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1 Problem #1: Generalized Gauss' Divergence Theorem.

Let $f(\mathbf{a}, \mathbf{r})$ satisfy the condition

$$f(c_1\mathbf{a}_1 + c_2\mathbf{a}_2, \mathbf{r}) = c_1 f(\mathbf{a}_1, \mathbf{r}) + c_2 f(\mathbf{a}_2, \mathbf{r}) \ , \tag{1}$$

where c_1 and c_2 are arbitrary constants, and suppose further that $f(\mathbf{a}, \mathbf{r})$ is a differentiable function of $\mathbf{r}$. Using Gauss' theorem in Cartesian coordinates, show that if V is an arbitrary volume, S its bounding surface, and $\mathbf{n}$ a unit normal to this surface, then

$$\oint_S f(\mathbf{n}, \mathbf{r}) dS = \int_V f(\nabla, \mathbf{r}) dV \ , \tag{2}$$

which is the generalized Gauss' divergence theorem. Here ∇ in the integrand on the RHS operates on $\mathbf{r}$ and lies on the left of all variables.

From Equation 1, we can write, in Cartesian coordinates,

$$f(\mathbf{n}, \mathbf{r}) = f(n_x\hat{\mathbf{x}} + n_y\hat{\mathbf{y}} + n_z\hat{\mathbf{z}}, \mathbf{r}) = n_x f(\hat{\mathbf{x}}, \mathbf{r}) + n_y f(\hat{\mathbf{y}}, \mathbf{r}) + n_z f(\hat{\mathbf{z}}, \mathbf{r}) = \mathbf{n} \cdot \mathbf{F} \ , \tag{3}$$

where the new vector has components defined as

$$F_i = f(\hat{\mathbf{e}}_i, \mathbf{r}) \ , \tag{4}$$

with $\hat{\mathbf{e}}_i \in \{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$. Similarly, we can write

$$f(\nabla, \mathbf{r}) = f(\partial_x\hat{\mathbf{x}} + \partial_y\hat{\mathbf{y}} + \partial_z\hat{\mathbf{z}}, \mathbf{r}) = \partial_x f(\hat{\mathbf{x}}, \mathbf{r}) + \partial_y f(\hat{\mathbf{y}}, \mathbf{r}) + \partial_z f(\hat{\mathbf{z}}, \mathbf{r}) = \nabla \cdot \mathbf{F} \ , \tag{5}$$

where ∂_i is the partial derivative operator with respect to the coordinate $w_i \in \{x, y, z\}$, and $\mathbf{F}$ is defined by Equation 4. Now, apply the divergence theorem to this vector,

$$\int_V \nabla \cdot \mathbf{F} dV = \oint_S \mathbf{F} \cdot \mathbf{n} \ , \tag{6}$$

which when we substitute in the relations in Equations 3 and 5, we get the result

$$\int_V f(\nabla, \mathbf{r}) dV = \oint_S f(\mathbf{n}, \mathbf{r}) dS \tag{7}$$

which proves Equation 2.

2 Problem #2: Electric Field Boundary Conditions.

In class, we derived the boundary conditions on the normal and tangential components of the $\mathbf{E}$ field. The boundary condition on the normal component was expressed as $(\mathbf{E}_2 - \mathbf{E}_1) \cdot \hat{\mathbf{n}} = \sigma$, where $\hat{\mathbf{n}}$ is in the direction of the normal to the surface from region 1 to region 2. Show that the boundary condition on the tangential component can be expressed as $\hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0$.

Consider a an electric field $\mathbf{E}$ and a surface S, near any point on this surface we can approximate the surface to be a plane, with normal vector $\hat{\mathbf{n}}$. The components of the electric field which are tangential to the surface are given by

$$\mathbf{E}_{||} = \mathbf{E} - (\hat{\mathbf{n}} \cdot \mathbf{E})\hat{\mathbf{n}} \ , \tag{8}$$

which is just the electric field minus the component of the electric field in the direction normal to the surface. Note that if we now cross this into the normal vector, this becomes

$$\hat{\mathbf{n}} \times \mathbf{E}_{||} = \hat{\mathbf{n}} \times \mathbf{E} - (\hat{\mathbf{n}} \cdot \mathbf{E})\hat{\mathbf{n}} \times \hat{\mathbf{n}} = \hat{\mathbf{n}} \times \mathbf{E} \ . \tag{9}$$

