right] - 4\frac{49y^2}{49} + 3y + i\left[8y\left(\frac{-y}{7} + \frac{25}{7}\right) - 3\left(\frac{-y}{7} + \frac{25}{7}\right)\right] \\ &= 4\left[\frac{-48y^2}{49} + \frac{25^2}{49} - \frac{2(25)y}{49}\right] + \frac{3(49)y}{49} + i\left[\frac{1}{7}\left(-8y^2 + (8)(25)y + 3y - 3(25)\right)\right] \\ &= \frac{1}{49}[-192y^2 + 53y + 2500] + \frac{i}{7}[-8y^2 + 203y - 75] \ . \end{aligned}$$

The integral then becomes

$$\int_{3+4i}^{4-3i} f(z)dz = \int_{3+4i}^{4-3i} f(x, y)(dx + idy) = \int_3^4 f_x(x)dx + i\int_4^{-3} f_y(y)dy \ , \tag{24}$$

and each integral can be evaluated separately. The x integral is

$$\int_3^4 f_x(x)dx = \frac{-192}{3}x^3 + \frac{1379}{2}x - 2425x + i\left[\frac{-56}{3}x^3 + \frac{197}{2}x^2\right]\Bigg|_3^4 = \frac{67}{2} - i\frac{7}{6} \ , \tag{25}$$

while the y integral is

$$\int_4^{-3} f_y(y)dy = \frac{1}{49}\left[\frac{-192}{3}y^3 + \frac{53}{2}y^2 + 2500y\right] + \frac{i}{7}\left[\frac{-8}{3}y^3 + \frac{203}{2}y^2 - 75y\right]\Bigg|_4^{-3} = -\frac{469}{2} + i\frac{49}{6} \ . \tag{26}$$

These values give the value of the integral,

$$\int_{3+4i}^{4-3i} (4z^2 - 3iz)dz = \left[\frac{67}{2} - i\frac{7}{6}\right] + i\left[-\frac{469}{2} + i\frac{49}{6}\right] = \frac{67-469i}{2} + \frac{-49-7i}{6} = \frac{76-707i}{3} \ . \quad (27)$$

3.2 Circle of Radius 5

In this case $z = 5e^{i\theta}$, the equation for a circle of radius 5 in the complex plane. The limits of integration are from $3+4i$ ($\theta_1 = \arctan[\frac{4}{3}]$) to $4-3i$ ($\theta_2 = \arctan[\frac{-3}{4}]$). For this z, the $f(z)$ becomes

$$\begin{aligned} f(z) = f(5e^{i\theta}) &= 4(5^2e^{2i\theta}) - 3i(5e^{i\theta}) \\ &= 100e^{2i\theta} - 15ie^{i\theta} = f(\theta), \end{aligned}$$

which makes the integral

$$\begin{aligned} \int_{3+4i}^{4-3i} (4z^2 - 3iz)dz = \int_{\theta 1}^{\theta 2} f(\theta) d[5e^{i\theta}] = \int_{\theta 1}^{\theta 2} (100e^{2i\theta} - 15ie^{i\theta})5ie^{i\theta} d\theta = \frac{500}{3i} ie^{3i\theta}\Big|_{\theta 1}^{\theta 2} - \frac{75}{2i} i^2 e^{2i\theta}\Big|_{\theta 1}^{\theta 2} \\ = \left[\frac{500}{3}e^{3i\theta} - \frac{75}{2}ie^{2i\theta}\right]_{\theta 1}^{\theta 2} = \frac{76-707i}{3} \ , \end{aligned}$$

where the final step was evaluated using MATHEMATICA.

4 Arfken 11.3.6.

Show that the integral

$$\int_0^{1+i} z^* dz \tag{28}$$

depends on the path for which the integral is evaluated. The function $f(z) = z^*$ is not analytic and therefore Cauchy's Integral Theorem does not apply. The two paths selected are (**1**) from the origin $(x, iy) = (0, 0)$ to (1,0) along the x-axis, then vertically from (1,0) to (1,i), and (**2**) from the origin to (0,1) along the y-axis, then horizontally from (0,1) to (1,i). The integral changes its form, when substituting $z = x + iy$, to

$$\int z^* dz = \int (x-iy)(dx+idy) = \int x(dx)-iy(dx)+x(idy)-iy(idy) = \int (x-iy)dx+\int (x-iy)idy \ . \tag{29}$$

This integral form can be used to calculate the path integral along each leg of both paths.

