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Contents

1 Arfken 11.5.7.

To find the Laurent expansion of

$$f(z) = \frac{ze^z}{z-1} \tag{1}$$

about z=1, the bare z terms can be rewritten as $z+1-1$, which allows $f(z)$ to be written as

$$f(z) = \frac{[z+1-1]e^{z+1-1}}{z-1} = e\left[\frac{z-1}{z-1} + \frac{1}{z-1}\right]e^{z-1} \ , \tag{2}$$

which after expanding the exponential becomes

$$f(z) = e\left[1 + \frac{1}{z-1}\right]\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} \ . \tag{3}$$

Writing out this series is

$$\frac{1}{1} + \frac{z-1}{1} + \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3!} + \dots \ , \tag{4}$$

which when multiplying through by a factor of $1/(z-1)$ is

$$\frac{1}{z-1} + \frac{1}{1} + \frac{z-1}{2} + \frac{(z-1)^2}{3!} + \dots = \sum_{n=-1}^{\infty}\frac{(z-1)^n}{|n|!} \ , \tag{5}$$

making the function of interest,

$$f(z) = e\left[\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} + \sum_{n=-1}^{\infty}\frac{(z-1)^n}{|n|!}\right] \ . \tag{6}$$

Adding and subtracting a $e/(z-1)$ term allows the first sum to be rewritten ,

$$f(z) = e\left[-\frac{1}{z-1} + \frac{1}{z-1} + \sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} + \sum_{n=-1}^{\infty}\frac{(z-1)^n}{|n|!}\right] \tag{7}$$

$$= e\left[-\frac{1}{z-1} + \sum_{n=-1}^{\infty}\frac{(z-1)^n}{|n|!} + \sum_{n=-1}^{\infty}\frac{(z-1)^n}{|n|!}\right] \ , \tag{8}$$

which gives the Laurent series of $f(z)$,

$$f(z) = -\frac{e}{z-1} + 2e\sum_{n=-1}^{\infty}\frac{(z-1)^n}{|n|!} \ . \tag{9}$$

2 Arfken 11.5.8.

To find the Laurent expansion of

$$f(z) = (z-1)e^{1/z} \tag{10}$$

about z=0, the $1/z$ term can be written z^{-1} and the exponential can be expanded,

$$f(z) = (z-1)\sum_{n=0}^{\infty} \frac{(z^{-1})^n}{n!} = (z-1)\sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = \sum_{n=0}^{\infty} \frac{z^{-n+1}}{n!} - \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} \ . \tag{11}$$

The first sum can be changed to go from $n = -1 \to \infty$, which would make the power of z in each sum the same,

$$\sum_{n=0}^{\infty} \frac{z^{-n+1}}{n!} = \sum_{n=-1}^{\infty} \frac{z^{-n}}{(n+1)!} = \frac{z^{-(-1)}}{1} + \sum_{n=0}^{\infty} \frac{z^{-n}}{(n+1)!} \ , \tag{12}$$

so the complete function can be written as

$$f(z) = z + \sum_{n=0}^{\infty} \frac{z^{-n}}{(n+1)!} - \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} \ , \tag{13}$$

which makes the Laurent series

$$f(z) = z + \sum_{n=0}^{\infty} \left[\frac{1}{(n+1)!} - \frac{1}{n!} \right] z^{-n} \ . \tag{14}$$

3 Arfken 11.6.11.

Consider the function

$$f_1(z) = \int_0^\infty e^{-zt} dt \ , \tag{15}$$

with t real. This function can be rewritten by using $z = x + iy$, as

$$f_1(z) = \int_0^\infty e^{-(x+iy)t} dt = \ , \tag{16}$$

which clearly is only convergent for $x > 0$. Therefore, the domain for which $f_1(z)$ exists, is Re(z) > 0. On this domain, f_1 can be evaluated,