We can apply this result to the interface of two regions to express the behavior of the tangential components of the electric field on either side of the interface. In region 1 we have electric field $\mathbf{E}_1$, and similarly for $\mathbf{E}_2$ and the normal vector always points from region one to region two. With this information and the relationship in Equation 9 we find

$$\hat{\mathbf{n}} \times \mathbf{E}_{||}^{(1)} = \hat{\mathbf{n}} \times \mathbf{E}_1 \quad \text{and} \quad \hat{\mathbf{n}} \times \mathbf{E}_{||}^{(2)} = \hat{\mathbf{n}} \times \mathbf{E}_2 \ , \tag{10}$$

which if we subtract the first from the second we get

$$\hat{\mathbf{n}} \times \mathbf{E}_{||}^{(2)} - \hat{\mathbf{n}} \times \mathbf{E}_{||}^{(1)} = \hat{\mathbf{n}} \times \mathbf{E}_2 - \hat{\mathbf{n}} \times \mathbf{E}_1 \tag{11}$$

$$\hat{\mathbf{n}} \times (\mathbf{E}_{||}^{(2)} - \mathbf{E}_{||}^{(1)}) = \hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) \ . \tag{12}$$

From class, the boundary conditions for the electric field are

$$E_{\perp}^{(2)} - E_{\perp}^{(1)} = \frac{\sigma}{\epsilon_0} \quad \text{and} \quad \mathbf{E}_{||}^{(2)} - \mathbf{E}_{||}^{(1)} = 0 \ , \tag{13}$$

so using Equation 13 on Equation 12 yields the desired result

$$0 = \hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) \ . \tag{14}$$

The second boundary condition can be quickly shown by considering a rectangular closed contour, perpendicular to the plane of the surface, that extends infinitesimally into each region. Now we can use the integral form of Faraday's law to write

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{\mathbf{n}} \ da \ , \tag{15}$$

which for constant $\mathbf{E}$ is equal to zero because there is no time-varying magnetic field. We can expand the contour integral to its four constituent line integrals and say the integrals along the infinitesimal legs do not contribute, or that

$$0 = \int_0^L \mathbf{E}_2 \cdot d\boldsymbol{\ell} - \int_0^L \mathbf{E}_1 \cdot d\boldsymbol{\ell} \tag{16}$$

where L is the length of the long arms of the contour. If we let the diretion of $d\boldsymbol{\ell}$ lie in the same direction as the projection of $\mathbf{E}$ into the plane of the interface between the two regions, then

$$0 = \int_0^L \mathbf{E}_2 \cdot d\boldsymbol{\ell} - L\mathbf{E}_{||}^{(1)} \ , \tag{17}$$

which is only true if $\mathbf{E}_{||}^{(2)}$ lies in the same direction as $\mathbf{E}_{||}^{(1)}$ with the same magnitude (so the length of the loop can be dropped). Thus proving the relevant (second) boundary condition given in Equation 13.

3 Problem #3: Self Capacitance of Ellipsoid of Revolution.

Use the derivation of the potential of a line charge that we did in class to show that the self-capacitance C of an ellipsoid of revolution, generated by rotating an ellipse about its major axis is given by

$$\frac{1}{C} = \frac{1}{\sqrt{a^2-b^2}} \ln \left| \frac{\sqrt{a^2-b^2}+a}{b} \right| \ . \tag{18}$$

Here a is the semi-major axis and b is the semi-minor axis of the ellipse.

Let us define elliptical cylindrical coordinates such that

$$x = c \cosh u \cos v \tag{19}$$
$$y = c \sinh u \sin v \tag{20}$$
$$z = z \ , \tag{21}$$

where c is the length from the origin to a focus. Consider now the shape of a constant u curve in the $x-y$ plane, while $v \in [0, 2\pi]$. When v maximizes the x-value, the y-value is zero (and vice-versa). Additionally, $\cosh u > \sinh u \ \forall \ u \in (0, \infty)$, so it must be that

$$a = c\cosh u \quad \text{and} \quad b = c \sinh u \ , \tag{22}$$

where a is the semi-major axis and b is the semi-minor axis. Using the identity for hyperbolic cosines and sines, we find $a^2 - b^2 = c^2$.

