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Contents

1 Problem #1: Interaction of Polarizable Molecules.

Show that the interaction between two polarizable molecules is attractive, and that it goes like the inverse sixth power of the distance between them.

Consider a single molecule, with no net dipole moment. In the presence of another polarizable molecule, a dipole moment can be spontaneously induced in each molecule. The induced dipole moment (akin to the polarization, just for one particle) of one molecule is proportional to the electric field of the other, and vice versa. The dipole moment of the first molecule, due to the electric field $\mathbf{E}_2$ produced by the second molecule[1] is

$$\mathbf{p}_1 = \epsilon_0 \chi_e \mathbf{E}_2 \ , \tag{1}$$

where χ_e is the electric susceptibility of the first molecule. The dipole moment $\mathbf{p}_2$ simultaneously[2] induced in the second molecule is what creates the electric field $\mathbf{E}_2$ felt by the first molecule (which induces its dipole moment). When the molecules are separated by a distance r, this electric field is

$$\mathbf{E}_2(r) = \frac{3\hat{\mathbf{n}}(\mathbf{p}_2 \cdot \hat{\mathbf{n}}) - \mathbf{p}_2}{4\pi\epsilon_0 r^3} \ , \tag{2}$$

where $\hat{\mathbf{n}}_{21}$ is a unit pointing from molecule two to molecule one. Since the molecules are mutually polarized, the dipole moment will always point along or anti-along the separation vector, so

$$|\mathbf{E}_2|(r) = \frac{|\mathbf{p}_2|}{2\pi\epsilon_0 r^3} \ . \tag{3}$$

The electrostatic energy of a dipole is

$$U = -\mathbf{p} \cdot \mathbf{E} \propto -|\mathbf{E}|^2 \ , \tag{4}$$

so the energy of the first molecule due to the second is

$$U = -\mathbf{p}_1 \cdot \mathbf{E}_2 = -\epsilon_0 \chi_e |\mathbf{E}_2|^2 \propto -\frac{|\mathbf{p}_2|^2}{r^6} \ , \tag{5}$$

which because the potential energy is negative, must be an attractive potential (*e.g.*, gravity). The dipole moments and therefore electric fields of both molecules must be identical (if their electric susceptibilities are), and the energies of each dipole is the same.

[1] Jackson, Classical Electrodynamics 3 Ed., Equation 4.36

[2] As the molecules are brought together, they both simultaneously induce slight dipole moments in each other which produce electric fields that induce a stronger dipole. At a distance r, when the system comes to equilibrium the first molecule has dipole moment $\mathbf{p}_1$ induced by electric field $\mathbf{E}_2$ of the second molecule, due to the dipole moment $\mathbf{p}_2$, which is induced in the second molecule by the electric field $\mathbf{E}_1$ produced by the first molecule, due to its dipole moment $\mathbf{p}_1$. This system is symmetric if the molecules are identical (having the same electric susceptibility), so the field and dipole moment produced and induced in one by the other, must be exactly the same as the field and dipole moment induced in the other by the first.

2 Problem #2: Polarizability of Electron in Harmonic Potential.

Determine quantum mechanically the polarizability of an electron in a one-dimensional harmonic potential.

For linear dielectrics, the electric field is always parallel to the electric displacement, so the electrostatic energy of a dipole is

$$U = -\mathbf{p} \cdot \mathbf{E} = -\epsilon_0 \chi |\mathbf{E}|^2 = -\epsilon_0 \chi \mathcal{E}^2 \ , \tag{6}$$

where χ is the polarizability, which can be determined by

$$\chi = -\frac{1}{2\epsilon_0} \frac{\partial^2 U}{\partial \mathcal{E}^2} \ . \tag{7}$$

Consider an electron in a one dimensional harmonic oscillator in the region of a uniform electric field $\mathcal{E}$ along the dimension of the oscillator. The Hamiltonian of this electron is

$$\mathscr{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 z^2 + e\mathcal{E} z \ , \tag{8}$$

where m and e are the electron mass and charge, and ω is the oscillation frequency. If we complete the square for the terms with z dependence, we enforce

$$Az^2 + Bz = C(z + z_0)^2 + D \ , \tag{9}$$

where A and B can be read off from Equation 8. Equating powers of z, we find

$$A = C \tag{10}$$
$$B = 2Cz_0 \tag{11}$$
$$0 = Cz_0^2 + D \ , \tag{12}$$

which give the results

$$C = \frac{1}{2} m\omega_2 \tag{13}$$
$$e\mathcal{E} = m\omega_2 z_0 \quad \Rightarrow \quad z_0 = \frac{e\mathcal{E}}{m\omega^2} \tag{14}$$
$$D = -\frac{1}{2} \frac{e^2 \mathcal{E}^2}{m\omega^2} \ , \tag{15}$$

allowing the Hamiltonian to be expressed as

$$\mathscr{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 \left(z + \frac{e\mathcal{E}}{m\omega^2} \right)^2 - \frac{1}{2} \frac{e^2 \mathcal{E}^2}{m\omega^2} \ . \tag{16}$$

