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Contents

1 Problem #1: Green's Function in Free Space.

In class, we developed the following expansion of the Green's function in free space:

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi)^3}\frac{1}{\epsilon_0}\int \mathrm{d}^3k \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^2} \tag{1}$$

Evaluate the integral by hand to show that the Green's function has its usual real space representation.

Let us define a coordinate system in which the z axis points along $\mathbf{R} \equiv \mathbf{r} - \mathbf{r}'$, the dot product in the exponential can then be written

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi)^3}\frac{1}{\epsilon_0}\int \mathrm{d}^3k \frac{e^{ikR\cos\theta}}{k^2}\ , \tag{2}$$

where θ is the angle from the z axis. In spherical coordinates where k is the radial component, the volume element is $\mathrm{d}^3k = k^2\sin\theta \mathrm{d}k\mathrm{d}\theta\mathrm{d}\phi$, with ϕ in the xy plane, measured from the x axis. The integral then becomes

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi)^3}\frac{1}{\epsilon_0}\int_0^{2\pi}\mathrm{d}\phi\int_0^\infty k^2\mathrm{d}k\int_0^\pi \sin\theta\mathrm{d}\theta\frac{e^{ikR\cos\theta}}{k^2}\ , \tag{3}$$

under a change of integration variable $x = \cos\theta \to \mathrm{d}x = -\sin\theta\mathrm{d}\theta$, the integral becomes

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi)^2}\frac{1}{\epsilon_0}\int_0^\infty \mathrm{d}k\int_{-1}^{+1} e^{ikRx}\mathrm{d}x = \frac{1}{(2\pi)^2}\frac{1}{\epsilon_0}\int_0^\infty \frac{1}{ikR}\left(e^{ikRx}\right)\Big|_{-1}^{+1}\mathrm{d}k\ . \tag{4}$$

We can break the complex exponential down:

$$e^{-ikR} - e^{ikR} = (\cos kR + i\sin kR) - (\cos kR - i\sin kR) = -2i\sin kR\ , \tag{5}$$

so the integral becomes

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi)^2}\frac{1}{\epsilon_0}\int_0^\infty (2)\frac{\sin kR}{kR}\mathrm{d}k\ , \tag{6}$$

note the limits of integration would normally be 1 to -1, but were swapped due to the negative in the differential of $\cos\theta$. Letting $\kappa = kR$, we see the integral is

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi^2}\frac{1}{\epsilon_0}\int_0^\infty \frac{\sin\kappa}{\kappa}\frac{\mathrm{d}\kappa}{R} = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{\mathbf{r}-\mathbf{r}'}\right)\left(\frac{2}{\pi}\right)\int_0^\infty \frac{\sin\kappa}{\kappa}\mathrm{d}\kappa\ . \tag{7}$$

If we define the free space Green's function as $G_{FS}(\mathbf{r},\mathbf{r}')$, the expression can be written

$$G(\mathbf{r},\mathbf{r}') = G_{FS}(\mathbf{r},\mathbf{r}')\left(\frac{2}{\pi}\right)\int_0^\infty \frac{\sin\kappa}{\kappa}\mathrm{d}\kappa\ . \tag{8}$$

The expression in Equation 1 simplifies to

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi)^3}\frac{1}{\epsilon_0}\int \mathrm{d}^3k\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^2} = G_{FS}(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\ , \tag{9}$$

if

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin\kappa}{\kappa}\mathrm{d}\kappa\ . \tag{10}$$

which we will show. Consider the function $f(z) = e^{iz}/z$ (note usinig Euler's rule $\Im f(z) = \sin(z)/z$), where $z = re^{i\theta}$, which has a simple pole at $z = 0$. We can express the differential of the complex variable z as $\mathrm{d}z = e^{i\theta}\mathrm{d}r + ire^{i\theta}\mathrm{d}\theta$. Now consider a contour C in the complex plane that consists of four curves. The curve C_+ runs along the positive real axis from R_0 to R_∞. The curve C_∞ is a semi-circle of radius R_∞ in the positive imaginary half-plane. The curve C_- lies along the negative real axis from $-R_\infty$ to $-R_0$ (since r ranges from 0 to ∞, when we integrate $\mathrm{d}r$ along this curve, the limits of integration are positive). The fourth curve, C_0 is a semi-circle of radius R_0 in the positive imaginary half-plane. The contour integral around C can be represented as the sum of the path integrals along $C_+,\ C_\infty,\ C_0,$ and C_- (see Figure 1):

