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Contents

1 Problem #1: Jackson 2.2.

Using the method of images, discuss the problem of a point charge q inside a hollow, grounded, conducting sphere of radius a.

1.1 Electric Potential.

Find the potential inside the sphere.

Consider a point charge at a distance $d < a$ from the origin, which will produce an image charge q' at a distance $d' > a$ from the origin along the same line. Along this line (that connects both charges and the origin), the potential at a distance r is given by $\phi(r) = k(q/|r-d| + q'/|r-d'|)$, where the coupling constant is $k = (4\pi\epsilon_0)^{-1}$. We can enforce the potential at $r = a$ is zero, so

$$0 = \frac{kq}{(a-d)} + \frac{kq}{(d'-a)} = k\left(\frac{q}{a(1-\frac{d}{a})} + \frac{q'}{d'(1-\frac{a}{d'})}\right) , \tag{1}$$

by inspection, we find $q/a = q'/d'$ and $d/a = a/d'$, which give

$$q' = -\frac{a}{d}q \quad \text{and} \quad d' = \frac{a^2}{d} , \tag{2}$$

which is the same as the solution to the problem with the charge located outside the sphere, as we expect.

Consider a point P on the plane which contains the origin, the charge, and its image (due to spherical symmetry, such a plane will always exist. This point is confined such that $\mathbf{r} = (a, \theta)$, in polar coordinates on this plane, where $\theta = 0$ is defined to be along the line containing the origin, the charge, and its image. If we write the potential in vector form for the potential inside the sphere we get

$$\phi(\mathbf{r}) = kq\left[\frac{1}{|\mathbf{r}-\mathbf{r}_1|} - \left(\frac{a}{d}\right)\frac{1}{|\mathbf{r}-\mathbf{r}_2|}\right] , \tag{3}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the vectors that point to the charge and its image, respectively. Using law of cosines to expand the vector magnitudes allows us to write the potential.

$$\phi(r,\theta) = kq\left[\frac{1}{\sqrt{r^2+d^2-2rd\cos\theta}} - \left(\frac{a}{d}\right)\frac{1}{\sqrt{r^2+(\frac{a^2}{d})^2-2r(\frac{a^2}{d})\cos\theta}}\right] . \tag{4}$$

We find that the potential at a point (r, θ) inside a grounded, conducting, hollow sphere produced by a point charge q located a distance d from the origin is

$$\phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+d^2-2rd\cos\theta}} - \frac{1}{\sqrt{\frac{d^2r^2}{a^2}+a^2-2rd\cos\theta}}\right] . \tag{5}$$

1.2 Charge Density.

Find the surface charge density.

The charge density is given by

$$\sigma = -\epsilon_0 \frac{\partial \phi}{\partial n}\bigg|_S = -\epsilon_0 \frac{\partial \phi}{\partial r}\bigg|_{r=a} = -\left(\frac{q}{4\pi}\right) \frac{d^2 - a^2}{a\left(a^2 - 2ad\cos\theta + d^2\right)^{3/2}} \ , \tag{6}$$

where the unit normal n points out of the sphere bounded by S.

1.3 Force.

Find the magnitude and direction of the force acting on q.

We can use the Coulomb force law to find the force between the charge and its image which is equivalent to the force felt on the charge by the grounded sphere. Using superposition and the values of q' and d', we find

$$F = \frac{1}{4\pi\epsilon_0} \frac{q\left(-\frac{a}{d}q\right)}{\left(d - \frac{a^2}{d}\right)^2} = -\frac{1}{4\pi\epsilon_0} \frac{(ad)q^2}{(d^2 - a^2)^2} \ , \tag{7}$$

which is attractive: the charge feels a net force towards the sphere along the line which connects the origin and the point charge.

1.4 Variations.

Is there any change in the solution if the sphere is kept at a fixed potential ϕ_0? If the sphere has a total charge Q on it?

Holding the sphere at a potential ϕ_0 only adds a constant term to the potential, Equation 5. This has no effect on the surface charge because the excess charge will evenly distribute itself across the surface to create this potential (additionally, taking the derivative causes this term to vanish). It also has no effect on the force, because both the charge and its image will react to it and due to superposition these constant terms will cancel.

If the charge has a constant charge Q, and we have shown there is an induced interior surface charge of $-q$, so there must be a charge of $Q+q$ induced on the outer face. This will raise the potential by an amount $(Q+q)(k/a)$ on the inside of the sphere, which just gets added to Equation 5. This does not change the system inside the sphere at all because the charge will arrange itself to maintain equilibrium.