This can be identified as a harmonic oscillator with energies

$$\epsilon_n = \hbar\omega(n + \tfrac{1}{2}) - \frac{1}{2} \frac{e^2 \mathcal{E}^2}{m\omega^2} \ , \tag{17}$$

and wave functions

$$\psi_n(z) = \psi_n^{\text{HO}} \left(z + \frac{e\mathcal{E}}{m\omega^2} \right) \ , \tag{18}$$

where $\psi_n^{\rm HO}(x)$ are the eigenfunctions of the simple one dimensional harmonic oscillator. We can find the expectation value of the dipole moment using quantum mechanics

$$\langle \mathbf{p} \rangle = -e \langle z \rangle = -e \int_{-\infty}^{\infty} \psi_n^{\rm HO*} \left(z + \frac{e\mathcal{E}}{m\omega^2} \right) z \psi_n^{\rm HO} \left(z + \frac{e\mathcal{E}}{m\omega^2} \right) {\rm d}z \ , \tag{19}$$

which is simply the expectation value of z. For the harmonic oscillator centered at zero this expectation value is zero, so we can say the expectation value for a shifted oscillator is the where the new coordinate (argument of wave function) is zero, so

$$\langle \mathbf{p} \rangle = -e \left(-\frac{e\mathcal{E}}{m\omega^2} \right) = \frac{e^2\mathcal{E}}{m\omega^2} \ , \tag{20}$$

using Jackson equation 4.67, we see the polarizability is

$$\gamma = \frac{e^2}{\epsilon_0 m\omega^2} \ . \tag{21}$$

Using the classical definition of polarizability given at the start of this problem we see

$$\frac{\partial^2}{\partial \mathcal{E}^2} \left[\hbar\omega(n + \tfrac{1}{2}) - \frac{1}{2} \frac{e^2\mathcal{E}^2}{m\omega^2} \right] = -\frac{e^2}{m\omega^2} \quad \Rightarrow \quad \chi = \frac{e^2}{\epsilon_0 m\omega^2} \ , \tag{22}$$

which is the exact same result as obtained quantum mechanically.

3 Problem #3: Dielectric Spherical Shell.

Consider a spherical dielectric shell of permittivity ϵ and inner and outer radii a and b ($b > a$) in a uniform external field $\mathbf{E}_0$. Determine the field inside the shell ($r < a$) as a function of ϵ, a, b. What happens as $\epsilon \to \infty$?

Consider the same geometry and coordinate system as in Jackson figure 4.6, but with the outer radius b and an interior cavity of radius a. Let us then define the regions: region 1 for $r < a$, region 2 for $a < r < b$, and region 3 for $r > b$. Using Jackson equation 4.49, we can define the potential in all three regions:

$$\phi_1 = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + B_\ell r^{-(\ell+1)} \right] P_\ell(\cos\theta) \tag{23}$$

$$\phi_2 = \sum_{\ell=0}^{\infty} \left[C_\ell r^\ell + D_\ell r^{-(\ell+1)} \right] P_\ell(\cos\theta) \tag{24}$$

$$\phi_3 = \sum_{\ell=0}^{\infty} \left[F_\ell r^\ell + G_\ell r^{-(\ell+1)} \right] P_\ell(\cos\theta) \ , \tag{25}$$

where the constants A, B, C, D, F, G will be determined by boundary conditions, and $P_\ell(x)$ are the Legendre polynomials. Since the potential in region one must be finite at the origin we can say $B_\ell = 0 \ \forall \ell \in \mathbb{Z}$. Similarly to the process in Jackson, we say[3] that $F_\ell = -E_0 \delta_{\ell 1}$, where $\delta_{\ell 1}$ is the Kronecker delta.