$$\oint_C f(z)\mathrm{d}z = \int_{R_0}^{R_\infty} \left[\frac{e^{iz}}{z}\mathrm{d}z\right]_{\theta=0} + \int_0^\pi \left[\frac{e^{iz}}{z}\mathrm{d}z\right]_{r=R_\infty} + \int_{R_\infty}^{R_0} \left[\frac{e^{iz}}{z}\mathrm{d}z\right]_{\theta=\pi} + \int_\pi^0 \left[\frac{e^{iz}}{z}\mathrm{d}z\right]_{r=R_0} \tag{11}$$

and taking the limit that $R_\infty \to \infty$ and $R_0 \to 0$. Let us label the four integrals as I_i where i is the contour named above, then we can evaluate the integrals one-by-one. The first integral, after noting $e^{i0} = 1$, is

$$I_{C_+} = \int_{R_0}^{R_\infty} \frac{e^{ir}}{r}\mathrm{d}r\ , \tag{12}$$

which is the sinc integral we are attempting to evaluate. The other integral along the real axis, using $e^{i\pi} = -1$, is

$$I_{C_-} = \int_{R_\infty}^{R_0} \frac{e^{i(-r)}}{(-r)}(-\mathrm{d}r) = \int_{-R_\infty}^{-R_0} \frac{e^{i\tilde{r}}}{\tilde{r}}\mathrm{d}\tilde{r}\ , \tag{13}$$

where $\tilde{r} \equiv -r$. Now for the angular integrals: the first of which is

$$I_{R_\infty} = \lim_{R_\infty\to\infty} \int_0^\pi \frac{\exp(iz)}{z}\mathrm{d}z = \int_0^\pi g(z)e^{iz}\mathrm{d}\theta\ , \tag{14}$$

which is of the form required to use Jorndan's lemma[1]. We investage the limit of $g(z)$:

$$\lim_{R_\infty\to\infty} g(z) = \lim_{R_\infty\to\infty} \frac{e^{-i\theta}}{R_\infty} = 0 \quad \forall\ \theta \in [0,\pi]\ , \tag{15}$$

so by Jordan's Lemma, the integral vanishes (see reference for footnote 1). The second is

$$I_{R_0} = \int_\pi^0 \frac{\exp(iR_0e^{i\theta})}{R_0e^{i\theta}}(iR_0e^{i\theta}\mathrm{d}\theta) = \int_\pi^0 \exp(R_0e^{i\theta})i\mathrm{d}\theta = -\int_0^\pi i\mathrm{d}\theta = -i\pi\ , \tag{16}$$

because in the limit $R_0 \to 0$, the expoential becomes unity. Combining the four integrals together, we get

$$\oint_C f(z)\mathrm{d}z = \int_{R_0}^{R_\infty} \frac{e^{ir}}{r}\mathrm{d}r + \int_{-R_\infty}^{-R_0} \frac{e^{ir}}{r}\mathrm{d}r - i\pi\ , \tag{17}$$

after renaming $\tilde{r}$ as r, and in the limit $R_0 \to 0$ and $R_\infty \to \infty$, the two integrals combine to yield an integral over the entire real line. Additionally, using Cauchy's integral theorem[2], since the countour C does not contain any singularities of $f(z)$, the sum of all residues in the contour is zero, and we have

$$i\pi = \int_{-\infty}^{\infty} \frac{e^{ir}}{r}\mathrm{d}r\ . \tag{18}$$

[1]Arfken. Mathematical Methods for Physicists, 7 ed. Page 528.
[2]Arfken. Mathematical Methods for Physicists, 7 ed. Page 478.

If we take the imaginary part of the above expression we get

$$\pi = \int_{-\infty}^{\infty} \frac{\Im(e^{ir})}{r} \mathrm{d}r = \int_{-\infty}^{\infty} \frac{\sin x}{x} \mathrm{d}x \ , \tag{19}$$

after renaming r as x. Since the sinc function is even $\sin(x)$ has the same sign as x, and is defined to be one at $x = 0$, we can split the integral into half space as

$$\pi = 2 \int_{0}^{\infty} \frac{\sin x}{x} \mathrm{d}x \ , \tag{20}$$

from which we get our result, Equation 10. Since this integral equals $\pi/2$, it exactly cancels the $2/pi$ in Equation 9, and we get the result wew expect: the Green's function for free space.

