Dylan J. Temples: Solution Set Eight

Northwestern University, Electrodynamics I
Wednesday, March 9, 2016

Contents

1 Problem #1: Toroidal Electromagnet with Gap.

Estimate the field strength H in the gap of the circular electromagnet shown in Figure 1. The electromagnet has N turns of wire carrying a current I wrapped around a magnetic material with circular cross-sectional area F_0, permeability μ and mean circumference $L - l$, where l is the width of the gap. You can assume constant and Homogeneous fields.

Let us define a coordinate x which is the distance along the mean circumference, with $x = 0 = L$ at one end of the gap. The positive direction is away from the gap, as indicated. If we consider the closed-loop integral from $x = 0$ to $x = L$:

$$\oint \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \oint \mathbf{H} \cdot \mathrm{d}\hat{\mathbf{x}} = \oint H \mathrm{d}x = I_{\mathrm{enc}} = NI \ , \tag{1}$$

if there are N total turns around this the loop. We can note that the magnetic field from a circular electromagnet is entirely in the axial direction, $\mathbf{H} = H\hat{\boldsymbol{\phi}}$. Therefore $\hat{\mathbf{x}} \parallel \mathbf{H}$, so the dot product results in the entire magnitude of the fields. Each loop contributes a current I independently. We can then express the integral as

$$\int_0^{L-l} H_i \mathrm{d}x + \int_{L-l}^{L} H_o \mathrm{d}x = NI \ , \tag{2}$$

where H_i is the magnetic field inside the electromagnet along the mean circumference and H_o is the magnetic field in the gap. Evaluating this, we see

$$NI = H_i(L - l) + H_o l \ . \tag{3}$$

If we approximate the gap as small, we can say the field in the gap also points in the axial direction, and the field normal to the faces of the gap, is the entire magnetic field. If we use the magnetostatic boundary conditions, we see

$$B_\perp^i = B_\perp^o \quad \Rightarrow \quad \mathbf{B}_i = \mathbf{B}_o \ . \tag{4}$$

We can replace the magnetic induction with the magnetic field by using the permeability of the material. In the electromagnet, the permeability is μ and in the gap it is μ_0, so

$$\mu \mathbf{H}_i = \mu_0 \mathbf{H}_o \quad \Rightarrow \quad H_i = \frac{\mu_0}{\mu} H_o \ . \tag{5}$$

If we insert this into Equation 3, we obtain

$$NI = H_o \left(\frac{\mu_0}{\mu}(L - l) + l \right) \quad \Rightarrow \quad H_o = \frac{NI}{\frac{\mu_0}{\mu}(L - l) + l} \ . \tag{6}$$

This is a reasonable result because in the limit that $\mu_0 \to \mu$ (the gap being filled with the same dielectric material as the rest of the electromagnet), we get the magnetic field due to a circular electromagnet:

$$H = \frac{NI}{L} = \frac{1}{\mu} B_{\mathrm{toroid}} \ , \tag{7}$$

as expected - we get the field in the gap region (now with permeability μ) to be the same as the field around the rest of the mean circumference. We get the same result if we take $l = 0$:

$$H = \frac{NI}{\frac{\mu_0}{\mu} L} = \frac{\mu}{\mu_0} \frac{NI}{L} = \frac{\mu}{\mu_0} B_{\mathrm{toroid}} \ , \tag{8}$$

but the in the gap region we must replace μ_0 by μ, and we get the same result.

2 Problem #2: Electromagnetic Momentum.

A solenoid of radius R with n turns per unit length carries a stationary current I. Two hollow cylinders of length l are fixed coaxially with it and are free to rotate along their axes (l is much shorter than the length of the solenoid). One cylinder of radius a is inside the coil ($a < R$) and carries a uniformly distributed charge Q. The outer cylinder of radius $b > R$ carries a charge $-Q$. If the current is switched off, the cylinders start to rotate. Calculate the resulting angular momentum of each cylinder, and show this angular momentum results from the electromagnetic field.