Using the boundary conditions on the tangential components of $\mathbf{E}$ and the normal component of $\mathbf{D}$ we obtain the conditions

$$-\frac{1}{a}\frac{\partial \phi_1}{\partial \theta}\bigg|_{r=a} = -\frac{1}{a}\frac{\partial \phi_2}{\partial \theta}\bigg|_{r=a} \quad \text{and} \quad -\epsilon_0 \frac{\partial \phi_1}{\partial r}\bigg|_{r=a} = -\epsilon \frac{\partial \phi_2}{\partial r}\bigg|_{r=a} \tag{26}$$

$$-\frac{1}{b}\frac{\partial \phi_2}{\partial \theta}\bigg|_{r=b} = -\frac{1}{b}\frac{\partial \phi_3}{\partial \theta}\bigg|_{r=b} \quad \text{and} \quad -\epsilon \frac{\partial \phi_2}{\partial r}\bigg|_{r=b} = -\epsilon_0 \frac{\partial \phi_3}{\partial r}\bigg|_{r=b} \ , \tag{27}$$

where the top line is the conditions for the surface at $r = a$ and the second line is the conditions for the surface $r = b$. Let us calculate these derivatives noting that

$$\frac{\partial}{\partial \theta} P_\ell(\cos\theta) = P_\ell^1(\cos\theta) \ , \tag{28}$$

where $P_\ell^1(x)$ is the ℓth associated Legendre function of first order. The derivatives of the potential in region one are then

$$\frac{\partial \phi_1}{\partial \theta} = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell^1(\cos\theta) \tag{29}$$

$$\frac{\partial \phi_1}{\partial r} = \sum_{\ell=0}^{\infty} A_\ell \ell r^{\ell-1} P_\ell(\cos\theta) \ , \tag{30}$$

[3]The electric field is $\mathbf{E}_0 = E_0 \mathbf{z} = E_0 r \cos\theta$, so the coefficient of the $r^1 P_1(\cos\theta)$ must be the electric field for large values of r. This imposes no condition on G_ℓ because the negative exponentials vanish for large r.

and region two:

$$\frac{\partial \phi_2}{\partial \theta} = \sum_{\ell=0}^{\infty} \left[C_\ell r^\ell + D_\ell r^{-(\ell+1)} \right] P_\ell^1(\cos\theta) \tag{31}$$

$$\frac{\partial \phi_2}{\partial r} = \sum_{\ell=0}^{\infty} \left[C_\ell \ell r^{\ell-1} - (\ell+1) D_\ell r^{-(\ell+2)} \right] P_\ell(\cos\theta) \ , \tag{32}$$

and region three:

$$\frac{\partial \phi_3}{\partial \theta} = E_0 r \sin\theta + \sum_{\ell=0}^{\infty} G_\ell r^{-(\ell+1)} P_\ell^1(\cos\theta) \tag{33}$$

$$\frac{\partial \phi_3}{\partial r} = -E_0 \cos\theta + \sum_{\ell=0}^{\infty} -(\ell+1) G_\ell r^{-(\ell+2)} P_\ell(\cos\theta) \ . \tag{34}$$

Starting with the first condition, we see

$$0 = \left(\frac{\partial \phi_1}{\partial \theta} - \frac{\partial \phi_2}{\partial \theta} \right) \Bigg|_{r=a} = \sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell^1(\cos\theta) - \sum_{\ell=0}^{\infty} \left[C_\ell a^\ell + D_\ell a^{-(\ell+1)} \right] P_\ell^1(\cos\theta) \ , \tag{35}$$

since the associated Legendre functions form an orthogonal basis, the coefficient of each term in the sum must vanish independently,

$$0 = A_\ell a^\ell - C_\ell a^\ell - D_\ell a^{-(\ell+1)} \quad \Rightarrow \quad A_\ell = C_\ell + D_\ell a^{-(2\ell+1)} \ . \tag{36}$$

Similarly, for the tangential condition at the $r = b$ surface,

$$\begin{aligned} 0 = \left(\frac{\partial \phi_2}{\partial \theta} - \frac{\partial \phi_3}{\partial \theta} \right) \Bigg|_{r=b} &= \sum_{\ell=0}^{\infty} \left[C_\ell b^\ell + D_\ell b^{-(\ell+1)} \right] P_\ell^1(\cos\theta) - \sum_{\ell=0}^{\infty} \left[-E_0 b^\ell \delta_{1\ell} + G_\ell b^{-(\ell+1)} \right] P_\ell^1(\cos\theta) \\ &= C_\ell b^\ell + D_\ell b^{-(\ell+1)} + E_0 b \delta_{1\ell} \sin\theta - G_\ell b^{-(\ell+1)} \\ \ell = 1 \ : \quad & 0 = C_1 b + D_1 b^{-2} + E_0 b - G_1 b^{-2} \quad \Rightarrow \quad G_1 = D_1 + (C_1 + E_0) b^3 \\ \ell \neq 1 \ : \quad & 0 = C_\ell b^\ell + D_\ell b^{-(\ell+1)} - G_\ell b^{-(\ell+1)} \quad \Rightarrow \quad C_\ell = (G_\ell - D_\ell) b^{-(2\ell+1)} \end{aligned}$$