2 Problem #2: Equipotential Surfaces of Two Infinite Wires.

Two infinitely long wires run parallel to the x, and carry linear charge densities of $+\lambda$ (for the wire at $y = a, z = 0$) and one at $-\lambda$ (for the wire at $y = -a, z = 0$). Show that the equipotential surfaces are right circular cylinders, and locate the axis and radius corresponding to a given potential ϕ_0.

Consider a point P in the $x = 0$ plane, the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are the distances from this point to the wire with positive line charge and to the wire with negative line charge, respectively:

$$|\mathbf{r}_1| = \sqrt{z^2 + (y-a)^2} \tag{8}$$

$$|\mathbf{r}_2| = \sqrt{z^2 + (y+a)^2} \; . \tag{9}$$

Additionally, the potential a distance r from an infinite line charge is $\phi(r) = (\lambda/2\pi\epsilon_0)\ln[r_0/r]$, where r_0 is some reference distance where the potential is zero. The potential at P is then the superposition of both line charges. If we assume both line charges have the same reference distance (a safe assumption if they have the same linear charge density), we can write

$$\phi(y,z) = \frac{\lambda}{2\pi\epsilon_0}\ln\left[\frac{|\mathbf{r}_2|}{|\mathbf{r}_1|}\right] = \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{z^2+y^2+a^2+2ya}{z^2+y^2+a^2-2ya}\right] \; . \tag{10}$$

If we are searching for surfaces of constant potential, we can set $\phi(y,z) = \phi_0$, where ϕ_0 is a constant. Furthermore, we can incorporate this with the rest of the constants on the right hand side, and then exponentiate the equation to find

$$C(z^2+y^2+a^2-2ya) = z^2+y^2+a^2+2ya \; , \tag{11}$$

where the constant $C = \exp[4\pi\epsilon\phi_0/\lambda]$. If we distribute and collect terms, we find

$$(C-1)z^2 + (C-1)y^2 + (C-1)a^2 - 2ya(C+1) = 0 \tag{12}$$

$$z^2 + y^2 - 2y\left[a\frac{C+1}{C-1}\right] + a^2 = 0 \; , \tag{13}$$

which can be noted to be the equation of a circle.

A circle of radius R in the $y-z$ plane, centered at y_0 can be expressed as $(y-y_0)^2 + z^2 - R^2 = 0$. If we expand the square and rearrange we get

$$z^2 + y^2 - 2yy_0 + (y_0^2 - R^2) = 0 \; , \tag{14}$$

which is of the same form as Equation 13, with

$$y_0 = a\frac{C+1}{C-1} \quad \text{and} \quad R^2 = y_0^2 - a^2 = a^2\left(\left[\frac{C+1}{C-1}\right]^2 - 1\right) = a^2\left(\frac{C^2+2C+1}{(C-1)^2} - 1\right) \; , \tag{15}$$

where y_0 is the axis and R is the radius of the circle. We can simplify the expression for the radius

$$R^2 = a^2\left(\frac{C^2+2C+1}{(C-1)^2} - \frac{C^2-2C+1}{(C-1)^2}\right) = a^2\frac{4C}{(C-1)^2} \tag{16}$$

We can extend this along the x axis and see the equipotential surfaces are cylinders with axes y_0 and radii R defined by

$$\begin{cases} y_0 & = a\frac{C+1}{C-1} \\ R & = \frac{2a\sqrt{C}}{C-1} \end{cases} \quad \text{with} \quad C = e^{4\pi\epsilon\phi_0/\lambda} \; . \tag{17}$$

3 Problem #3.

3.1 Jackson 2.12.

Find the Green's function for the two-dimensional potential problem with the potential specified on the surface of a cylinder of radius b, and show that the solution inside the cylinder is given by Poisson's integral:

$$\phi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\phi(b,\theta')\frac{b^2-r^2}{b^2+r^2-2br\cos(\theta'-\theta)}d\theta' \ . \tag{18}$$

Consider the scenario where there is an infinite line charge $+\lambda$ running parallel to the cylinder axis. Then in a plane that intersects the cylinder perpendicular to its axis, a point P inside the cylinder is defined by $\mathbf{r} = (r,\theta)$ with $r < b$; the location the line charge intersects this plane is $\mathbf{r}_1 = (r_1, \theta')$. The image line charge, of magnitude $-\lambda$, created by this line charge is located at $\mathbf{r}_2 = (r_2, \theta')$, because it lays along the same ray as the line charge. We can then define the distance from P to the line charge and its image:

$$s_1 = |\mathbf{r}-\mathbf{r}_1| = \sqrt{r^2+r_1^2-2rr_1\cos(\theta-\theta')} \tag{19}$$

$$s_2 = |\mathbf{r}-\mathbf{r}_2| = \sqrt{r^2+r_2^2-2rr_2\cos(\theta-\theta')} \ . \tag{20}$$