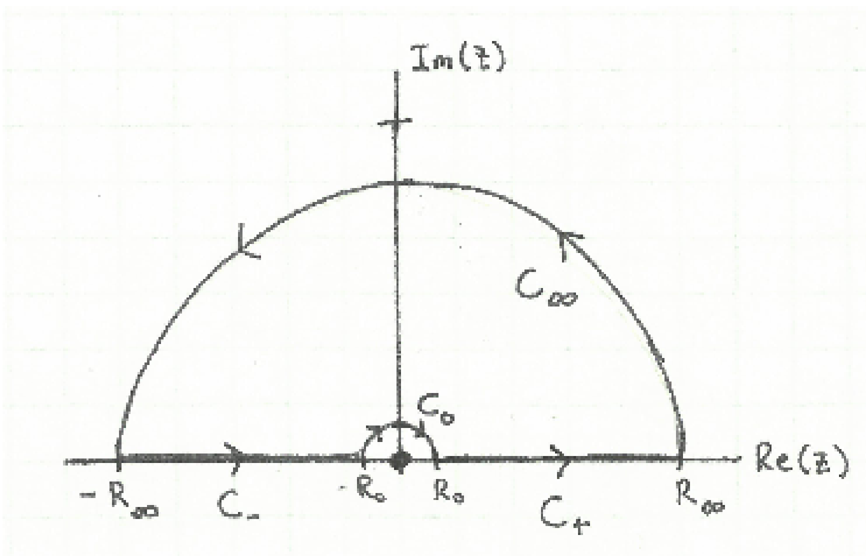


Figure 1: Complex contour used to integrate sinc(x) in problem #1.

2 Problem #2: Angled Capacitor.

By choosing a suitable complex potential, calculate the capacitance of the radial capacitor shown in Figure 2b, which consists of two metal plates of width $b-a$ at an angle θ_0 with respect to each other, extending to $\pm\infty$ in the direction perpendicular to the plane.

The plates extend infinitely into and out of the page, so we will define a polar coordinate system $\{r,\theta\}$ with the origin at the center of the circles of radius a and b. The lower plate makes and angle α with the axis θ is measured from. We can express the rays that make the bottom (z_1) and top (Z_2) plates in complex form as

$$z_1 = re^{i\alpha} \tag{21}$$

$$z_2 = re^{i(\alpha+\theta_0)} \ , \tag{22}$$

for r on the interval $[a,b]$. Let us define a complex potential $w \equiv \log z$, which we can equivalently express as $w = u(r,\theta) + iv(r,\theta)$. We can transform z_1 and z_2 to w_1 and w_2, as

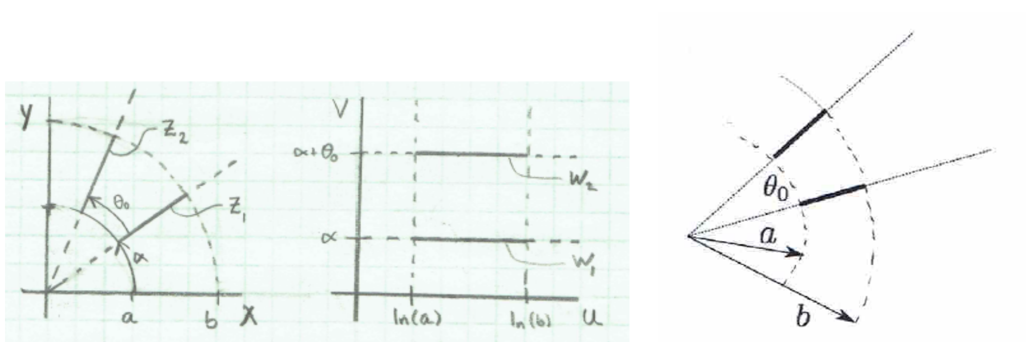
$$w_1 = \log z_1 = \ln r + i\alpha \tag{23}$$