Let us define a coordinate system with the z axis along the central axis of the cylinders, where $z = 0$ is the plane perpendicular to the axis that intersects the inner and outer cylinders halfway down their length l. The only magnetic field in the system is interior to the solenoid ($r < R$), so the Poynting vector only exists in this region (the poynting vector carries the momentum of electromagnetic fields). The magnetic induction in this region is simply

$$\mathbf{B} = \mu_0 In\hat{\mathbf{z}} \ , \tag{9}$$

where n is the number of turns per unit length, and I is the current running through the solenoid (in a direction such that it produces a field in the positive z direction). For $r < a$ there is no electric field (no enclosed charge), and since the magnetic field only exists for $r < R$, we are only interested in the electric field in the region $a < r < R$. By Gauss' law, the electric field in this region is

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\epsilon_0} \ , \tag{10}$$

because the for $r > a$ the entire charge of the inner cylinder is contained within the Gaussian surface. The electric field outside created by the charged cylinder points radially outward, so the Gaussian surface is a cylinder at $r > a$ with a length l. The only flux is through the tube face (not the endcaps), so

$$E(2\pi rl) = \frac{Q}{\epsilon_0} \quad \Rightarrow \quad \mathbf{E} = \frac{Q}{2\pi\epsilon_0 l}\frac{1}{r}\hat{\mathbf{r}} \ , \tag{11}$$

which is valid in the region $a < r < R$ (really valid to b) and $-l/2 \leq z \leq l/2$, for all φ, in cylindrical coordinates. Using Jackson equation 6.117, we can write the linear momentum density of the electromagnetic field in this volume as

$$\mathbf{g} = \mu_0\epsilon_0 \mathbf{E} \times \mathbf{H} = \epsilon_0 \mathbf{E} \times \mathbf{B} \ , \tag{12}$$

where $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is the Poynting vector. In cylindrical coordinates, the Poynting vector in the specfied region is

$$\mathbf{E} \times \mathbf{B} = \mu_0 \mathbf{S} = \frac{\mu_0 InQ}{2\pi\epsilon_0 l}\frac{1}{r}\hat{\mathbf{r}} \times \hat{\mathbf{z}} = -\frac{\mu_0 InQ}{2\pi\epsilon_0 l}\frac{1}{r}\hat{\boldsymbol{\varphi}} \ . \tag{13}$$

Using the definition of angular momentum, we can write the angular momentum density of the electromagnetic field in this region as

$$\boldsymbol{\ell} \equiv \mathbf{r} \times \mathbf{g} = \epsilon_0 \mathbf{r} \times \mathbf{E} \times \mathbf{B} = (\epsilon_0 r)\left(-\frac{\mu_0 InQ}{2\pi\epsilon_0 l}\right)\frac{1}{r}(\hat{\mathbf{r}} \times \hat{\boldsymbol{\varphi}}) = -\frac{\mu_0 InQ}{2\pi l}\hat{\mathbf{z}} \ . \tag{14}$$

To find the total electromagnetic angular momentum, we must integrate this over the entire volume in which the Poynting vector exists. This is equivalent to simply multiplying to the total volume because the angular momentum density is a constant:

$$\mathbf{L}_{\mathrm{EM}} = -\frac{\mu_0 InQ}{2\pi l}\left(\pi(R^2 - a^2)l\right)\hat{\mathbf{z}} = -\frac{1}{2}\mu_0 nIQ(R^2 - a^2)\hat{\mathbf{z}} \ . \tag{15}$$

We now consider turning off the current so that over some period of time (regardless of how short) the current decreases from I to zero, and as such reduces the magnetic field inside the solenoid linearly with the current. This changing magnetic field gives rise to an induced electric field $\mathbf{E}_I$ defined by

$$\boldsymbol{\nabla} \times \mathbf{E}_I = -\frac{\partial \mathbf{B}}{\partial t} \ , \tag{16}$$

by Faraday's law of induction. If we integrate these over a surface S, we see

$$\int_S \boldsymbol{\nabla} \times \mathbf{E}_I \cdot \mathrm{d}\mathbf{S} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} \tag{17}$$

$$\oint_C \mathbf{E}_I \cdot \mathrm{d}\mathbf{l} = -\frac{\partial}{\partial t} \Phi_M \ , \tag{18}$$

where C is the contour bounding the surface S and Φ_M is the magnetic flux through this surface. If we assume there is only an axial component of the electric field, the line element will always be parallel to the field. For this surface and electric field, the left-hand side of the above equation is