Changing gears now, we found in class the potential created by a total line charge q over length $2c$, in these coordinates is

$$\phi(u,v) = \frac{q}{2c} \ln \left| \frac{1+\cosh u}{1-\cosh u} \right| = \frac{q}{2c} \ln \left| \frac{\cosh u + 1}{\cosh u - 1} \right| \ , \tag{23}$$

note that for a constant u the potential is constant, in the elliptical cylindrical coordinates this is equivalent to an ellipsoid of revolution with the same potential across its surface. This ellipsoid would have a focal length of c as, so the ellipse with constant potential is defined by the semi-major and -minor axes given by Equation 22, from this we can write

$$\phi = \frac{q}{2c} \ln \left| \frac{\frac{a}{c}+1}{\frac{a}{c}-1} \right| = \frac{q}{2c} \ln \left| \frac{a+c}{a-c} \right| \ , \tag{24}$$

and multiplying the argument of the logarithm with its conjugate yields

$$\phi = \frac{q}{2c} \ln \left| \frac{a+c}{a-c} \left(\frac{a+c}{a+c} \right) \right| = \frac{q}{2c} \ln \left| \frac{(a+c)^2}{a^2-c^2} \right| = \frac{1}{2} \frac{q}{\sqrt{a^2-b^2}} \ln \left| \frac{(a+\sqrt{a^2-b^2})^2}{b^2} \right| \ , \tag{25}$$

from the results of Equation 22. We can now take the coefficient of one half and apply it as an exponent to the argument of the logarithm, taking the square root, with result

$$\phi = \frac{q}{\sqrt{a^2-b^2}} \ln \left| \frac{\sqrt{a^2-b^2}+a}{b} \right| \ , \tag{26}$$

which is the equation of an ellipse with constant potential of semi-major axis a and semi-minor axis b. We can now compare this surface to the potential at infinity ϕ_∞, which we will define to be zero, to find the self-capacitance C of this surface,

$$\frac{1}{C} = \frac{\Delta\phi}{q} = \frac{\phi - \phi_\infty}{q} = \frac{1}{\sqrt{a^2 - b^2}} \ln \left| \frac{\sqrt{a^2 - b^2} + a}{b} \right| \ , \tag{27}$$

which proves Equation 18 because the argument of the logarthim is positive-definite.

4 Problem #4: Force Between Point Charge and Grounded Sphere.

Show that the potential due to a point charge q placed a distance d from the center of a grounded conducting sphere of radius $a < d$ is given by the contribution of the charge q plus the contribution of an image charge $q' = -(a/d)q$ located a distance $d' = a^2/d$ from the origin along the line from q to the center of the sphere. Find the force between the charge q and the sphere.

The image charge must be located inside the sphere, because it cannot be in the region we are solving for the potential. Additionally, due to the spherical symmetry of the problem, the image charge must lay along the same line that connects the charge q with the origin. We consider an arbitrary charge q' at a distance d' from the origin, and ignore the sphere. This creates the potential

$$V(r) = \frac{kq}{|d-r|} + \frac{kq'}{|d'-r|} \ , \tag{28}$$

at a distance r along the same line the charges lie on. We know the potential on the sphere is zero because it is grounded, so

$$V(a) = 0 = \frac{kq}{d-a} + \frac{kq'}{a-d'} \ , \tag{29}$$

where both denominators are positive because the image charge is inside the radius of the sphere $d' < a$. We can factor a length scale out of both denominators, and divide through by the coupling constant k,

$$0 = \frac{q}{d(1-\frac{a}{d})} + \frac{q'}{a(1-\frac{d'}{a})} \ . \tag{30}$$

Clearly this equation is satisfied if

$$\frac{q}{d} = -\frac{q'}{a} \quad \text{and} \quad \frac{a}{d} = \frac{d'}{a} \tag{31}$$

are simultaneously true. We can solve these conditions for the primed parameters and see that

$$q' = -\frac{a}{d}q \quad \text{and} \quad d' = \frac{a^2}{d} \ . \tag{32}$$

Given this we can find the force between the sphere and the charge using the Coulomb force law with the image charge,

$$F = \frac{kqq'}{(d-d')^2} = -kq^2\frac{\frac{a}{d}}{(d-\frac{a^2}{d})^2} = -kq^2\frac{\frac{a}{d}}{(d^2-a^2)^2\frac{1}{d^2}} = -\frac{k(ad)q^2}{(d^2-a^2)^2} \ , \tag{33}$$

this force is always attractive, and in the radial direction $\hat{\mathbf{r}}$ towards the center of the sphere from the point charge.