$$f_1(z) = \int_0^\infty e^{-zt} dt = -\frac{1}{z} e^{-zt}\Big|_0^\infty = 0 - \left(-\frac{1}{z}\right) e^{-z(0)} = \frac{1}{z} \ , \tag{17}$$

which is defined everywhere but $z = 0$. So $f_2(z) = 1/z$ is therefore an analytic continuation of $f_1(z)$ over the entire z-plane except $z = 0$. This function can be rewritten as

$$\frac{1}{z} = \frac{1}{z - i + i} = \left(\frac{1}{-i}\right) \frac{1}{-(z+i)/i + 1} = \left(\frac{1}{-i}\right) \frac{1}{1 - (z+i)/i} = i \frac{1}{1 - (z+i)/i} \tag{18}$$

this function can be expanded around $z = -i$ as

$$f_3(z) = i \sum_{n=0}^{\infty} \left[\frac{z+i}{i}\right]^n = i \sum_{n=0}^{\infty} (i)^{-n} (z+i)^n \ , \tag{19}$$

which converges on $|(z+i)/i| < 1$, which if the modulus of i is brought to the other side, and noting that $|i| = 1$ implies that $f_3(z)$ converges for $|(z + i)| < 1$

4 Arfken 11.7.1.

To calculate residues, the formulas on page 510 of Arfken can be used:

$$(11.66)\ a_{-1} = \lim_{z \to z_0} \left((z - z_0) f(z) \right) \tag{20}$$

$$(11.68)\ a_{-1} = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[\frac{d^{n-1}}{dz^{n-1}} \left((z - z_0)^n f(z) \right) \right] , \tag{21}$$

where the first equation is for simple poles, and the second is for poles of order n. Secondarily, note that $z^2 + a^2$ can be factored to be $(z + ia)(z - ia)$, which has zeros at $\pm ia$. Note that for this case the residue of one pole can be conjugated to find the residue for the other. Because there are no poles higher than order 2 in this problem the Equation 21 becomes

$$a_{-1} = \lim_{z \to z_0} \left[\frac{d}{dz} \left((z - z_0)^2 f(z) \right) \right] . \tag{22}$$

4.1 $f(z) = \frac{1}{z^2+a^2}$

As noted above this function has two simple poles at $z_0 = \pm ia$. To find the residues, Equation 20 is used,

$$\text{Res}(ia) = \lim_{z \to ia} \left(\frac{(z - ia)}{(z - ia)(z + ia)} \right) = \frac{1}{2ia} \tag{23}$$

$$\text{Res}(-ia) = \lim_{z \to -ia} \left(\frac{(z - (-ia))}{(z - ia)(z + ia)} \right) = -\frac{1}{2ia} . \tag{24}$$

4.2 $f(z) = \frac{1}{(z^2+a^2)^2}$

The denominator of this function is squared, so it has two poles of order 2 at $z_0 = \pm ia$. To find the residues, Equation 22 is used,

$$\text{Res}(ia) = \lim_{z \to ai} \left[\frac{d}{dz} \left(\frac{(z - ia)^2}{(z - ia)^2 (z + ia)^2} \right) \right] = \lim_{z \to ia} -2(z + ia)^{-3} = \frac{-2}{8i^3 a^3} = \frac{1}{4a^3 i} \tag{25}$$

$$\text{Res}(-ia) = \lim_{z \to -ai} \left[\frac{d}{dz} \left(\frac{(z - (-ia))^2}{(z - ia)^2 (z + ia)^2} \right) \right] = \lim_{z \to -ia} -2(z - ia)^{-3} = \frac{2}{8i^3 a^3} = -\frac{1}{4a^3 i} . \tag{26}$$