$$|\mathbf{E}_I|(2\pi r) \ , \tag{19}$$

Consider a circular surface of radius $r > R$ perpendicular to the axis of the cylinders, then the right-hand side of Equation 17 is

$$-\frac{\partial}{\partial t}\Phi_M = -(\pi R^2 \hat{\mathbf{z}}) \cdot \frac{\partial}{\partial t}\mathbf{B} = -\mu_0 \pi n R^2 \frac{\partial I}{\partial t} \hat{\boldsymbol{\varphi}} \ . \tag{20}$$

so we get

$$\mathbf{E}_I(r,t) = -\frac{\mu_0 \pi n R^2}{2\pi r} \hat{\boldsymbol{\varphi}} = -\frac{1}{2}\mu_0 n \frac{R^2}{r} \frac{\partial I}{\partial t} \hat{\boldsymbol{\varphi}} \ . \tag{21}$$

This electric field induces a torque on the charged cylinder of radius b, given by

$$\boldsymbol{\tau}_b(t) = b\hat{\mathbf{r}} \times (-Q)\mathbf{E}_I(b,t) = \frac{1}{2}\mu_0 n Q R^2 \frac{\partial I}{\partial t} \hat{\mathbf{z}} \ , \tag{22}$$

and angular momentum is related to torque by $\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \tau$. Therefore the angular momentum of the outer cylinder is

$$\mathbf{L}_b = \int_0^t \boldsymbol{\tau}_b(t)\mathrm{d}t = \frac{1}{2}\mu_0 n Q R^2 \int_0^t \frac{\partial I}{\partial t} \mathrm{d}t \ \hat{\mathbf{z}} \tag{23}$$

$$= \frac{1}{2}\mu_0 n Q R^2 [I(t) - I(0)]\hat{\mathbf{z}} = -\frac{1}{2}\mu_0 n Q R^2 I \hat{\mathbf{z}} \ , \tag{24}$$

because the current is at full strength I at $t = 0$ and after a time t the current has been totally turned off.

Similarly, for $r < R$, the right hand side of Equation 17 gives

$$-(\pi r^2 \hat{\mathbf{z}}) \cdot \frac{\partial}{\partial t}\mathbf{B} = -\mu_0 \pi n r^2 \frac{\partial I}{\partial t} \hat{\boldsymbol{\varphi}} \ , \tag{25}$$

yielding

$$\mathbf{E}_I(r,t) = -\frac{1}{2}\mu_0 n r \frac{\partial I}{\partial t} \hat{\boldsymbol{\varphi}} \ . \tag{26}$$

The torque on the inner conductor is

$$\boldsymbol{\tau}_a = a\hat{\mathbf{r}} \times Q\mathbf{E}_I(a,t) = -\frac{1}{2}\mu_0 n Q a^2 \frac{\partial I}{\partial t}\hat{\boldsymbol{\varphi}} \ , \tag{27}$$

yielding an angular momentum of

$$\mathbf{L}_a = \int_0^t \boldsymbol{\tau}_a(t)\mathrm{d}t = -\frac{1}{2}\mu_0 n Q a^2 \int_0^t \frac{\partial I}{\partial t}\mathrm{d}t \ \hat{\mathbf{z}} \tag{28}$$

$$= -\frac{1}{2}\mu_0 n Q a^2 [I(t) - I(0)]\hat{\mathbf{z}} = \frac{1}{2}\mu_0 n Q a^2 I \hat{\mathbf{z}} \ . \tag{29}$$

If we sum the results, the total angular momentum of the system after the current has been completely switched off, is

$$\mathbf{L}_{\text{mech}} = \frac{1}{2}\mu_0 n I Q(a^2 - R^2)\hat{\mathbf{z}} = -\frac{1}{2}\mu_0 n I Q(R^2 - a^2)\hat{\mathbf{z}} \ , \tag{30}$$

which we notice is equal in sign and magnitude to the angular momentum of the electromagnetic field that was applied before the current was removed.