5 Problem #5: Variations of Previous Problem.

Consider two variations of Problem #4.

5.1 Charged Sphere.

If instead of the sphere being grounded, we consider an ungrounded sphere with a total charge Q, determine the potential at a point (r, θ) outside the sphere.

In the previous case the sphere had some total charge that was distributed about the surface to create an image charge q' to balance the forces created by the point charge q[1]. Now if the sphere has total charge Q, it will still use the same amount to create the image charge in the exact same location, and the remaining charge will distribute itself uniformly about the surface. This allows the amount of charge left over after the image charge to be treated as a point charge at the center of the sphere. Remember from the previous problem the charge q' is located at a distance a^2/d in the $\hat{\mathbf{d}}$ direction, so

$$\mathbf{d}' = \left(\frac{a^2}{d}\right)\frac{\mathbf{d}}{d} \,. \tag{34}$$

From this we can write down the potential at a location $\mathbf{r}$ from the three contributions, ϕ_q, $\phi_{q'}$, $\phi_{Q-q'}$,

$$\phi(\mathbf{r}) = \phi_q(\mathbf{r}) + \phi_{q'}(\mathbf{r}) + \phi_{Q-q'}(\mathbf{r}) = \frac{kq}{|\mathbf{r}-\mathbf{d}|} - \frac{akq}{d|\mathbf{r}-\frac{a^2}{d^2}\mathbf{d}|} + \frac{k(Q+\frac{a}{d}q)}{|\mathbf{r}|} \,, \tag{35}$$

where the first two terms are the result from the previous section in vector notation. Note that $\mathbf{d}$ is the vector from the origin to the location of the point charge, which has magnitude d.

We can use the law of cosines to represent the magnitude of the difference of vectors in terms of their respective magnitudes and relative angle θ,

$$\phi(r,\theta) = k\left[\frac{q}{\sqrt{r^2+d^2-2rd\cos\theta}} - \left(\frac{a}{d}\right)\frac{q}{\sqrt{r^2+\left(\frac{a^2}{d^2}\right)^2 d^2 - 2r\left(\frac{a^2}{d^2}\right)d\cos\theta}} + \frac{Q+\frac{a}{d}q}{r}\right] \tag{36}$$

$$= kq\left[\frac{1}{\sqrt{r^2+d^2-2rd\cos\theta}} - \frac{1}{\sqrt{\left(\frac{d}{a}\right)^2\left[r^2+\left(\frac{a^4}{d^2}\right)-2r\left(\frac{a^2}{d^2}\right)d\cos\theta\right]}} + \frac{\frac{Q}{q}+\frac{a}{d}}{r}\right] \tag{37}$$

$$= kq\left[\frac{1}{\sqrt{r^2+d^2-2rd\cos\theta}} - \frac{1}{\sqrt{\left(\frac{d}{a}\right)^2 r^2 + a^2 - 2rd\cos\theta}} + \frac{\frac{Q}{q}+\frac{a}{d}}{r}\right] \,, \tag{38}$$

where r is the radial coordinate of the point at which we are interested in the potential and θ is the normal polar angle (if we define the x axis to be the line along which the center of the sphere and point charge lie). Note that this system is azimuthally symmetrical so the φ coordinate does not enter into the expression.

[1]See Jackson page 61.

5.2 Arbitrary Constant Potential.

The next variation is a point charge q is kept in the vicinity of a conducting sphere whose surface is kept at a potential ϕ_0. As before, determine the potential $\phi(r, \theta)$ outside the sphere.