Now consider the normal component conditions on the surface at $r = a$, since there is no free charge anywhere, we have

$$0 = \left(\epsilon \frac{\partial \phi_2}{\partial \theta} - \epsilon_0 \frac{\partial \phi_1}{\partial \theta} \right) \Bigg|_{r=a} \tag{37}$$

$$= \epsilon \sum_{\ell=0}^{\infty} \left[C_\ell \ell a^{\ell-1} - (\ell+1) D_\ell a^{-(\ell+2)} \right] P_\ell(\cos\theta) - \epsilon_0 \sum_{\ell=0}^{\infty} A_\ell \ell a^{\ell-1} P_\ell(\cos\theta) \ , \tag{38}$$

since the Legendre polynomials form an orthogonal basis, the coefficient of each term in the sum must vanish independently:

$$0 = \epsilon C_\ell \ell a^{\ell-1} - \epsilon(\ell+1) D_\ell a^{-(\ell+2)} - \epsilon_0 A_\ell \ell a^{\ell-1} \tag{39}$$

$$= \frac{\ell}{\ell+1} \left(\epsilon C_\ell - \epsilon_0 A_\ell \right) - \epsilon D_\ell a^{-(2\ell+1)} \ . \tag{40}$$

The final boundary condition is

$$
\begin{aligned}
0 &= \left(\epsilon \frac{\partial \phi_2}{\partial \theta} - \epsilon_0 \frac{\partial \phi_3}{\partial \theta} \right) \Bigg|_{r=b} \\
&= \epsilon \sum_{\ell=0}^{\infty} \left[C_\ell \ell b^{\ell-1} - (\ell+1) D_\ell b^{-(\ell+2)} \right] P_\ell(\cos\theta) - \epsilon_0 \sum_{\ell=0}^{\infty} \left[-E_0 \delta_{1\ell} b^{\ell-1} - (\ell+1) G_\ell b^{-(\ell+2)} \right] P_\ell(\cos\theta) \\
&= \epsilon C_\ell \ell b^{\ell-1} - \epsilon(\ell+1) D_\ell b^{-(\ell+2)} + \epsilon_0 E_0 \delta_{1\ell} + \epsilon_0 (\ell+1) G_\ell b^{-(\ell+2)}.
\end{aligned}
$$

For $\ell = 1$ this becomes

$$
0 = \epsilon C_1 - 2\epsilon D_1 b^{-3} + \epsilon_0 E_0 + 2\epsilon_0 G_1 b^{-3} \ , \tag{41}
$$

and for $\ell \neq 1$, we have

$$
0 = \epsilon C_\ell \ell b^{2\ell+1} + (\ell+1)(\epsilon_0 G_\ell - \epsilon D_\ell) \ , \tag{42}
$$

after multiplying through by $b^{\ell+2}$.

In summary, the boundary conditions (in the order: tangential at $r = a$, tangential at $r = b$, normal at $r = a$, and normal at $r = b$) for $\ell = 1$ are

$$
\begin{aligned}
0 &= (C_1 - A_1)a^3 + D_1 && (43) \\
0 &= D_1 - G_1 + (C_1 + E_0) b^3 && (44) \\
0 &= (\epsilon_0 A_1 - \epsilon C_1) a^3 + 2\epsilon D_1 && (45) \\
0 &= \epsilon C_1 - 2\epsilon D_1 b^{-3} + \epsilon_0 E_0 + 2\epsilon_0 G_1 b^{-3} \ . && (46)
\end{aligned}
$$

The boundary conditions for $\ell \neq 1$ are

$$
\begin{aligned}
0 &= (C_\ell - A_\ell) a^{2\ell+1} + D_\ell && (47) \\
0 &= D_\ell - G_\ell + C_\ell b^{2\ell+1} && (48) \\
0 &= (\epsilon_0 A_\ell - \epsilon C_\ell) \ell a^{2\ell+1} + \epsilon D_\ell (\ell+1) && (49) \\
0 &= \epsilon C_\ell \ell b^{2\ell+1} + (\ell+1)(\epsilon_0 G_\ell - \epsilon D_\ell) \ . && (50)
\end{aligned}
$$

Using Mathematica to solve for the constants A_1, C_1, D_1, G_1, we get the results

$$
\begin{aligned}
A_1 &= -\frac{1}{f} \left(9 E_0 b^3 \epsilon \epsilon_0 \right) && (51) \\
C_1 &= -\frac{1}{f} \left(3 E_0 b^3 \epsilon_0 \left(2\epsilon + \epsilon_0 \right) \right) && (52) \\
D_1 &= -\frac{1}{f} \left(3 E_0 a^3 b^3 \left(\epsilon - \epsilon_0 \right) \epsilon_0 \right) && (53) \\
G_1 &= \frac{1}{f} \left(E_0 b^3 \left(\epsilon - \epsilon_0 \right) \left(2\epsilon + \epsilon_0 \right) \left(b^3 - a^3 \right) \right) && (54) \\
\text{with} \quad f &= 2\epsilon^2 \left(b^3 - a^3 \right) + \epsilon_0 \epsilon \left(4a^3 + 5b^3 \right) + 2\epsilon_0^2 \left(b^3 - a^3 \right) \ . && (55)
\end{aligned}
$$