4.1 Path 1

On the line from (0,0) to (1,0), the value of y is always zero, so dy is zero. Equation 33 becomes

$$\int (x-iy)dx + \int (x-iy)idy = \int_0^1 xdx = \frac{1}{2} \ , \tag{30}$$

which will be added to the value of the integral over the second leg of the path, to get the value of the integral along the total path. The second leg has x constant, equal to 1, so dx is zero. Along this leg, y ranges from 0 to 1 which makes Equation 33 into

$$\int (x-iy)dx + \int (x-iy)idy = \int_0^1 (1-iy)idy = \int_0^1 (i+y)dy = iy + \frac{1}{2}y^2 \bigg|_0^1 = i + \frac{1}{2} \ , \tag{31}$$

which gives the total value of $\int_0^{1i} z^* dz$ along path (**1**) to be $i + 1$.

4.2 Path 2

On the line from (0,0) to (0,1), the value of x is always zero, so dx is zero. Equation 33 becomes

$$\int (x-iy)dx + \int (x-iy)idy = \int_0^1 -iyidy = \int_0^1 ydy = \frac{1}{2} \ , \tag{32}$$

which will be added to the value of the integral over the second leg of the path, to get the value of the integral along the total path. The second leg has y constant, equal to 1, so dy is zero. Along this leg, y ranges from 0 to 1 which makes Equation 33 into

$$\int (x-iy)dx + \int (x-iy)idy = \int_0^1 (x-1i)dx = \frac{1}{2}x^2 - ix \bigg|_0^1 = \frac{1}{2} - i \ , \tag{33}$$

which gives the total value of $\int_0^{1i} z^* dz$ along path (**2**) to be $1 - i$. This is clearly not equivalent to the integral along path (**1**), so the integral of $f(z) = z^*$ does depend on path.

5 Arfken 11.4.7.

Evaluate the integral

$$\oint_C \frac{\sin^2 z - z^2}{(z-a)^3} dz \ , \tag{34}$$

where the contour encircles point a. Begin by noting this integral has the same form as Arfken Equation 11.33, which implies,

$$\frac{2\pi i}{2} f''(a) = \oint_C \frac{f(z)}{(z-a)^3} dz \ . \tag{35}$$

To calculate this, the second derivative of $f(z)$ is needed,

$$f(z) = \sin^2 z - z^2 \tag{36}$$

$$f'(z) = 2\sin z \cos z - 2z \tag{37}$$

$$f''(z) = (2\cos^2 z) - (2\sin^2 z) - 2 \ , \tag{38}$$

so that $f''(a) = 2(\cos^2 a - \sin^2 a) - 2$. Using the double angle formula this becomes $f''(a) = 2\cos(2a) - 2$, which gives the value of the integral,

$$\oint_C \frac{\sin^2 z - z^2}{(z-a)^3} dz = 2\pi i(\cos(2a) - 1) \ . \tag{39}$$

6 Arfken 11.4.8.

Evaluate the integral

$$\oint^C \frac{dz}{z(2z+1)} \ , \tag{40}$$

where the contour is the unit circle. The function $g(z) = (2z^2 + z)^{-1}$ is not in the correct form to use Cauchy's Integral Formula, but it can be made into the correct form by using a partial fraction decomposition,

$$\frac{1}{z(2z+1)} = \frac{A}{z} + \frac{B}{2z+1} = \frac{A(2z+1)+Bz}{z(2z+1)}. \tag{41}$$

Which gives the equation $2Az + A + Bz = 1$, which says $A = 1$ and $B = -2A$. Now the integral becomes

$$\oint^C \frac{dz}{z(2z+1)} = \oint_C dz\left(\frac{1}{z} + \frac{-2}{2z+1}\right) = \oint_C dz\left(\frac{1}{z} + \frac{-1}{z+\frac{1}{2}}\right) = \oint_C \frac{dz}{z} + \oint_C \frac{-dz}{z+\frac{1}{2}} \ , \tag{42}$$

for which Cauchy's Integral Formula does apply. For the left term $f_1(z) = 1$ and for the right $f_2(z) = -1$. The singularities of this function are located at $z_0 = 0$ and $z_0 = \frac{1}{2}$, both of which are inside the unit circle contour. Applying Cauchy's Integral Formula gives,

$$\oint^C \frac{dz}{z(2z+1)} = 2\pi f_1(z) + 2\pi f_2(z) = 2\pi(1) + 2\pi(-1) = 0 \ . \tag{43}$$