We can now write the potential at P using the superposition of the line charges

$$V_P(r,\theta) = \frac{\lambda}{2\pi\epsilon_0}\left[\ln\frac{r_1}{s_1} - \ln\frac{r_2}{s_2}\right] = \frac{\lambda}{2\pi\epsilon_0}\left[\ln\frac{s_2}{s_1} + \ln\frac{r_1}{r_2}\right] \ , \tag{21}$$

where s_1 and s_2 are the reference distances for each line charge, which we have taken to be the distance to the origin. Furthermore if we define $r' = r_1$, and assume the image line charge is at the inverse position of this relative to a circle of radius b, we find $r_2 = b^2/r'$. If we insert all this into Equation 21, and pull down the square root to be a coefficient of one half, we find

$$V_P(r,\theta) = \frac{\lambda}{2\pi\epsilon_0}\ln\left[\left(\frac{r^2+\left(\frac{b^2}{r'}\right)^2-2r\frac{b^2}{r'}\cos(\theta-\theta')}{r^2+r'^2-2rr'\cos(\theta-\theta')}\right)^{1/2}\left(\frac{r'}{b}\right)^2\right] \ . \tag{22}$$

We can now define the Green's function for this geometry, which is how the system responds to the addition of the fundamental unit of charge, in this case $\lambda = 4\pi\epsilon_0$. We can take how the system reacts to the introduction of an arbitrary charge at an arbitrary point V_P, and set our value of lambda to see

$$G(\mathbf{r},\mathbf{r}') = \ln\left[\left(\frac{r^2+\left(\frac{b^2}{r'}\right)^2-2r\frac{b^2}{r'}\cos(\theta-\theta')}{r^2+r'^2-2rr'\cos(\theta-\theta')}\right)^{1/2}\left(\frac{r'}{b}\right)^2\right] \ . \tag{23}$$

We can find the potential inside our surface using

$$\Phi(r,\theta) = -\frac{1}{4\pi}\oint_S \phi(b,\theta')\frac{\partial G}{\partial r}da \ , \tag{24}$$

where the unit normal points radially outward (positive). If we use MATHEMATICA to take the derivative and and evaluate it at $r = b$ (the surface S), we find

$$\Phi(r,\theta) = -\frac{1}{4\pi}\oint_S \phi(b,\theta')\frac{2(r'^2 - b^2)}{b\,(b^2 + r'^2 - 2br'\cos(\theta-\theta'))}(bd\theta) \ . \tag{25}$$

We can cancel the factors of b, and rename r' as r and θ as θ'. This is acceptable because of the reciprocity of the Green's function, we can swap the field term and the source term, as long as we're consistent about changing which set of coordinates the derivative is with respect to and which variable we evaluate at the surface. Also note that the cosinne is symmetric under swapping the variables. This yields the expression

$$\Phi(r,\theta) = \frac{1}{2\pi}\oint_S \phi(b,\theta')\frac{b^2 - r^2}{b^2 + r^2 - 2br\cos(\theta-\theta')}d\theta' \ , \tag{26}$$

which proves Equation 18.

3.2 Jackson 2.13.

Two halves of a long conducting cylinder of radius b are separated by a small gap, and are kept at different potentials V_1 and V_2. Show that the potential inside is given by

$$\phi(r,\theta) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2br}{b^2-r^2}\cos\theta\right) \ , \tag{27}$$

where θ is measured from a plane perpendicular to the plane through the gap.

Let us consider a cross-sectional plane of the cylinder, with the origin at the axis of the cylinder; using polar coordinates the surface has coordinates (b,ψ), where ψ is an angle measured from the plane that separates the two halves of different potentials. We can express our boundary conditions in this coordinate system as

$$\phi(b,\psi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{2}\mathrm{sign}(\psi) \ , \tag{28}$$

where

$$\mathrm{sign}(\psi) = \begin{cases} -1 & -\pi < \psi < 0 \\ +1 & 0 < \psi < \pi \end{cases} \ . \tag{29}$$