Using Mathematica to solve for the coefficients for arbitrary ℓ, shows the only solution is if the constants are $A_\ell = C_\ell = D_\ell = G_\ell = 0$, which is not an interesting solution. The potentials in each region are then

$$\phi_1 = -\frac{1}{f}\left(9E_0 b^3 \epsilon\epsilon_0\right) r P_1(\cos\theta) \tag{56}$$

$$\phi_2 = -\frac{1}{f}\left[\left(3E_0 b^3 \epsilon_0 \left(2\epsilon + \epsilon_0\right)\right) r + \left(3E_0 a^3 b^3 \left(\epsilon - \epsilon_0\right)\epsilon_0\right)\frac{1}{r^2}\right] P_1(\cos\theta) \tag{57}$$

$$\phi_3 = -E_0 r\cos\theta + \frac{1}{f}\left[\left(E_0 b^3 \left(\epsilon - \epsilon_0\right)\left(2\epsilon + \epsilon_0\right)\left(b^3 - a^3\right)\right)\frac{1}{r^2}\right] P_1(\cos\theta)\ , \tag{58}$$

where $P_1(\cos\theta) = \cos\theta$. The electric field in the interior cavity is

$$\mathbf{E}_1 = -\nabla\phi_1 = -\left(\frac{\partial\phi_1}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\phi_1}{\partial\theta}\hat{\boldsymbol{\theta}}\right) \tag{59}$$

$$= 9E_0 b^3\epsilon\epsilon_0\left[\cos\theta\frac{\partial}{\partial r}\frac{r}{f}\hat{\mathbf{r}} + \frac{\partial}{\partial\theta}\frac{\cos\theta}{f}\hat{\boldsymbol{\theta}}\right] \tag{60}$$

$$= \frac{9E_0 b^3\epsilon\epsilon_0}{2\epsilon^2\left(b^3 - a^3\right) + \epsilon_0\epsilon\left(4a^3 + 5b^3\right) + 2\epsilon_0^2\left(b^3 - a^3\right)}\left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right)\ . \tag{61}$$

Consider the coefficient in the above expression in the limit $\epsilon \to \infty$. From a qualitative standpoint, this limit makes it so that the electric field cannot penetrate the dielectric material, and the electric field inside must be zero. This seems to say that as the dielectric constant tends to infinity, the dielectric material behaves like a perfect conductor. Mathematically, we can show that in this limit the electric field inside the cavity is zero, just as in a perfect conductor:

$$\lim_{\epsilon\to\infty}\mathbf{E}_1 \lim_{\epsilon\to\infty} \propto \frac{\alpha\epsilon}{\beta\epsilon^2 + \gamma\epsilon + \delta} = \lim_{\epsilon\to\infty}\frac{\alpha}{\beta\epsilon + \gamma} = 0\ , \tag{62}$$

by L'Hôpital's rule. This proves the electric field inside the cavity is zero as the dielectric constant tends to infinity.

4 Problem #4: Dielectric Infinite Cylinder.

By solving Laplace's equation, derive the electric field inside and outside an infinitely long uniform dielectric cylinder of permittivity $\epsilon = \epsilon_r \epsilon_0$ placed in a uniform electric field $\mathbf{E}_0$ oriented perpendicular to the cylinder's axis. Determine the depolarization factors. Based in your results, what do you expect the field inside the cylinder to be if the external field were aligned along the axis of the cylinder.

For an infinite cylinder of uniform dielectric in a uniform electric field, the potential everywhere in space can have no z dependence. Furthermore, we will assume a separable solution to Laplace's equation such that

$$\nabla^2 \Phi(\rho, \phi) = \nabla^2 R(\rho) Y(\phi) = 0 \ , \tag{63}$$

in cylindrical coordinates this is

$$0 = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho} + \rho \frac{\partial^2}{\partial \rho^2}\right] R(\rho)Y(\phi) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} R(\rho) Y(\phi) \tag{64}$$

$$= \frac{1}{\rho R}\left[\frac{\partial R}{\partial \rho} + \rho \frac{\partial^2 R}{\partial \rho^2}\right] + \frac{1}{Y\rho^2}\frac{\partial^2 Y}{\partial \phi^2} \tag{65}$$