This is the exact same scenario as section 5.1, but with the charge $Q - q'$ replaced with the charge given by a constant potential ϕ_0[2],

$$Q_{\phi_0} = \frac{1}{k}\phi_0 a \ , \tag{39}$$

so the potential (from Equation 35) is

$$\phi(r, \theta) = \frac{kq}{|\mathbf{r} - \mathbf{d}|} - \frac{akq}{d|\mathbf{r} - \frac{a^2}{d^2}\mathbf{d}|} + \frac{k(\frac{1}{k}\phi_0 a)}{|\mathbf{r}|} \ , \tag{40}$$

which can be expanded using the law of cosines as was done previously,

$$\phi(r, \theta) = kq \left[\frac{1}{\sqrt{r^2 + d^2 - 2rd\cos\theta}} - \frac{1}{\sqrt{\left(\frac{d}{a}\right)^2 r^2 + a^2 - 2rd\cos\theta}} \right] + \frac{\phi_0 a}{r} \ , \tag{41}$$

using the same coordinate system described in section 5.1.

[2]See Jackson page 62.

6 Problem #6: Ensemble of Point Charges.

If an ensemble of charges q_i $(i = 1, 2)$ sitting at the postions $(r_i, \theta_i, \varphi_i)$ create the potential $\phi(r, \theta, \varphi)$ at the point (r, θ, φ), show that the potential originating from the ensemble of charges $q_i' = (a/r_i)q_i$, inverted with respect to a sphere of radius $a < r_i$ and thus sitting at the positions $(a^2/r_i, \theta_i, \varphi_i)$ is given by $\phi'(r, \theta, \varphi) = (a/r)\phi(a^2/r, \theta, \phi)$.

We begin by writing the potential at a location $\mathbf{r}(r, \theta, \varphi)$ due to an ensemble of charges q_i at positions $\mathbf{r}_i(r_i, \theta_i, \varphi_i)$,

$$\phi(\mathbf{r}) = \sum_i \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|} \ , \tag{42}$$

which by the law of cosines is

$$\phi(\mathbf{r}) = \phi(r, \theta, \varphi) = \sum_i q_i \left[|\mathbf{r}|^2 + |\mathbf{r}_i|^2 + 2|\mathbf{r}||\mathbf{r}_i| \cos\gamma \right]^{-1/2} = \sum_i q_i \left[r^2 + r_i^2 + 2rr_i \cos\gamma \right]^{-1/2} \ , \tag{43}$$

where γ is the angle between the charges position and the position of interest. Now consider the potential from an ensemble of charges q_i', which are related to the previous ensemble by $q_i' = (a/r_i)q_i$, and their locations are inverted with respect to a sphere $r_i' = a^2/r_i$, with the same angular components $\varphi_i' = \varphi_i$ and $\theta_i' = \theta_i$. This is given by

$$\phi'(\mathbf{r}) = \phi'(r, \theta, \varphi) = \sum_i \frac{a}{r_i}(q_i) \frac{1}{|\mathbf{r} - \mathbf{r}_i'|} = \sum_i \frac{a}{r_i}(q_i) \left[r^2 + \left(\frac{a^2}{r_i}\right)^2 + 2r\frac{a^2}{r_i}\cos\gamma \right]^{-1/2} \ , \tag{44}$$

we can pull a factor of $1/r$ out of the square root,

$$\phi'(r, \theta, \varphi) = \sum_i \frac{a}{r_i}(q_i)(r^2)^{-1/2} \left[1 + \left(\frac{a^2}{rr_i}\right)^2 + 2\frac{a^2}{rr_i}\cos\gamma \right]^{-1/2} \ , \tag{45}$$

and pull in a factor of $1/r_i$,

$$\phi'(r, \theta, \varphi) = \sum_i \frac{a}{r}(q_i)(r_i^2)^{-1/2} \left[1 + \left(\frac{a^2}{rr_i}\right)^2 + 2\frac{a^2}{rr_i}\cos\gamma \right]^{-1/2} \tag{46}$$

$$= \frac{a}{r} \sum_i q_i \left[r_i^2 + \left(\frac{a^2}{r}\right)^2 + 2\frac{a^2}{r} r_i \cos\gamma \right]^{-1/2} \ . \tag{47}$$

By comparison with Equation 43, we see that this expression becomes

$$\phi'(r, \theta, \varphi) = \frac{a}{r}\phi\left(a^2/r, \theta, \varphi\right) \ , \tag{48}$$

which can be interpreted as the potential due to the image charges relative to a sphere of radius a.