Note that $\phi(b,\psi) = V_1$ for $\psi \in [0,\pi]$ and $\phi(b,\psi) = V_2$ for $\psi \in [-\pi,0]$, which are the boundary conditions. Furthermore, we can assume the potential at any point interior of the cylinder has a potential given by the sum of the average potential and some function that varies with r and ψ. In cylindrical coordinates, we can assume a Fourier series for the angular behavior and a power series for the radial behavior,

$$\phi(r,\psi) = \frac{V_1+V_2}{2} + \sum_{n=0}^{\infty} r^{\pm n}\left(a_n\sin n\psi + b_n\cos n\psi\right) \ , \tag{30}$$

which can be constrained for physical reasons. The function must be finite at $r = 0$, so negative values of n are not allowed. Additionally, the potential is not symmetric about $\psi = 0$ ($\phi(r,\psi) \neq \phi(r,-\psi)$), so we want the varying function to be asymmetric about this point as well, so $b_n = 0$.

The potential is symmetric about $\psi = \pi/2$, so that $\phi(r,\psi) = \phi(r, \pi - \psi)$, which implies there are only odd values of n[1]. To determine the coefficients a_n, let us impose the boundary condition by evaluating Equation 30 at $r = b$ and equating it with Equation 28,

$$\sum_{n \text{ odd}}^{\infty} a_n b^n \sin n\psi = \frac{V_1 - V_2}{2} \text{sign}(\psi) \ . \tag{31}$$

If we exploit the orthogonality condition of the sine function, we find

$$\int_{-\pi}^{\pi} d\psi \sin(m\psi) \sum_{n \text{ odd}}^{\infty} a_n b^n \sin(n\psi) = \int_{-\pi}^{\pi} d\psi \sin(m\psi) \frac{V_1 - V_2}{2} \text{sign}(\psi) \tag{32}$$

$$\sum_{n \text{ odd}}^{\infty} a_n b^n \int_{-\pi}^{\pi} d\psi \sin(m\psi)\sin(n\psi) = \frac{V_1 - V_2}{2} \int_{-\pi}^{\pi} d\psi |\sin(m\psi)| \ , \tag{33}$$

where the absolute value comes from noting that sine and the sign function always have the same sign on the domain $\theta \in [-\pi, \pi]$. We can replace the integral on the right hand side by an two times an integral from zero to π of just sine, and use the orthogonality condition on the left hand side to get

$$\sum_{n \text{ odd}}^{\infty} a_n b^n (\pi \delta_{mn}) = (V_1 - V_2) \int_0^{\pi} \sin(m\psi) d\psi \tag{34}$$

$$a_n b^n \pi = 2\frac{V_1 - V_2}{n} \ , \tag{35}$$

which can be rearranged to get the values of a_n. We are now free to write the expression for the potential as

$$\phi(r,\psi) = \frac{V_1 + V_2}{2} + 2\frac{V_1 - V_2}{\pi} \sum_{n \text{ odd}} \left(\frac{r}{b}\right)^n \frac{\sin n\psi}{n} \ . \tag{36}$$

Let us define a complex number $z = (r/b)e^{i\psi}$, and note $\text{Im}[e^{i\theta}] = \sin\theta$, so we can express the sum in the above expression as

$$\sum_{n \text{ odd}} \left(\frac{r}{b}\right)^n \frac{\text{Im}[e^{in\psi}]}{n} = \text{Im}\left[\sum_{n \text{ odd}} \left(\frac{r}{b}e^{i\psi}\right)^n \frac{1}{n}\right] = \text{Im}\left[\sum_{n \text{ odd}} \frac{z^n}{n}\right] = \frac{1}{2}\text{Im}\left[\ln \frac{1+z}{1-z}\right] \ , \tag{37}$$

using the identity for the sum[2]. Let us now simplify the argument of the logarithm by multiplying the top and bottom by the complex conjugate of the bottom

$$\frac{1+z}{1-z} = \frac{1 + re^{i\psi}/b}{1 - re^{i\psi}/b} = \frac{(1 + re^{i\psi}/b)(1 - re^{-i\psi}/b)}{1 + (r/b)^2 - (r/b)(e^{i\psi} + e^{-i\psi})} = \frac{1 + (r/b)(e^{i\psi} - e^{-i\psi}) - (r/b)^2)}{1 + (r/b)^2 - 2(r/b)\cosh\psi} \tag{38}$$

$$= \frac{1 - (r/b)^2 + 2i(r/b)\sin\psi}{1 + (r/b)^2 - 2(r/b)\cos\psi} \ . \tag{39}$$

[1]Let us show odd n satisfy our symmetry condition $\phi(r,\psi) = \phi(r, \pi - \psi)$
$\sin(n\psi) \to \sin[(2m+1)\psi] = \sin[(2m+1)(\pi - \psi)]$
$= \sin[2m\pi + \pi - (2m+1)\psi]$ note m is an integer so $2m\pi$ does not change the value of the sine
$= \sin[\pi - n\psi] = \sin[n\psi]$ which is just the reference angle. So for odd n, $\phi(r,\psi) = \phi(r, \pi - \psi)$. Note this would not be true for even $n = 2m$ because the lone π term would not be there so you would get $\sin(n\psi) = \sin(-n\psi)$ for even n

[2]Schaum's Outlines, Mathematical Handbook of Formulas and Tables. Equation 22.18.