$$= \frac{1}{R}\left[\rho\frac{\partial R}{\partial \rho} + \rho^2 \frac{\partial^2 R}{\partial \rho^2}\right] + \frac{1}{Y}\frac{\partial^2 Y}{\partial \phi^2} \ , \tag{66}$$

after multiplying the equation by ρ^2. We are left with the two equations

$$k^2 R = \rho^2 R'' + \rho R' \tag{67}$$

$$k^2 Y = -Y'' \ , \tag{68}$$

where k is an arbitrary constant. The polar equation has sine and cosine solutions:

$$Y(\phi) = C \sin(k\phi) + D \sin(k\phi) \ , \tag{69}$$

and we will assume a power series solution[4] for the radial equation,

$$R(\rho) = \sum_{n=-\infty}^{\infty} a_n \rho^n \ , \tag{70}$$

where a_n are constants. If we substitute this ansatz into the radial differential equation, we obtain

$$0 = \rho^2 \sum_{n=-\infty}^{\infty} n(n+1) a_n \rho^{n-2} + \rho \sum_{n=-\infty}^{\infty} n a_n \rho^{n-1} + \sum_{n=-\infty}^{\infty} (-k^2) a_n \rho^n \tag{71}$$

$$0 = \sum_{n=-\infty}^{\infty} \left(n(n+1) + n - k^2\right) a_n \rho^n = \sum_{n=-\infty}^{\infty} \left(n^2 - k^2\right) a_n \rho^n \ , \tag{72}$$

due to the form of the ansatz, each term must be zero independently, and if we enforce that $a_n \neq 0$, it must be that

$$0 = n^2 - k^2 \quad \Rightarrow \quad k = \pm n \ , \tag{73}$$

[4] Since each order of derivative appears next to an equivalent power of ρ, a power series of the indicated form will satisfy this ODE. Taking n derivative reduces the power of ρ by n, then multiplying it by a factor of ρ^n ensures that each power of ρ will appear in multiple terms and each term in the infinite sum is allowed to cancel exactly.

so it must be that since n is any integer, $k \in \mathbb{Z}$. We can then express the power series solution as

$$R(\rho) = a_0 + \sum_{n=1}^{\infty} a_n \rho^n + \sum_{n=1}^{\infty} a_{-n} \rho^{-n} \ , \tag{74}$$

but we are free to set the zero of our potential such that[5] $a_0 = 0$. The potential everywhere, without enforcing boundary conditions is

$$\Phi(\rho,\phi) = \sum_{n=1}^{\infty} \rho^n \left(b_n \sin n\phi + c_n \cos n\phi\right) + \sum_{n=1}^{\infty} \rho^{-n} \left(b_n \sin n\phi + c_n \cos n\phi\right) \ , \tag{75}$$

where the a_n were incorporated in the new coefficients b_n, c_n. The potential inside the cylinder must be finite at the origin, and also finite as $\rho \to \infty$, so we can separate the potential into separate expressions for the regions inside and outside the cylinder surface,

$$\Phi_{\text{in}}(\rho,\phi) = \sum_{n=1}^{\infty} \rho^n \left(b_n \sin n\phi + c_n \cos n\phi\right) \tag{76}$$

$$\Phi_{\text{out}}(\rho,\phi) = -E_0 \rho \cos\phi + \sum_{n=1}^{\infty} \rho^{-n} \left(d_n \sin n\phi + f_n \cos n\phi\right) \tag{77}$$

$$= \sum_{n=1}^{\infty} -E_0 \rho^n \delta_{1n} \cos n\phi + \rho^{-n} \left(d_n \sin n\phi + f_n \cos n\phi\right) \ , \tag{78}$$

where the first term in Φ_{out} is due to the far field potential needing to be equivalent to that due to the uniform magnetic field. Here we have set the electric field along the x axis $\mathbf{E}_0 = E_0 \rho \cos\phi$, and the x axis is taken at $\phi = 0$. The cylinder has radius r. Using the boundary conditions for the tangential component of $\mathbf{E}$ and the normal component of the electric displacement $\mathbf{D}$, we must have

$$0 = \left(\frac{\partial \Phi_{\text{in}}}{\partial \phi} - \frac{\partial \Phi_{\text{out}}}{\partial \phi}\right)\bigg|_{\rho=s} \tag{79}$$

$$0 = \left(\epsilon \frac{\partial \Phi_{\text{in}}}{\partial \rho} - \epsilon_0 \frac{\partial \Phi_{\text{out}}}{\partial \rho}\right)\bigg|_{\rho=r} \ , \tag{80}$$