4.3 $f(z) = \frac{z^2}{(z^2+a^2)^2}$

The denominator of this function is squared, so it has two poles of order 2 at $z_0 = \pm ia$. To find the residues, Equation 22 is used,

$$\text{Res}(ia) = \lim_{z \to ai} \left[\frac{d}{dz} \left(\frac{z^2 (z - ia)^2}{(z - ia)^2 (z + ia)^2} \right) \right] = \lim_{z \to ia} \left[-\frac{2z^2}{(z + ia)^3} + \frac{2z}{(z + ia)^3} \right] = -\frac{i}{4a} \tag{27}$$

$$\text{Res}(-ia) = \lim_{z \to ai} \left[\frac{d}{dz} \left(\frac{z^2 (z - (-ia))^2}{(z - ia)^2 (z + ia)^2} \right) \right] = \lim_{z \to -ia} \left[-\frac{2z^2}{(z - ia)^3} + \frac{2z}{(z - ia)^3} \right] = \frac{i}{4a} . \tag{28}$$

4.4 $f(z) = \frac{\sin(1/z)}{z^2+a^2}$

This function has two simple poles at $z_0 = \pm ia$. To find the residues at $\pm ia$, Equation 20 is used,

$$\text{Res}(ia) = \lim_{z\to ia}\left(\frac{\sin(1/z)(z-ia)}{(z-ia)(z+ia)}\right) = \frac{\sin(-i/a)}{2ia} = -\frac{\sinh 1/a}{2a} \tag{29}$$

$$\text{Res}(-ia) = \lim_{z\to -ia}\left(\frac{\sin(1/z)(z-(-ia))}{(z-ia)(z+ia)}\right) = -\frac{\sin(i/a)}{2ia} = -\frac{\sinh 1/a}{2a}\ . \tag{30}$$

This function also has an essential singularity at $z_0 = 0$, whose residue can be found by Laurent expanding the numerator and denominator and looking for the coefficient of the $\frac{1}{z}$ term. The expansion of the denominator has a pole at $z = \pm ia$, so for $|z| < a$ the expansion is

$$\frac{1}{z^2+a^2} = \frac{1}{a^2\left((z/a)^2+1\right)} = \frac{1}{a^2}\sum_{n=0}^{\infty}\left(\frac{z^2}{a^2}\right)^n = \sum_{n=0}^{\infty} z^{2n}a^{-2n-2}\ , \tag{31}$$

where the sum was introduced as in Arfken Example 11.5.1, and using the binomial expansion. When $|z| > a$, the expansion is

$$\frac{1}{z^2+a^2} = \frac{1}{z^2\left(1+(a/z)^2\right)} = \frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{a^2}{z^2}\right)^n = \sum_{n=0}^{\infty} a^{2n}z^{-2n-2}\ . \tag{32}$$

Now expand the numerator using a Taylor series,

$$\sin(1/z) = \sum_{n=0}^{\infty}(-1)^n\frac{(1/z)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{-(2n+1)}}{(2n+1)!}\ . \tag{33}$$

Using these expansions, the complete series representing $f(z)$ is

$$\frac{\sin(1/z)}{z^2+a^2} = \left[\sum_{n=0}^{\infty} z^{2n}a^{-2n-2} + \sum_{n=0}^{\infty} a^{2n}z^{-2n-2}\right]\sum_{n=0}^{\infty}(-1)^n\frac{z^{-(2n+1)}}{(2n+1)!}\ . \tag{34}$$

The z^{-1} term will come from the positive powers of z from the $1/(z^2+a^2)$ series multiplied by the sine series:

$$\sum_{n=0}^{\infty}(-1)^n\frac{z^{-(2n+1)}}{(2n+1)!}z^{2n}a^{-2n-2} = \sum_{n=0}^{\infty}\frac{(-1)^n a^{-2(n+1)}}{(2n+1)!}z^{2n-2n-1} \tag{35}$$

$$= \frac{1}{za}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{a}\right)^{2n+1} \tag{36}$$

$$= \left[\frac{1}{a}\sin(1/a)\right]\frac{1}{z}\ , \tag{37}$$

which says

$$\text{Res}(0) = a_{-1} = \left[\frac{1}{a}\sin(1/a)\right]\ . \tag{38}$$

4.5 $f(z) = \frac{ze^{iz}}{z^2+a^2}$

This function has two simple poles at $z_0 = \pm ia$. To find the residues, Equation 20 is used,