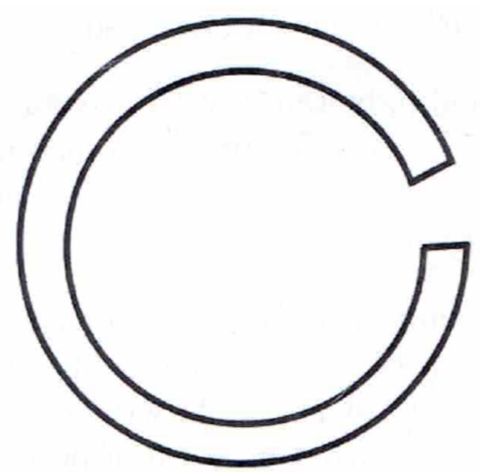


Figure 1: The geometry of the electromagnet in problem one.

3 Problem #3: Type I Superconductor.

A Type I superconductor is a material which is a perfect diamagnet, in that below a certain critical field H_c, the magnetic field $\mathbf{B}$ inside the superconductor is zero. If the magnitude of the field inside the superconductor $H_i > H_c$, the superconductor becomes a normal metal, in which state its permeability $\mu \sim \mu_0$. Consider a superconducting sphere of radius a in a uniform magnetic field $\mathbf{B}_0$ inside and outside the superconductor.

3.1 Permeability of Superconductor.

For $H_i < H_c$, what is the permeability μ of the superconductor?

If the magnetic field inside the superconductor is H_i, the magnetic induction is given by

$$\mathbf{B} = \mu \mathbf{H}_i \ . \tag{31}$$

However, for fields of magnitude less than H_c, the magnetic induction inside is zero, but the magnetic field is non-zero, so the permeability must be zero. We can interpret this as a superconductor (in its superconducting state) having no capacity to support or transmit magnetic inductances.

3.2 Fields of Superconductor.

For the case above, determine $\mathbf{H}$ and $\mathbf{B}$ inside and outside the superconductor.

We can immediately write $\mathbf{B}_{\text{in}} = 0$, by the logic above. Then, using Jackson equation 5.112, we can write the magnetization of the sphere as

$$\mathbf{M} = -\frac{3}{2\mu_0}\mathbf{B}_0 \ , \tag{32}$$

and inser that into the second expression of Jackson equation 5.112 to get the magnetic field inside the superconductor:

$$\mathbf{H}_{\text{in}} = \frac{1}{\mu_0}\mathbf{B}_0 + \frac{1}{2\mu_0}\mathbf{B}_0 = \frac{3}{2\mu_0}\mathbf{B}_0 \ , \tag{33}$$

which we note is consistent with Jackson equation 5.115, because in the uperconducting state $\mu = 0$. Using the magnetization, we can write the magnetic dipole moment of the sphere as

$$\mathbf{m} = \frac{4}{3}\pi a^3 \mathbf{M} = -\frac{2\pi a^3}{\mu_0}\mathbf{B}_0 \ . \tag{34}$$

Now we can write the magnetic induction outside the sphere as the sum of the applied field and the field due to the dipole moment of the sphere:

$$\mathbf{B}_{\text{out}}(\mathbf{x}) = \mathbf{B}_0 + \mathbf{B}_{\text{dipole}} = \mathbf{B}_0 + \frac{\mu_0}{4\pi}\left[\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m}) - \mathbf{m}}{|\mathbf{x}|^3}\right] \ , \tag{35}$$

where $\hat{\mathbf{r}}$ is a unit vector, such that $\mathbf{x} \equiv r\hat{\mathbf{r}}$. The system is azimuthally symmetric, so we can reduce this problem to that of a plane. We can use a polar coordinate system with θ measured from the z axis, with r as the radial coordinate. Let the y axis be perpendicular to z in a right-handed coordinate system. The unit vector is then

$$\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{z}} + \sin\theta\hat{\mathbf{y}} \ , \tag{36}$$

so that

$$\hat{\mathbf{n}} \cdot \mathbf{m} = -\frac{2\pi a^3}{\mu_0} B_0 \cos\theta \equiv -\mathcal{M}\cos\theta \ . \tag{37}$$