Furthermore, note that $\ln(z) = \ln(\alpha e^{i\phi}) = \ln(\alpha) + i\phi \rightarrow \text{Im}[\ln(z)] = \text{phase}(z) = \phi$, and in the representation $z = x+iy$, we find that $\tan\phi = y/x$. We can therefore right $\text{Im}[\ln(z)] = \tan^{-1}(x/y)$, but we must write Equation 39 in the appropriate form

$$\frac{1}{1+(r/b)^2 - 2(r/b)\cos\psi}\left[\left(1-\frac{r^2}{b^2}\right) + i\left(2\frac{r}{b}\sin\psi\right)\right] \quad \Rightarrow \quad \frac{y}{x} = \frac{2r\sin\psi}{b\left(1-\frac{r^2}{b^2}\right)} \ . \tag{40}$$

If we simplify this result, and thread all the pieces from Equation 36 through 40 together, we get the result

$$\phi(r,\psi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left[\frac{2br\sin\psi}{b^2-r^2}\right] \ , \tag{41}$$

which proves Equation 27. We will note that we defined our coordinate ψ as zero at the plane that separates the two halves, while θ is measured from the plane perpendicular to that, so $\sin\psi = \cos\theta$, and the result of this is consistent with the expected solution.

4 Problem #4: Jackson 2.7.

Consider a potential in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane ($z = 0$) and at infinity.

4.1 Green's Function.

Write down the appropriate Green's function $G(\mathbf{r}, \mathbf{r}')$.

Consider a point charge q at a position (x_0, y_0, z_0) for $z_0 > 0$, which has an image charge of $-q$ at $(x_0, y_0, -z_0)$. Let the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ be the distances from a point P in half-space (x, y, z), so that

$$|\mathbf{r}_1| = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \tag{42}$$

$$|\mathbf{r}_2| = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2} \ , \tag{43}$$

and the potential at P is

$$V_P = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{|\mathbf{r}_1|} - \frac{1}{|\mathbf{r}_2|} \right] \ , \tag{44}$$

so the Green's function is how the system responds to unit charge $q = 4\pi\epsilon_0$,

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} - \frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2}} \ . \tag{45}$$

4.2 Cylindrical Coordinates.

If the potential on the plane $z = 0$ is specified to be ϕ_0 inside a circle of radius a centered at the origin, and $\phi = 0$ outside that circle, find an integral expression for the potential at the point P specified in terms of cylindrical coordinates (r, θ, z).

If we use polar coordinates on the $z = 0$ plane, we can consider the distance from a point $(x_0, y_0, 0) \to (r_0, \theta_0)$ to a point $(x, y, 0) \to (r, \theta)$, denoted by the vector $\boldsymbol{\rho}$. We can get the magnitude of $\boldsymbol{\rho}$ using the law of cosines,

$$|\boldsymbol{\rho}|^2 = r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0) \ , \tag{46}$$

which allows us to write the magnitude of the distance vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of ρ and θ instead of x and y,

$$|\mathbf{r}_1|^2 = r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0) + (z - z_0)^2 \tag{47}$$

$$|\mathbf{r}_2|^2 = r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0) + (z + z_0)^2 \ , \tag{48}$$

so the Green's function in polar coordinates is these results inserted in Equation 44 with q equal to the unit charge. We are interested in solving for the potential in half-space, so the unit normal is $n = -z_0$. We can use the Green's function to to find the potential using Jackson Equation 1.44 with $\rho(\mathbf{r}') = 0$ because there is no charge density in the volume of interest. The partial derivative of the Green's function with respect to the unit normal is

$$\left.\frac{\partial G}{\partial n}\right|_S = \left.\frac{\partial G}{\partial(-z_0)}\right|_{z_0=0} = -\left[\frac{z - z_0}{(|\mathbf{r}_1|)^{3/2}} + \frac{z + z_0}{(|\mathbf{r}_2|)^{3/2}} \right]_{z_0=0} \ , \tag{49}$$