$$\text{Res}(ia) = \lim_{z\to ia}\left(\frac{ze^{iz}(z-ia)}{(z-ia)(z+ia)}\right) = \frac{iae^{i(ia)}}{2ia} = \frac{e^{-a}}{2} \tag{39}$$

$$\text{Res}(-ia) = \lim_{z\to -ia}\left(\frac{ze^{iz}(z-(-ia))}{(z-ia)(z+ia)}\right) = \frac{-iae^{i(-ia)}}{-2ia} = \frac{e^{a}}{2}\ . \tag{40}$$

4.6 $f(z) = \frac{ze^{iz}}{z^2-a^2}$

This function has two simple poles at $z_0 = \pm a$. To find the residues, Equation 20 is used,

$$\text{Res}(a) = \lim_{z\to a}\left(\frac{ze^{iz}(z-a)}{(z-a)(z+a)}\right) = \frac{ae^{i(a)}}{2a} = \frac{e^{ia}}{2} \tag{41}$$

$$\text{Res}(-a) = \lim_{z\to -a}\left(\frac{ze^{iz}(z-(-a))}{(z-a)(z+a)}\right) = \frac{-ae^{i(-a)}}{-2a} = \frac{e^{-ia}}{2}\ . \tag{42}$$

4.7 $f(z) = \frac{e^{iz}}{z^2-a^2}$

This function has two simple poles at $z_0 = \pm a$. To find the residues, Equation 20 is used,

$$\text{Res}(a) = \lim_{z\to a}\left(\frac{e^{iz}(z-a)}{(z-a)(z+a)}\right) = \frac{e^{i(a)}}{2a} = \frac{e^{ia}}{2a} \tag{43}$$

$$\text{Res}(-a) = \lim_{z\to -a}\left(\frac{e^{iz}(z-(-a))}{(z-a)(z+a)}\right) = \frac{e^{i(-a)}}{-2a} = -\frac{e^{-ia}}{2a}\ . \tag{44}$$

4.8 $f(z) = \frac{z^{-k}}{z+1}$ for $0 < k < 1$

This function has one simple pole at $z_0 = -1$. To find the residues, Equation 20 is used,

$$\text{Res}(-1) = \lim_{z\to -1}\left(\frac{z^{-k}(z-(-1))}{(z+1)}\right) = (-1)^{-k} = (e^{i\pi})^{-k} = e^{-i\pi k}\ . \tag{45}$$

5 Arfken 11.7.4.

The principal value of the integral

$$P = \fint_0^\infty \frac{x^{-p}}{x-1}dx \ , \tag{46}$$

can be found by splitting the integral at $x = 1$, by introducing a small parameter δ. Ignoring the limit as the small parameter approaches zero, for now, this is denoted by $\bar{P}$,

$$\bar{P} = \int_0^{1-\delta} \frac{x^{-p}}{x-1}dx + \int_{1+\delta}^\infty \frac{x^{-p}}{x-1}dx = -\int_0^{1-\delta} x^{-p}\frac{1}{1-x}dx + \int_{1+\delta}^\infty x^{-p}\frac{1/x}{1-1/x}dx \tag{47}$$

$$= -\int_0^{1-\delta} x^{-p}\sum_{n=0}^\infty x^n dx + \int_{1+\delta}^\infty x^{-p}\frac{1}{x}\frac{1}{1-(1/x)}dx = -\sum_{n=0}^\infty \int_0^{1-\delta} x^{n-p}dx + \int_{1+\delta}^\infty x^{-p-1}\sum_{n=0}^\infty \left(\frac{1}{x}\right)^n \tag{48}$$

$$= -\sum_{n=0}^\infty \int_0^{1-\delta} x^{n-p}dx + \sum_{n=0}^\infty \int_{1+\delta}^\infty x^{-n-p-1} \tag{49}$$