again since there is no free charge. The first condition gives

$$0 = \sum_{n=1}^{\infty} r^n \left(b_n n \cos n\phi - c_n n \sin n\phi\right) + E_0 n r \sin\phi - \sum_{n=1}^{\infty} a^{-n} \left(d_n n \cos n\phi - f_n n \sin n\phi\right) \tag{81}$$

$$= \sum_{n=1}^{\infty} \left(-r^n c_n n + E_0 n r^n \delta_{1n} + r^{-n} f_n n\right) \sin n\phi + \sum_{n=1}^{\infty} \left(r^n b_n n - r^{-n} d_n n\right) \cos n\phi \tag{82}$$

$$= \sum_{n=1}^{\infty} \left(-r^{2n} c_n + E_0 r^{2n} \delta_{1n} + f_n\right) n \sin n\phi + \sum_{n=1}^{\infty} \left(r^{2n} b_n - d_n\right) n \cos n\phi \ , \tag{83}$$

since both sines and cosines are form an orthogonal basis, their coefficients in each term must vanish:

$$0 = -r^{2n} c_n + E_0 r^{2n} \delta_{1n} + f_n \quad \Rightarrow \quad f_n = r^{2n}(c_n + E_0 \delta_{1n}) \tag{84}$$

$$0 = r^{2n} b_n - d_n \quad \Rightarrow \quad d_n = r^{2n} b_n \ . \tag{85}$$

[5] Especially because we are interested in the electric field, not the potential, so the derivatives with respect to r vanish.

The second boundary condition gives

$$0 = \epsilon \sum_{n=1}^{\infty} n r^{n-1} \left(b_n \sin n\phi + c_n \cos n\phi \right) + \epsilon_0 \sum_{n=1}^{\infty} -E_0 n r^{n-1} \delta_{1n} \cos n\phi + n r^{-(n+1)} \left(d_n \sin n\phi + f_n \cos n\phi \right)$$
$$= \sum_{n=1}^{\infty} \left[\epsilon r^{n-1} b_n + \epsilon_0 r^{-(n+1)} d_n \right] n \sin n\phi + \sum_{n=1}^{\infty} \left[\epsilon r^{n-1} c_n + \epsilon_0 E_0 r^{n-1} \delta_{1n} + \epsilon_0 r^{-(n+1)} f_n \right] n \cos n\phi$$

again the coefficient in each term of the sum must vanish independently:

$$0 = \epsilon r^{n-1} b_n + \epsilon_0 r^{-(n+1)} d_n \quad \Rightarrow \quad d_n = -r^{2n} \frac{\epsilon}{\epsilon_0} b_n \tag{86}$$
$$0 = \epsilon r^{n-1} c_n + \epsilon_0 E_0 r^{n-1} \delta_{1n} + \epsilon_0 r^{-(n+1)} f_n \quad \Rightarrow \quad f_n = -r^{2n} \left(\frac{\epsilon}{\epsilon_0} c_n + E_0 \delta_{1n} \right) \ , \tag{87}$$

The boundary conditions can be summarized as

$$f_n = r^{2n} (c_n + E_0 \delta_{1n}) \tag{88}$$
$$d_n = r^{2n} b_n \tag{89}$$
$$d_n = -r^{2n} \frac{\epsilon}{\epsilon_0} b_n \tag{90}$$
$$f_n = -r^{2n} \left(\frac{\epsilon}{\epsilon_0} c_n + E_0 \delta_{1n} \right) \ . \tag{91}$$

For Equations 89 and 90 to be consistent we must have $\epsilon = -\epsilon_0$. We cannot have negative permittivities, so this result is unphysical. The only way these conditions are consistent is if $d_n = b_n = 0 \ \forall n \in \mathbb{Z}$. Extending this logic to Equations 88 and 91, for $n \neq 1$, we obtain the same inconsistency as before, so the only nonzero constants are c_1 and f_1. We now have

$$f_1 = r^2 (c_1 - E_0) \tag{92}$$
$$f_1 = -r^2 \left(\frac{\epsilon}{\epsilon_0} c_1 + E_0 \right) \ , \tag{93}$$

which can be solved for the results

$$c_1 = -2E_0 \frac{\epsilon_0}{\epsilon + \epsilon_0} \tag{94}$$
$$f_1 = E_0 r^2 \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \ , \tag{95}$$

which we can insert into the expressions for the potentials inside and outside the cylinder to obtain

$$\Phi_{\text{in}}(\rho, \phi) = -2E_0 \frac{\epsilon_0}{\epsilon + \epsilon_0} \rho \cos \phi \tag{96}$$
$$\Phi_{\text{out}}(\rho, \phi) = -E_0 \rho \cos \phi + E_0 r^2 \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \frac{1}{\rho} \cos \phi \ . \tag{97}$$