so using the specified equation from Jackson we find

$$\Phi(r,\theta,z) = -\frac{1}{4\pi}\oint_{S_0} \phi(r_0,\theta_0,z_0)\frac{\partial G}{\partial n} da_0 \tag{50}$$

$$= -\frac{\phi_0}{4\pi}\int_0^a \int_0^{2\pi} \left.\frac{\partial G}{\partial n}\right|_{z=0} r_0 dr_0 d\theta_0 \tag{51}$$

$$= \frac{\phi_0}{4\pi}\int_0^a \int_0^{2\pi} \frac{2z}{\left(r^2+r_0^2+z^2-2\cos(\theta-\theta_0)rr_0\right)^{3/2}} r_0 dr_0 d\theta_0 \tag{52}$$

$$= \frac{\phi_0 z}{2\pi}\int_0^a \int_0^{2\pi} \frac{r_0 dr_0 d\theta_0}{\left(r^2+r_0^2+z^2-2\cos(\theta-\theta_0)rr_0\right)^{3/2}} \ , \tag{53}$$

note the integral was carried out only over the region in which $\phi(r_0,\theta_0,z_0)$ was nonzero. This integral can be evaluated to find the potential anywhere in half-space, due to the constant potential from the disk centered at the origin on the $z=0$ plane.

4.3 Potential along Cylinder Axis.

Show that along the axis of the cylinder, $(r=0)$ the potential is given by

$$\phi = \phi_0\left(1-\frac{z}{\sqrt{a^2+z^2}}\right) \ . \tag{54}$$

If we are interested on what the potential will be along the axis of the cylinder (z-axis), we set $r=0$ to obtain

$$\Phi(\theta,z) = \frac{\phi_0 z}{2\pi}\int_0^a \int_0^{2\pi} \frac{r_0 dr_0 d\theta_0}{\left(r_0^2+z^2\right)^{3/2}} = \phi z \int_0^a \frac{r_0 dr_0}{\left(r_0^2+z^2\right)^{3/2}} \ . \tag{55}$$

This integral can be evaluated using a substitution of $u = r_0^2+z^2$, so $du/dr_0 = 2r_0$, which makes the integral

$$\Phi(\theta,z) = \frac{\phi z}{2}\int_{z^2}^{a^2+z^2} \frac{du}{u^{3/2}} = \frac{\phi z}{2}\left(\frac{2}{\sqrt{z^2}} - \frac{2}{\sqrt{a^2+z^2}}\right) \ , \tag{56}$$

so we find the potential along the cylinder axis in half space to be

$$\Phi(\theta,z) = \phi\left(1-\frac{z}{\sqrt{a^2+z^2}}\right) \ , \tag{57}$$

which proves Equation 54.

4.4 Large Distance Expansion.

Show that at large distances ($r^2+z^2 \gg a^2$), the potential can be expanded in powers of $(r^2+z^2)^{-1}$, and that the leading terms are

$$\phi = \frac{\phi_0 a^2}{2}\frac{z}{(r^2+z^2)^{3/2}}\left[1-\frac{3a^2}{4(r^2+z^2)} + \frac{5(3r^2a^2+a^4)}{8(r^2+z^2)^2} + \dots\right] \ . \tag{58}$$

We will begin from Equation 53, but will divide top and bottom by the factor $(r^2+z^2)^{3/2}$, which yields

$$\Phi(r,\theta,z) = \frac{\phi_0 z}{2\pi}(r^2+z^2)^{-3/2}\int_0^a\int_0^{2\pi}\frac{r_0dr_0d\theta_0}{\left(1+\frac{r_0^2-2\cos(\theta-\theta_0)rr_0}{r^2+z^2}\right)^{3/2}} \tag{59}$$

$$= \frac{\phi_0 z}{2\pi}(r^2+z^2)^{-3/2}\int_0^a\int_0^{2\pi} r_0dr_0d\theta_0\left(1+\frac{r_0^2-2\cos(\theta-\theta_0)rr_0}{r^2+z^2}\right)^{-3/2} . \tag{60}$$

We can use the Taylor series $(1+x)^n = 1+nx+\frac{n(n-1)}{2}x^2+\ldots$ to write the factor as

$$1-\frac{3}{2}\left[\frac{r_0^2-2\cos(\theta-\theta_0)rr_0}{r^2+z^2}\right]+\frac{\frac{3}{2}\frac{5}{2}}{2}\left[\frac{r_0^2-2\cos(\theta-\theta_0)rr_0}{r^2+z^2}\right]^2+\ldots \tag{61}$$