$$w_2 = \log z_2 = \ln r + i(\alpha+\theta_0) \ , \tag{24}$$

for r still on the interval $[a,b]$. In $u-v$ space, w_1 and w_2 are line segments parallel to the u axis of length $\ln b - \ln a$, separated by a distance $(\alpha+\theta_0)-\theta_0$. We can view this as a parallel plate capacitor in uv space, see Figure 2a. Using the well known formula for a parallel plate capacitor: $C = \epsilon_0 A/d$, where A is the area of a plate. If we consider a parallel plate capacitor of separation d with plates of a finite width ℓ, and zero thickness, but being infinite into and out of the page, then the capacitance per unit length (along the infinite dimension) is $c_{pp} = \epsilon_0 \ell/d$. In uv space, the width of the plates is $\ell = \ln b - \ln a = \ln(b/a)$, and are separated by a distance of θ_0. Putting the pieces together we see that

$$c = \epsilon_0 \frac{\ln(b/a)}{\theta_0} \ . \tag{25}$$

Normally capacitance has units of ϵ_0 times length, but since we are only calculating the capacitance in a plane the units should only be that of ϵ_0, which the solution has.



(a) Conversion from the xy plane to the uv plane for problem #2.

(b) Angled capacitor geometry for problem #2.

Figure 2: Diagrams relevant to problem #2.

3 Problem #3: Dielectric Spherical Capacitor.

A spherical capacitor whose electrodes have radii a and b is filled with a dielectric whose permittivity is given by $\epsilon(r) = \epsilon_0(a^2/r^2)$, where r is the distance form the center. Show that the capacitance of this capacitor is equal to the capacitance of a plane parallel capacitor with a homogeneous dielectric with permittivity ϵ_0, electrode area $4\pi a^2$ and distance between electrodes equal to $b - a$ (neglect edge effects).

If we charge the electrode of radius a with free charge Q, then the equivalent of Gauss' law for the displacement field is

$$\int \mathbf{D} \cdot \mathrm{d}\mathbf{A} = Q \ , \tag{26}$$

we will consider a Gaussian sphere of radius $a < r < b$, so the above equation becomes

$$D(4\pi r^2) = Q \ , \tag{27}$$

because the displacement field is normal to the surface, so

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}} \ . \tag{28}$$

The electric field (related to displacement field by $\mathbf{D} = \epsilon \mathbf{E}$)[3] at a distance r from the center is

$$\mathbf{E} = \frac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}} = \frac{Q}{4\pi\epsilon_0 a^2}\hat{\mathbf{r}} \ , \tag{29}$$

so it is constant at every distance between the plates. We can find the potential difference from $r = a$ to $r = b$ by the line integral

$$\Delta V = V_a - V_b = -\int_b^a \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{Q}{4\pi\epsilon_0 a^2}\int_b^a \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\mathrm{d}r = \frac{Q}{4\pi\epsilon_0 a^2}(b-a) \ . \tag{30}$$

From the definition of capacitance, $Q = C\Delta V$, we have

$$C = \frac{Q}{\Delta V} = \frac{4\pi\epsilon_0 a^2}{b-a} \ , \tag{31}$$

which we can compare to a parallel plate capacitor filled with a dielectric of permittivity ϵ_0:

$$C_{pp} = \frac{\epsilon_0 A}{d} = \frac{\epsilon_0(4\pi a^2)}{b-a} \ , \tag{32}$$

which is exactly the same as the result found for the spherical capacitor with the given permittivity.

[3] Jackson, Classical Electrodynamics, 3 ed. Equation 4.37.

4 Problem #4: Step-Function Dielectric.

The permittivity of a medium filling the space between the electrodes of a spherical capacitor whose radii are a and b is given by

$$\epsilon(r) = \begin{cases} \epsilon_1 & a \leq r < c \\ \epsilon_2 & c \leq r \leq b \end{cases} , \tag{33}$$

where $\epsilon_1,\ \epsilon_2$ are constants, and $a < c < b$. Find the capacitance C of the capacitor, and any bound charges in the region $a < r < b$.

4.1 Capacitance.

This geometry is equivalent to two capacitors in series: the capacitance of the whole sphere is really just two spherical capacitors where one electrode of each is the same. Therefore the total capacitance is

$$\frac{1}{C_{tot}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{\Delta V_1}{Q} + \frac{\Delta V_2}{Q} , \tag{34}$$

where again Q is the free charge on the plate of radius a, and ΔV_1 and ΔV_2 are the voltage drops from a to c and c to b, respectively. We can find the voltage drop from a to c in a similar fashion as the previous problem. Gauss's law for the electric displacement in the region $a < r < c$ gives

$$\mathbf{D}_1 = \frac{Q}{4\pi r^2}\hat{\mathbf{r}} , \tag{35}$$

the same way as the previous problem. The electric field in this region is simply,