The corresponding electric fields are given by the negative gradient of the potential:

$$\mathbf{E}_{\text{in}} = -\nabla \Phi_{\text{in}} = -\left(\frac{\partial \Phi_{\text{in}}}{\partial \rho} \hat{\boldsymbol{\rho}} + \frac{1}{\rho} \frac{\partial \Phi_{\text{in}}}{\partial \phi} \hat{\boldsymbol{\phi}} \right) \tag{98}$$
$$= 2E_0 \frac{\epsilon_0}{\epsilon + \epsilon_0} \cos \phi \hat{\boldsymbol{\rho}} - 2E_0 \frac{\epsilon_0}{\epsilon + \epsilon_0} \sin \phi \hat{\boldsymbol{\phi}} \tag{99}$$
$$= 2E_0 \frac{\epsilon_0}{\epsilon + \epsilon_0} \left(\cos \phi \hat{\boldsymbol{\rho}} - \sin \phi \hat{\boldsymbol{\phi}} \right) = 2E_0 \frac{\epsilon_0}{\epsilon + \epsilon_0} \hat{\mathbf{x}} \ . \tag{100}$$

note that this is along the same direction as the original electric field $\mathbf{E}_0$. The potential outside is

$$\mathbf{E}_{\text{out}} = -\nabla\Phi_{\text{out}} = -\left(\frac{\partial\Phi_{\text{out}}}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\Phi_{\text{out}}}{\partial\phi}\hat{\boldsymbol{\phi}}\right) \tag{101}$$

$$= \left[E_0\cos\phi + E_0\frac{r^2}{\rho^2}\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\cos\phi\right]\hat{\boldsymbol{\rho}} + \left[-E_0\sin\phi + E_0\frac{r^2}{\rho^2}\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\frac{r^2}{\rho^2}\sin\phi\right]\hat{\boldsymbol{\phi}} \tag{102}$$

$$= E_0\left[\left(\frac{r^2}{\rho^2}\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}+1\right)\cos\phi\hat{\boldsymbol{\rho}} + \left(\frac{r^2}{\rho^2}\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}-1\right)\sin\phi\hat{\boldsymbol{\phi}}\right] \ . \tag{103}$$

For the case in which the electric field was aligned along the axis of the cylinder, I would assume the electric field inside would be exactly the same as that outside. Since the cylinder is infinite in extent, any polarization caused by the electric field in a single plane (perpendicular to the axis of the cylinder) will be canceled out by the planes in front and behind it.

Depolarization factors N_i are geometrical constants determined by the polarization along each axis. In Cartesian coordinates the sum of depolarization factors must be one: $1 = N_x + N_y + N_z$. Additionally, since the cylinder is infinite in extent in the z direction we must have $N_z = 0$. Through symmetry in the $x-y$ plane, it must be that $N_x = N_y = \frac{1}{2}$. This can be shown mathematically:

We can write the electric field inside the dielectric as

$$\mathbf{E}_{\text{in}} = 2E_0\frac{1}{\frac{\epsilon}{\epsilon_0}+1}\hat{\mathbf{x}} = E_0\frac{1+1+\frac{\epsilon}{\epsilon_0}-\frac{\epsilon}{\epsilon_0}}{\frac{\epsilon}{\epsilon_0}+1}\hat{\mathbf{x}} = E_0\left(\frac{1+\frac{\epsilon}{\epsilon_0}}{\frac{\epsilon}{\epsilon_0}+1}+\frac{1-\frac{\epsilon}{\epsilon_0}}{\frac{\epsilon}{\epsilon_0}+1}\right)\hat{\mathbf{x}} \tag{104}$$

$$= E_0\hat{\mathbf{x}} - E_0\left(\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\right)\hat{\mathbf{x}} \ , \tag{105}$$

while the polarization of the dielectric is

$$\mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E} = (\epsilon-\epsilon_0)\mathbf{E} = 2E_0\epsilon_0\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\hat{\mathbf{x}} \ , \tag{106}$$

so if we write the electric field inside the dielectric in terms of the polarization we see

$$\mathbf{E}_{\text{in}} = \mathbf{E}_0 - \frac{1}{2}\frac{\mathbf{P}}{\epsilon_0} \ , \tag{107}$$

which we note that the coefficient of $-\mathbf{P}/\epsilon_0$ is the depolarization coefficient[6], in this case it is $N_x = 1/2$ as claimed previously. If we repeat the calculation with the initial electric field aligned along the y axis we find the same result, $N_y = 0$. If the electric field is aligned along the axis of the cylinder, we have previously said $\mathbf{E}_{\text{in}} = \mathbf{E}_0$, so the coefficient of the polarization is zero: $N_z = 0$, as expected.

[6]Garg, Classical Electrodynamics in a Nutshell. Section 96, page 351.