$$= -\sum_{n=0}^\infty \frac{x^{n-p+1}}{n-p+1}\bigg|_{1-\delta} + \sum_{n=0}^\infty \left[-\frac{x^{-n-p}}{n+p}\right]\bigg|_{1+\delta}^\infty \tag{50}$$

$$= -\sum_{n=0}^\infty \frac{(1-\delta)^{n-p+1}}{n-p+1} + \sum_{n=0}^\infty \frac{(1+\delta)^{-n-p}}{n+p} \tag{51}$$

$$= -\sum_{n=0}^\infty \left[\frac{(1-\delta)^{n-p+1}}{n-p+1} - \frac{(1+\delta)^{-n-p}}{n+p}\right] \ . \tag{52}$$

Taking the limit as the small parameter approaches zero is

$$P = \lim_{\delta\to 0}\bar{P} = \sum_{n=0}^\infty \left[\frac{-1}{n-p+1} + \frac{1}{n+p}\right] \ , \tag{53}$$

under a change of variable in the first sum such that $m = n + 1$ and isolate the first term in the seconod sum, this becomes

$$P = \sum_{m=1}^\infty \frac{-1}{m-p} + \frac{1}{p} + \sum_{n=1}^\infty \frac{1}{n+p} = \frac{1}{p} + \sum_{n=1}^\infty \left[\frac{1}{n+p} - \frac{1}{n-p}\right] \ , \tag{54}$$

after changing the dummy index m back to n. Simplifying this some more makes it

$$P = \frac{1}{p} + \sum_{n=1}^\infty \left[\frac{n-p-n-p}{n^2-p^2}\right] = \frac{1}{p} + 2p\sum_{n=1}^\infty \frac{1}{p^2-n^2} \ . \tag{55}$$

Note the pole expansion of $\cot z$ given by Arfken Equation 11.81 can be written

$$\cot(z) = \frac{1}{z} + 2z\sum_{n=1}^\infty \frac{1}{z^2-(n\pi)^2} \ \Rightarrow \ \cot(p\pi) = \frac{1}{p\pi} + 2p\pi\sum_{n=1}^\infty \frac{1}{(p\pi)^2-(n\pi)^2} = \frac{1}{p\pi} + \frac{2p}{\pi}\sum_{n=1}^\infty \frac{1}{p^2-n^2} \ , \tag{56}$$

from this it is easy to see

$$\fint_0^\infty \frac{x^{-p}}{x-1}dx = \pi\cot(\pi p) \ , \tag{57}$$

which differs from Arfken by a factor of -1.

6 Arfken 11.7.8.

6.1 Secant Pole Expansion

Using the Mittag-Leffler theorem the pole expansion of $\sec z$ can be written as

$$\sec z = \sec 0 + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z - z_n} + \frac{1}{z_n} \right) \ . \tag{58}$$

Secant has poles at odd multiples of $\pi/2$, so that $z_0 = \pm(2n+1)\pi/2$, for integer n; each pole is a simple pole. The residues can be calculated using L'Hôpital's rule,

$$b_n = \lim_{z \to (2n+1)\pi/2} \frac{z - (2n+1)\pi/2}{\cos z} = \lim_{z \to (2n+1)\pi/2} -\frac{1}{\sin z} = (-1)^{n+1} \ , \tag{59}$$

for $n = 0$, the residue must be -1, hence the $n+1$. However, the positive and negative poles of the same absolute value have opposite sign residues, so the sums are subtracted instead of added, making Equation 58 (following the method in Arfken) into

$$\sec z = 1 + \sum_{n=0}^{N} (-1)^{n+1} \left(\frac{1}{z - \frac{m\pi}{2}} + \frac{1}{\frac{m\pi}{2}} \right) - \sum_{n=0}^{N} (-1)^{n+1} \left(\frac{1}{z + \frac{m\pi}{2}} + \frac{1}{-\frac{m\pi}{2}} \right) \ , \tag{60}$$