$$\mathbf{E}_1 = \frac{Q}{4\pi\epsilon_1 r^2}\hat{\mathbf{r}} , \tag{36}$$

where ϵ_1 is the permittivity of the dielectric in this region. The voltage drop across this region is

$$\Delta V_1 = V_a - V_c = -\int_c^a \frac{Q}{4\pi\epsilon_1 r^2}\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\mathrm{d}r = \frac{Q}{4\pi\epsilon_1}\left(\frac{1}{a} - \frac{1}{c}\right) = \frac{Q}{4\pi\epsilon_1}\left(\frac{c-a}{ac}\right) . \tag{37}$$

This is just the potential drop across a spherical capacitor filled with a constant dielectric, of inner radius a and outer radius b. So by symmetry, the potential drop across a spherical capacitor of inner radius c and outer radius b is

$$\Delta V_2 = \frac{Q}{4\pi\epsilon_2}\left(\frac{1}{c} - \frac{1}{b}\right) = \frac{Q}{4\pi\epsilon_2}\left(\frac{b-c}{bc}\right) , \tag{38}$$

where ϵ_2 is the constant permittivity of the region. From Equation 34, the capacitance of the spherical capacitor of inner radius a and outer radius b, with a step-function dielectric is

$$\frac{1}{C_{tot}} = \frac{1}{4\pi}\left(\frac{1}{\epsilon_1}\frac{c-a}{ac} + \frac{1}{\epsilon_2}\frac{b-c}{bc}\right) = \frac{1}{4\pi}\frac{\epsilon_2 bc(c-a) + \epsilon_1 ac(b-c)}{abc^2} , \tag{39}$$

so the total capacitance is

$$C_{tot} = \frac{4\pi abc\epsilon_1\epsilon_2}{a\epsilon_1(b-c) + b\epsilon_2(c-a)} . \tag{40}$$

In the special case of $\epsilon_1 = \epsilon_2 \equiv \epsilon_0$, this expression should reduce to that of a single spherical capacitor of inner radius a and outer radius b filled with a dielectric of constant permittivity ϵ_0:

$$C_{tot} = \frac{4\pi abc\epsilon_0^2}{a\epsilon_0(b-c) + b\epsilon_0(c-a)} = 4\pi\epsilon_0 \frac{abc}{-ac+bc} = 4\pi\epsilon_0 \frac{ab}{b-a} \ , \tag{41}$$

which is the capacitance of a spherical capacitor[4].

4.2 Bound Charges.

The bound charges in a dielectric are given[5] by

$$\rho_b(r) = -\boldsymbol{\nabla} \cdot \mathbf{P} \tag{42}$$

$$\sigma_b = -\left(\hat{\mathbf{n}} \cdot [\mathbf{P}_2 - \mathbf{P}_1]\right)\Big|_S \tag{43}$$

where ρ_b is the volume bound charge density in the dielectric bulk, σ_b is the surface bound charge density, $\hat{\mathbf{n}}$ is a unit normal to the boundary S from region 1 to 2, and $\mathbf{P}$ is the polarization. The polarization is given[6] by

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E} = \epsilon \mathbf{E} - \epsilon_0 \mathbf{E} \ , \tag{44}$$

using the results from the previous section, the polarization in each region is

$$\mathbf{P}_1 = \frac{Q}{4\pi r^2}\hat{\mathbf{r}} - \epsilon_0 \frac{Q}{4\pi\epsilon_1 r^2}\hat{\mathbf{r}} = \frac{Q}{4\pi r^2}\left(1 - \frac{\epsilon_0}{\epsilon_1}\right)\hat{\mathbf{r}} \tag{45}$$

$$\mathbf{P}_2 = \frac{Q}{4\pi r^2}\hat{\mathbf{r}} - \epsilon_0 \frac{Q}{4\pi\epsilon_2 r^2}\hat{\mathbf{r}} = \frac{Q}{4\pi r^2}\left(1 - \frac{\epsilon_0}{\epsilon_2}\right)\hat{\mathbf{r}} \ . \tag{46}$$