We can perform the integrals term-by-term, starting with the first term

$$\int_0^a\int_0^{2\pi} r_0dr_0d\theta_0\,(1) = \pi a^2 . \tag{62}$$

The second integral, evaluated in Mathematica, is

$$-\frac{3}{2}\int_0^a\int_0^{2\pi} r_0dr_0d\theta_0\left(\frac{r_0^2-2\cos(\theta-\theta_0)rr_0}{r^2+z^2}\right) = -\frac{3}{2(r^2+z^2)}\frac{\pi a^4}{2} = -(\pi a^2)\frac{3a^2}{4(r^2+z^2)} , \tag{63}$$

and the third, evaluated in Mathematica, is

$$\frac{15}{8}\int_0^a\int_0^{2\pi} r_0dr_0d\theta_0\left(\frac{r_0^2-2\cos(\theta-\theta_0)rr_0}{r^2+z^2}\right)^2 = \frac{15}{8(r^2+z^2)^2}\frac{1}{3}\pi a^4\left(a^2+3r^2\right) \tag{64}$$

$$= (\pi a^2)\frac{5a^2(3r^2+a^2)}{8(r^2+z^2)^2} . \tag{65}$$

If we combine these three integrals with Equation 60, we see the potential in the $a^2 \ll r^2+z^2$ limit is (to order $(r^2+z^2)^2$) given by the expansion

$$\Phi(r,\theta,z) = \frac{1}{2}\frac{\phi_0 z a^2}{(r^2+z^2)^{3/2}}\left[1-\frac{3a^2}{4(r^2+z^2)}+\frac{5a^2(3r^2+a^2)}{8(r^2+z^2)^2}\right] , \tag{66}$$

which proves Equation 58.

5 Problem #5: Potential of Hollow Cylinder.

Consider a cylinder of radius a and height L, and with no charge inside the cylinder. One endcap and the lateral sides of the cylinder are held at the potential ϕ_0; on the other endcap, the potential vanishes (you can consider this endcap to be separated from the rest of the cylinder by a thin insulating spacer). Solve Laplace's equation to find the potential ϕ inside the cylinder, evaluating all required integrals.

We need to solve Laplace's equation $\nabla^2\Phi = 0$ in cylindrical coordinates where we assume $\Phi(\rho, \varphi, z)$ is a separable solution $\phi = R(\rho)\Theta(\varphi)\mathcal{Z}(z)$. In cylindrical coordinates we have the equation

$$\nabla^2\Phi = \left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}\right] R(\rho)\Theta(\varphi)\mathcal{Z}(z) = 0 \tag{67}$$

$$0 = R''\Theta\mathcal{Z} + \frac{1}{\rho}R'\Theta\mathcal{Z} + \frac{1}{\rho^2}R\Theta''\mathcal{Z} + R\Theta\mathcal{Z}' \tag{68}$$

$$-k^2 + k^2 = \frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} + \frac{1}{\rho^2}\frac{\Theta''}{\Theta} + \frac{\mathcal{Z}''}{Z}\ , \tag{69}$$

where k is some real constant. We can immediately isolate a differential equation for $\mathcal{Z}(z)$

$$\mathcal{Z}'' = k^2 z \quad \Rightarrow \quad \mathcal{Z}(z) \begin{cases} \cosh kz \\ \sinh kz \end{cases} . \tag{70}$$

If we continue separating variables, we find

$$-\nu^2 + \nu^2 = \rho^2\frac{R''}{R} + \rho\frac{R'}{R} + \frac{\Theta''}{\Theta} + \rho^2 k^2\ , \tag{71}$$

which isolates the angular dependence:

$$\Theta'' = -\nu^2\Theta \quad \Rightarrow \quad \Theta(\varphi) \begin{cases} \cos\nu\varphi \\ \sin\nu\varphi \end{cases} . \tag{72}$$

We are left with the radial equation

$$\nu^2 = \rho^2 k^2 + \rho^2\frac{R''}{R} + \rho\frac{R'}{R} \quad \Rightarrow \quad 0 = \rho^2 R'' + \rho R' + (\rho^2 k^2 - \nu^2)R\ , \tag{73}$$