Alternatively we can express $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$, so that

$$\mathbf{m} = -\mathcal{M}\hat{\mathbf{z}} = -\mathcal{M}\left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right) \ , \tag{38}$$

so we see that the numerator in the dipole term is

$$3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m}) - \mathbf{m} = -3\mathcal{M}\cos\theta\hat{\mathbf{r}} + \mathcal{M}\left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right) \tag{39}$$

$$= -2\mathcal{M}\cos\theta\hat{\mathbf{r}} - \mathcal{M}\sin\theta\hat{\boldsymbol{\theta}} \ . \tag{40}$$

Using this we can write the magnetic induction outside the sphere as

$$\mathbf{B}_{\text{out}}(\mathbf{x}) = B_0\left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right) - \frac{\mu_0}{4\pi r^3}\left[2\mathcal{M}\cos\theta\hat{\mathbf{r}} + \mathcal{M}\sin\theta\hat{\boldsymbol{\theta}}\right] \tag{41}$$

$$= \left(B_0 - 2\frac{\mu_0\mathcal{M}}{4\pi r^3}\right)\cos\theta\hat{\mathbf{r}} - \left(B_0 + \frac{\mu_0\mathcal{M}}{4\pi r^3}\right)\sin\theta\hat{\boldsymbol{\theta}} \ , \tag{42}$$

and noting

$$\frac{\mu_0}{2\pi r^3}\mathcal{M} = \frac{B_0}{2}\left(\frac{a}{r}\right)^3 \ , \tag{43}$$

we obtain the result

$$\mathbf{B}_{\text{out}}(r,\theta) = B_0\left(1 - \left(\frac{a}{r}\right)^3\right)\cos\theta\hat{\mathbf{r}} - B_0\left(1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right)\sin\theta\hat{\boldsymbol{\theta}} \ , \tag{44}$$

and using the definition of the magnetic field in terms of the magnetic induction yields

$$\mathbf{H}_{\text{out}}(r,\theta) = \frac{\mathbf{B}_{\text{out}}(r,\theta)}{\mu_0} = \frac{B_0}{\mu_0}\left(1 - \left(\frac{a}{r}\right)^3\right)\cos\theta\hat{\mathbf{r}} - \frac{B_0}{\mu_0}\left(1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right)\sin\theta\hat{\boldsymbol{\theta}} \ , \tag{45}$$

3.3 Superconductor in Super-Critical Fields.

What happens to the fields if B_0 is increased so that the H field inside the superconductor exceeds H_c? How do you think the superconductor resolves the resulting logical inconsistency?

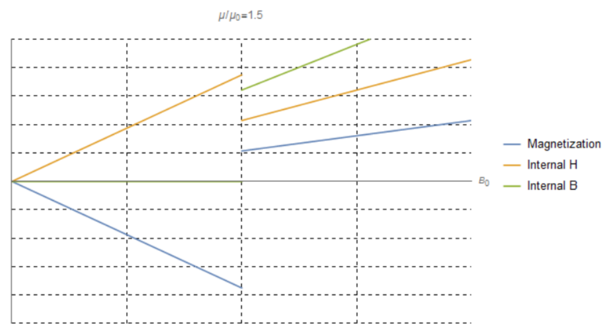
As the applied magnetic induction $\mathbf{B}_0$ increases in magnitude, the magnitude of the internal magnetic field H_i increases linearly, and the magnetization decreases (increasingly negative) linearly as well (see Equation 32). As the applied field increases, there is still no $\mathbf{B}$ field inside the conductor until the magntiude of the internal field surpasses H_c. At this point, the permeability of the sphere goes from 0 to some finite μ with the same order of magnitude as μ_0. The discontinuous jump in permeability gives rise to a discontinuity of the magnetic induction inside the sphere. The magnetization of the sphere in its superconducting state is given by Equation 32, while the magnetization of a normally-conducting sphere in an external magnetic field is given by Jackson equation 5.115, which combine to give

$$\mathbf{M} = \frac{3}{\mu_0}\mathbf{B}_0\left[-\frac{1}{2}\Theta(B_c - B_0) + \frac{n-1}{n+2}\Theta(B_0 - B_c)\right] \ , \tag{46}$$