where $m = 2n+1$. Combing the terms in the sums yields

$$\sec z = 1 + \sum_{n=0}^{N} (-1)^{n+1} \left(\frac{1}{z - \frac{m\pi}{2}} + \frac{1}{\frac{m\pi}{2}} - \frac{1}{z + \frac{m\pi}{2}} - \frac{1}{-\frac{m\pi}{2}} \right) \tag{61}$$

$$= 1 + \sum_{n=0}^{N} (-1)^{n+1} \left(\frac{z - \frac{m\pi}{2} - z - \frac{m\pi}{2}}{z^2 - \left(\frac{m\pi}{2}\right)^2} + \frac{2}{\frac{m\pi}{2}} \right) \tag{62}$$

$$= 1 + \sum_{n=0}^{N} (-1)^{n+1} \left(\frac{-m\pi}{z^2 - \left(\frac{m\pi}{2}\right)^2} + \frac{4}{m\pi} \right) \tag{63}$$

$$= 1 + \sum_{n=0}^{N} (-1)^{n+1} \left(\frac{-m\pi}{z^2 - \left(\frac{m\pi}{2}\right)^2} \right) + \sum_{n=0}^{N} (-1)^{n+1} \left(\frac{4}{m\pi} \right) \ , \tag{64}$$

Evaluating the sum with no z dependence in the limit $N \to \infty$, using MATHEMATICA yields,

$$\sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{4}{m\pi} \right) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{4}{(2n+1)\pi} = -1 \ , \tag{65}$$

making the expression for secant, in the limit $N \to \infty$, into

$$\sec(z) = \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{-m\pi}{z^2 - \left(\frac{m\pi}{2}\right)^2} \right) = \pi \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{-(2n+1)}{z^2 - \left(\frac{m\pi}{2}\right)^2} \right) \tag{66}$$

$$= \pi \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{2n+1}{\left(\frac{(2n+1)\pi}{2}\right)^2 - z^2} \right) \ , \tag{67}$$

which is exactly the form of Arfken Equation 11.82.

6.2 Cosecant Pole Expansion

Cosecant has simple poles at integer multiples of π, so $z_0 = n\pi$, with residues given by (using L'Hôpital's rule)

$$b_n = \lim_{z\to n\pi} \frac{z-n\pi}{\sin z} = \lim_{z\to n\pi} \frac{1}{\cos z} = (-1)^n \ , \tag{68}$$

for $n = 0$ the pole must be $+1$, hence the power of n. Following the method in Arfken used for cotangent, the expansion becomes

$$\csc(z) - \frac{1}{z} = \sum_{n=1}^{N} (-1)^n \left[\frac{1}{z-n\pi} + \frac{1}{n\pi} + \frac{1}{z+n\pi} + \frac{1}{-n\pi} \right] \ . \tag{69}$$

Adding the $1/z$ term removes the simple pole at $z = 0$ because for small z,

$$\csc(z) - \frac{1}{z} = \frac{1}{\sin(z)} - \frac{1}{z} = \frac{1}{z} - \frac{1}{z} = 0 \ . \tag{70}$$

Now Equation 69 becomes

$$\csc(z) - \frac{1}{z} = \sum_{n=1}^{N} (-1)^n \left[\frac{1}{z-n\pi} + \frac{1}{z+n\pi} \right] = \sum_{n=1}^{N} (-1)^n \left[\frac{2z}{z^2-(n\pi)^2} \right] \ , \tag{71}$$

so that

$$\csc(z) = \frac{1}{z} + \lim_{N\to\infty} \sum_{n=1}^{N} (-1)^n \left[\frac{2z}{z^2-(n\pi)^2} \right] \tag{72}$$

$$= \frac{1}{z} - 2z \sum_{n=1}^{N} (-1)^{n+1} \left[\frac{1}{z^2-(n\pi)^2} \right] \ , \tag{73}$$

which is exactly the form of Arfken Equation 11.83.