7 Problem #7: Green's Function Technique - Conducting Sphere.

Using the Green's function technique, show that the potential outside a conducting sphere of radius a with a potential distribution $\phi(a, \theta, \varphi)$ on its surface is given by

$$\phi(\mathbf{r}) = \frac{1}{4\pi} \oint_S \phi(a, \theta', \varphi') \frac{a(r^2 - a^2)}{(r^2 + a^2 - 2ar\cos\gamma)^{3/2}} d\Omega' \ . \tag{49}$$

Assume the charge density outside the sphere is zero.

The potential from a point charge and its image which satisfy the homogeneous boundary conditions on the surface of the conducting sphere of radius a (and at infinity) is simply the Green's function for the appropriate type of boundary conditions (Dirichlet in this case)[3]. From the defintion of the Green's function,

$$\nabla_{\mathbf{r}'} G(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}') \ , \tag{50}$$

we can use the potential given by Equation 28, in vector notation, with $q \to 1/k$. Substituting in the relations for the primed parameters $\mathbf{d}'$ and q' (of the image charge), we get

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{(a/d)}{|\mathbf{r} - \frac{a^2}{d^2}\mathbf{r}'|} \ , \tag{51}$$

where $\mathbf{r}$ is the coordinate position at which the potential is being evaluated and $\mathbf{r}'$ is the location of the unit source, with $|\mathbf{r}'| = d$. Using the law of cosines, as in the previous problems, this can be written in spherical coordinates (with no φ dependence because the system is rotationally invariant) as

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{\sqrt{r^2 + d^2 - 2rd\cos\gamma}} - \frac{(a/d)}{\sqrt{r^2 + \left(\frac{a^2}{d^2}\right)^2 d^2 - 2r\left(\frac{a^2}{d^2}\right) d\cos\gamma}} \tag{52}$$

$$= \frac{1}{\sqrt{r^2 + d^2 - 2rd\cos\gamma}} - \frac{1}{\sqrt{\left(\frac{d^2}{a^2}\right) r^2 + a^2 - 2rd\cos\gamma}} \ . \tag{53}$$

Using this method with Dirichlet boundary conditions, the potential at $\mathbf{r}$ is given by Jackson Equation 1.44. This formula requires the partial deriviative of the Green's function with respect to a normal vector pointing away from the volume of interest. We are interested in the region outside the conducting sphere (but within the infinite spherical shell) so the normal vector points inwards from the point charge,

$$\mathbf{n} = -\mathbf{r}' \quad \Rightarrow \quad n = |\mathbf{n}| = -|\mathbf{r}'| = -d \ , \tag{54}$$

so the derivative is

$$\left.\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n}\right|_{|\mathbf{r}'|=a} = -\left.\frac{\partial G}{\partial d}\right|_{d=a} = -\left[\frac{r^2 - a^2}{a\left(a^2 - 2ar\cos(\gamma) + r^2\right)^{3/2}}\right] \ , \tag{55}$$

evaluated in Mathematica. There is no charge density outside the sphere which is our volume of interest, therefore $\rho(\mathbf{r}')$ is zero in this volume, so the first integral in Jackson Equation 1.44 is zero, which yields the result

$$\phi(\mathbf{r}) = -\frac{1}{4\pi} \oint_S \phi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} \ dA' = \frac{1}{4\pi} \oint_S \phi(r', \theta', \varphi') \frac{r^2 - a^2}{a\left(a^2 - 2ar\cos(\gamma) + r^2\right)^{3/2}} \ dA \ , \tag{56}$$

[3]See Jackson page 64.

but since the surface is at a constant r' coordinate, we can write this as

$$\phi(\mathbf{r}) = \frac{1}{4\pi} \oint_S \phi(a, \theta', \varphi') \frac{r^2 - a^2}{a\,(a^2 - 2ar\cos\gamma + r^2)^{3/2}}\ (a^2 d\Omega) \tag{57}$$

$$= \frac{1}{4\pi} \oint_S \phi(a, \theta', \varphi') \frac{a(r^2 - a^2)}{(a^2 - 2ar\cos\gamma + r^2)^{3/2}}\ d\Omega\ , \tag{58}$$

which shows Equation 49.