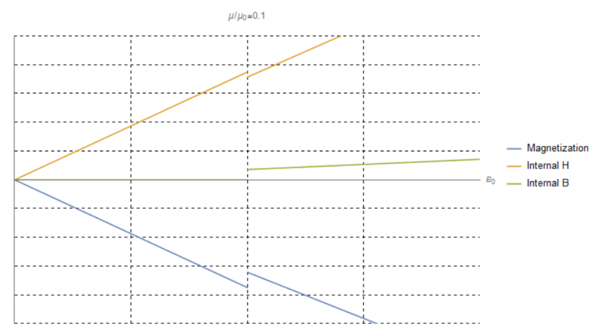
where $\mu/\mu_0 = n$, and B_c is the critical applied field such that $H_c = 3/(2\mu_0)B_c$, and $\Theta(x)$ is the Heaviside step function. We can use this, and the second expression in Jackson equation 5.112, to find the value of the internal magnetic field $\mathbf{H}$ inside the sphere for any applied field. The magnetic induction inside the sphere is

$$\mathbf{B}_{\text{in}} = \left[\mathbf{B}_0 + \frac{2\mu_0}{3}\mathbf{M}\right]\Theta(B_0 - B_c)\ . \tag{47}$$

The magnitude of these fields, and magnetization (all point in $\pm z$ direction), are plotted in Figure 2, with $n = 1.5$ and $\mu_0 = 1$. We see the discontinuous jump in the permeability of the sphere causes a discontinuous jump in the internal magnetic induction.



(a) Permeability of non-superconducting state greater than the permeability of free space.



(b) Permeability of non-superconducting state less than the permeability of free space.

Figure 2: Magnetization and magnetic fields inside a spherical superconductor as a function of applied external field. Note the discontinuous jumps occur when the applied magnetic field increases the magnitude of the magnetic field inside the conductor past some critical value.

If we examine the internal H field, we see there is a drop in magnitude when the sphere transitions out of its superconducting state. This drop in magnitude puts $H_{\text{in}} < H_c$, so it should return to its superconducting state - so every time the magnetic field is increased past a critical value, the magnetization shifts and drops the field below the critical value, so it should still be superconducting -which means the magnetization could not have changed! There is some logical inconsistency here, because in practice there is definitely a magnitude of applied magnetic field which will cause a superconductor to transition to a non-superconducting phase. The drop in H_{in} occurs for any value of μ, except $\mu = \mu_0$, in this case there is no discontinuity of the H field, and it linearly passes the critical value. However, this would mean the superconductor makes a phase transition to become equivalent to a vacuum when its superconducting state is killed, which I don't think is the case.

4 Problem #4: Self-Inductance of a Long Solenoid.

Determine the self-inductance of a long solenoid.

Consider a cylindrical coordinate system with the z axis aligned with the axis of a solenoid of radius R, with n turns per unit length. This solenoid carries a current I such that it creates a magnetic induction in the positive z direction, given by

$$\mathbf{B} = \mu_0 n I \hat{\mathbf{z}} \ . \tag{48}$$

This field is only created for $r < R$, while for $r > R$ there is no field, note r is the radial coordinate. This is easy to show using Ampere's law: consider a square Amperian loop with two edges of unit length parallel to the axis, one inside the solenoid and one outside the solenoid. These lines are connected by parallel infinitesimal lengths perpendicular to the axis. The contour C around this loop is taken such that the line element inside the solenoid points in the positive z direction. Ampere's law gives

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_{\mathrm{enc}} \ , \tag{49}$$

and we have claimed there are no magnetic field lines in the radial direction, or outside the solenoid. Therefore

$$|\mathbf{B}| = \mu_0 n I \ , \tag{50}$$

and since it was parallel to the line element, the magnetic field is also in the positive z direction.

Consider the surface S which is a circular cross-section of the solenoid, which has area πR^2, and its normal points in the positive z direction. The magnetic flux through this surface is then

$$\Phi_M = \pi \mu_0 R^2 n I \ . \tag{51}$$

If we give the solenoid a total length d, there are nd loops, each of which contributes one factor of the flux given above. The inductance L is a geometrical quantity which is related to the magnetic flux by $\Phi_{\mathrm{tot}} = LI$, for the long solenoid, the inductance is simply

$$L = \pi \mu_0 R^2 n^2 d \ . \tag{52}$$

5 Problem #5: Mutual Inductance of Coaxial, Displaced Current Loops.

Calculate the mutual inductance of two coaxial loops of wire of radius a and b whose centers are separated by a distance d. Approximate the result (which will likely be in terms of elliptic integrals) in the limit when $a \sim b \gg d$.