Therefore, the distribution of volume bound charge is

$$\rho_b(r, \epsilon) = -\boldsymbol{\nabla} \cdot \frac{Q}{4\pi r^2}\left(1 - \frac{\epsilon_0}{\epsilon}\right)\hat{\mathbf{r}} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{Q}{4\pi r^2}\left(1 - \frac{\epsilon_0}{\epsilon}\right)\right] = 0 \ , \tag{47}$$

in spherical coordinates, so there is no bound charge in the bulk. This is because the dielectric is uniform through the region. At the interface of the two dielectrics, there must be a bound charge that satisfies Equation 43,

$$\sigma_b^{(c)} = -\left(\hat{\mathbf{r}} \cdot \frac{Q}{4\pi r^2}\left[\left(1 - \frac{\epsilon_0}{\epsilon_2}\right) - \left(1 - \frac{\epsilon_0}{\epsilon_1}\right)\right]\hat{\mathbf{r}}\right)\Bigg|_{r=c} = \frac{Q\epsilon_0}{4\pi c^2}\left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1}\right) \ . \tag{48}$$

[4] Liao, Dourmashkin, Belcher. Visualizing E& M: 5. Capacitors, Equation 5.2.11.
[5] Jackson, Classical Electrodynamics, 3 ed. Page 156.
[6] Jackson, Classical Electrodynamics, 3 ed. Equation 4.34.

5 Problem #5: Dielectric Parallel Plate Capacitor.

A plane parallel capacitor is filled with a dielectric whose permittivity is given by $\epsilon = \epsilon_0 (x+a)/a$, where a is the distance between the electrodes, S is the area of the plates, and the x axis is perpendicular to the plates, with one plate at $x = 0$ and the other at $x = a$. Neglecting edge effects, find the capacitance C and the distribution of bound charges when a potential difference V is applied between the plates.

5.1 Capacitance.

Consider a Gaussian pillbox of face area α that extends infinitesimally above and below the plate at $x = 0$. Using Gauss' law for the electric displacement, we see

$$\int \mathbf{D} \cdot (\alpha \hat{\mathbf{x}}) \mathrm{d}x = \sigma \alpha \ , \tag{49}$$

where σ is the surface charge due to the potential V, so $\mathbf{D} = \sigma \hat{\mathbf{x}}$, and therefore

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{\sigma}{\epsilon_0} \frac{a}{x+a} \hat{\mathbf{x}} \ . \tag{50}$$

The potential drop from the plate at $x = a$ to this plate is

$$V = -\int_0^a \mathbf{E} \cdot \hat{\mathbf{x}} \mathrm{d}x = -\int_0^a \frac{\sigma}{\epsilon_0} \frac{a}{x+a} \mathrm{d}x = -\frac{\sigma a}{\epsilon_0} \ln(x+a) \bigg|_0^a = -\frac{\sigma a}{\epsilon_0} \ln 2 \ , \tag{51}$$

but since the surface charge is just the total free charge Q per unit area, we have

$$C = \frac{Q}{\Delta V} = \frac{S\sigma}{\frac{\sigma a}{\epsilon_0} \ln 2} = \frac{\epsilon_0 S}{a \ln 2} \ . \tag{52}$$

5.2 Bound Charges.

From Equation 44, the polarization is

$$\mathbf{P} = \left(\sigma - \sigma \frac{a}{x+a} \right) \hat{\mathbf{x}} = \left(\frac{x}{x+a} \right) \sigma \hat{\mathbf{x}} \ , \tag{53}$$

and the bound volume charge is

$$\rho_b(x) = -\boldsymbol{\nabla} \cdot \left(\frac{x}{x+a} \right) \sigma \hat{\mathbf{x}} = -\frac{\sigma a}{(x+a)^2} \ . \tag{54}$$

If there is a potential V maintained across the capacitor, from Equation 51 we see

$$\rho_b(x) = \frac{\epsilon_0 V}{\ln 2} \frac{1}{(x+a)^2} \ , \tag{55}$$

so the bound charge density varies as x^{-2} across the dielectric bulk. The bound surface charges are

$$\sigma_b^{(0)} = -\left(\frac{x}{x+a} \right) \sigma \hat{\mathbf{x}} \cdot \hat{\mathbf{x}} \bigg|_{x=0} = 0 \tag{56}$$

$$\sigma_b^{(a)} = -\left(\frac{x}{x+a} \right) \sigma \hat{\mathbf{x}} \cdot \hat{\mathbf{x}} \bigg|_{x=a} = -\frac{\sigma}{2} = \frac{\epsilon_0 V}{2a \ln 2} \ . \tag{57}$$