which is just Bessel's equation, so the solutions are Bessel functions of the first ($J_\nu(kr)$) and second kind ($Y_\nu(kr)$). However, our domain of interest contains the origin and the Bessel functions of the second kind are divergent at the origin, they cannot be allowed solutions. Furthermore, we know there can be no angular dependence on the potential due to the cylindrical symmetry, therefore $\Theta(\varphi)$ must equal some nonzero constant. This means $\nu = 0$, so we only have one Bessel function in our solution, $J_0(kr)$. Therefore the boundary conditions of pure geometry have constrained our solution to be the zeroth Bessel function of the first time times a series of hyperbolic sines and cosines. If we consider the potential dependent boundary conditions we must note which endcap is held at ground. Let the endcap on the $z = 0$ plane, centered at the origin be held at ground, while the rest of the cylinder is at ϕ_0. This is equivalent to shifting the arbitrary zero point of potential and saying the entire cylinder is held at ground and the bottom endcap is held at $-\phi_0$. We therefore have the conditions

$$\mathcal{S}(z = 0) = -\phi_0 \qquad \mathcal{S}(z = L) = 0\ , \tag{74}$$

where S is the series which is added to the constant potential. We want the series to be $-\phi_0$ at $z = 0$ so when it is summed to the constant potential term, the total potential is zero on the bottom endcap, which is the boundary condition for the actual problem (no shifted potentials). We can only accept hyperbolic sine solutions for the $\mathcal{Z}$ solution because the value of hyperbolic cosine is never negative or zero, which is required for either potential setup (original or shifted). Additionally, for $\rho = a \to R = 0$, so $k = \chi_{0m}/a$ where χ_{0m} is the mth zero of the zeroth Bessel function of the first kind. Therefore, we can write the potential inside the cylinder as

$$\Phi(\rho, \varphi, z) = \phi_0 + \sum_{m=1}^{\infty} a_m J_0\left(\frac{\chi_{0m}}{a} r\right) \sinh\left(\frac{\chi_{0m}}{a}(z - L)\right) \ , \tag{75}$$

and we can apply the boundary condition,

$$\Phi(\rho, \varphi, 0) = 0 = \phi_0 + \sum_{m=1}^{\infty} a_m J_0\left(\frac{\chi_{0m}}{a} r\right) \sinh\left(-\frac{\chi_{0m}}{a} L\right) \tag{76}$$

$$\phi_0 = \sum_{m=1}^{\infty} a_m J_0\left(\frac{\chi_{0m}}{a} r\right) \sinh\left(\frac{\chi_{0m}}{a} L\right) \ . \tag{77}$$

To determine the coefficients a_m we can again exploit an orthogonality condition of Bessel functions,

$$\int_0^a \phi_0 J_0\left(\frac{\chi_{0n}}{a} r\right) r dr = \sum_{m=1}^{\infty} a_m \sinh\left(\frac{\chi_{0m}}{a} L\right) \int_0^a J_0\left(\frac{\chi_{0n}}{a} r\right) J_0\left(\frac{\chi_{0m}}{a} r\right) r dr \tag{78}$$

$$= \sum_{m=1}^{\infty} a_m \sinh\left(\frac{\chi_{0m}}{a} L\right) \left[\frac{a^2}{2} J_1^2(\chi_{0n}) \delta_{mn}\right] \tag{79}$$

$$= a_n \sinh\left(\frac{\chi_{0n}}{a} L\right) \left[\frac{a^2}{2} J_1^2(\chi_{0n})\right] \tag{80}$$

$$a_n = \frac{2\phi_0}{a^2 J_1^2(\chi_{0n}) \sinh\left[\frac{\chi_{0n}}{a} L\right]} \int_0^a J_0\left(\frac{\chi_{0n}}{a} r\right) r dr \ . \tag{81}$$

We can evaluate this integral by defining $w = (\chi_{0n}/a) r$, and $dw/dr = \chi_{0n}/a$, so

$$\int_0^a J_0\left(\frac{\chi_{0n}}{a} r\right) r dr = \int_{w=0}^{w=\chi_{0n}} J_0(u) \left(\frac{a}{\chi_{0n}} u\right) \left(\frac{a}{\chi_{0n}} du\right) = \frac{a^2}{\chi_{0n}^2} \int_0^{\chi_{0n}} J_0(u) u \ du \tag{82}$$

$$= \frac{a^2}{\chi_{0n}^2} (\chi_{0n} J_1(\chi_{0n})) \ . \tag{83}$$

Combining this back into Equation 75, we find

$$\Phi(\rho, \varphi, z) = \phi_0 + \sum_{m=1}^{\infty} \frac{2\phi_0}{\sinh\left[\frac{\chi_{0m}}{a} L\right]} \frac{J_0\left(\frac{\chi_{0m}}{a} r\right) \sinh\left(\frac{\chi_{0m}}{a}(z - L)\right)}{\chi_{0m}} \ , \tag{84}$$

which is the potential at a point inside the cylinder of radius a.