Consider a Cartesian coordinate system $(\hat{\mathbf{x}}_1, \hat{\mathbf{y}}_1, \hat{\mathbf{z}}_1)$, in which there is a ring of radius a, centered at the origin. If we define a radial coordinate r_1 and an angle from θ_1 from $\hat{\mathbf{x}}_1$, we can express a position in this coordinate system as

$$\mathbf{x}_1 = r_1 \cos\theta_1 \hat{\mathbf{x}}_1 + r_1 \sin\theta_1 \hat{\mathbf{y}}_1 + z_1 \hat{\mathbf{z}}_1 \ . \tag{53}$$

A line element around the path C_1 of the ring is given by

$$\mathrm{d}\mathbf{l}_1 = a\cos\theta_1 \mathrm{d}\theta_1 \hat{\mathbf{x}}_1 + a \sin\theta_1 \mathrm{d}\theta_1 \hat{\mathbf{y}}_1 \ . \tag{54}$$

Consider another coordinate system $(\hat{\mathbf{x}}_2, \hat{\mathbf{y}}_2, \hat{\mathbf{z}}_2)$, with radial coordinate r_2 and angle θ_2 with the x_2 axis. There is a ring of radius b centered at the origin; a line element around this path C_2 is given by

$$\mathrm{d}\mathbf{l}_2 = b\cos\theta_2 \mathrm{d}\theta_2 \hat{\mathbf{x}}_2 + b \sin\theta_2 \mathrm{d}\theta_2 \hat{\mathbf{y}}_2 \ . \tag{55}$$

These coordinates are aligned respect to each other such that $\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}_2$. The z axes point along the same direction, but are separated by a distance d. So the distance to a point on C_1 from a point on C_2 is given by

$$\mathbf{x}_{12} = (x_1 - x_2)\hat{\mathbf{x}} + (y_1 - y_2)\hat{\mathbf{y}} + d = d + (a\cos\theta_1 - b\cos\theta_2)\hat{\mathbf{x}} + (a\sin\theta_1 - b\sin\theta_2)\hat{\mathbf{y}} \tag{56}$$

$$|\mathbf{x}_{12}|^2 = d^2 + (a\cos\theta_1 - b\cos\theta_2)^2 + (a\sin\theta_1 - b\sin\theta_2)^2 = a^2 + b^2 + d^2 - 2ab\cos(\theta_1 - \theta_2) \ . \tag{57}$$

We can use the Neumann double integral formula[1] to write the mutual inductance of the two wire loops:

$$M \equiv M_{21} = M_{12} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{\mathrm{d}\mathbf{l}_1 \cdot \mathrm{d}\mathbf{l}_2}{|\mathbf{x}_{12}|} \ . \tag{58}$$

If we insert the expression for $|\mathbf{x}_{12}|$, and evaluate the dot product, we see

$$M = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{ab(\cos\theta_1 \cos\theta_2 \mathrm{d}\theta_1 + \sin\theta_1 \sin\theta_2)\mathrm{d}\theta_1 \mathrm{d}\theta_2}{\sqrt{a^2 + b^2 + d^2 - 2ab\cos(\theta_1 - \theta_2)}} \tag{59}$$

$$= \frac{\mu_0}{4\pi} \int_{\theta_1 = 0}^{2\pi} \int_{\theta_2 = 0}^{2\pi} \frac{ab\cos(\theta_1 - \theta_2)\mathrm{d}\theta_1 \mathrm{d}\theta_2}{\sqrt{a^2 + b^2 + d^2 - 2ab\cos(\theta_1 - \theta_2)}} \ . \tag{60}$$

Let us define a relative angle $\psi = \theta_1 - \theta_2$, so we have

$$M = \frac{\mu_0}{4\pi} \int_{\theta_1 = 0}^{2\pi} \int_{\theta_2 = 0}^{2\pi} \frac{ab\cos\psi \mathrm{d}\theta_1 \mathrm{d}\theta_2}{\sqrt{a^2 + b^2 + d^2 - 2ab\cos\psi}} \ . \tag{61}$$

We can use $\mathrm{d}\psi = \mathrm{d}\theta_2$ (we only vary θ_2 for the first integral) then this integral becomes

$$M = \frac{\mu_0}{4\pi} \oint_{\theta_1} \left[\oint_{\psi} \frac{ab\cos\psi \mathrm{d}\psi}{\sqrt{a^2 + b^2 + d^2 - 2ab\cos\psi}} \right] \mathrm{d}\theta_1 \ , \tag{62}$$

[1] Jackson uses this form to calculate the force between to wire loops in equation 5.10. The inductance is a purely geometric quantity, which requires we remove the interaction of the currents, and a factor of $\mathbf{x}_{12}/|\mathbf{x}_{12}|^2$ to get correct dimensions. Compare this to Jackson problem 5.33, or see Wikipedia for this form.

and now we note there is no dependence on θ_1, so we may integrate it out, picking up a factor of 2π:

$$M = \frac{\mu_0}{2} ab \int_0^{2\pi} \frac{\cos\psi}{\sqrt{a^2+b^2+d^2-2ab\cos\psi}} \mathrm{d}\psi \ . \tag{63}$$

We can write the remaining integral in terms of the complete elliptic integrals[2] $K(m)$ [first type] and $E(m)$ [second type] as

$$\int_0^{2\pi} \frac{\cos\psi}{\sqrt{\alpha-\beta\cos\psi}} = 4\frac{\sqrt{\alpha+\beta}}{\beta}\left[\left(1-\frac{\nu^2}{2}\right)K(\nu)-E(\nu)\right] \ , \tag{64}$$

with

$$\alpha = a^2+b^2+d^2 \tag{65}$$

$$\beta = 2ab \tag{66}$$

$$\nu = \sqrt{\frac{2\beta}{\alpha+\beta}} = \sqrt{\frac{4ab}{a^2+b^2+d^2+2ab}} = \sqrt{\frac{4ab}{(a+b)^2+d^2}} \ . \tag{67}$$

So the mutual inductance of the wire loops is given by

$$M = \frac{\mu_0 ab}{2}\frac{4\sqrt{\alpha+\beta}}{\beta}\left[\left(1-\frac{\nu^2}{2}\right)K(\nu)-E(\nu)\right] \tag{68}$$

$$= \mu_0\sqrt{(a+b)^2+d^2}\left[\left(1-\frac{\nu^2}{2}\right)K(\nu)-E(\nu)\right] \tag{69}$$

$$= \mu_0\frac{\sqrt{4ab}}{\nu}\left[\left(1-\frac{\nu^2}{2}\right)K(\nu)-E(\nu)\right] = \mu_0\sqrt{ab}\left[\left(\frac{2}{\nu}-\nu\right)K(\nu)-\frac{2}{\nu}E(\nu)\right] \ , \tag{70}$$

which is exact. If we are interested in the limit $a \sim b \equiv R \gg d$, we can approximate the mutual inductance. In this case $\alpha = 2R^2 = \beta$, so the mutual inductance is

$$M \simeq 2\mu_0 R\left[\frac{1}{2}K(\nu)-E(\nu)\right] \ , \tag{71}$$

but the complete elliptic integral of the first kind diverges at $\nu = 1$, while $E(1) = 1$. The Taylor series expansion for $K(\nu)$ is only valid for arguments strictly less than one. If we express the complete elliptic integral of the first kind as

$$K(\nu) = \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{1-\nu^2\sin^2 x}} = \int_{1-\nu^2}^{1} \frac{\mathrm{d}y}{2\sqrt{y(1-y)(y-1+\nu^2)}} \ , \tag{72}$$

after a change of variables such that $y = 1-\nu^2\sin^2\theta$. Let us insert a small parameter δ such that $\nu = 1-\delta$, so the integrand becomes

$$\left[y(1-y)(y-1+(1-\delta)^2)\right]^{-1/2} = \frac{1}{\sqrt{y}}\left[y-2\delta+\delta^2-y^2+2\delta y-\delta^2 y\right]^{-1/2} \ . \tag{73}$$

[2]Arfken, Mathematical Methods for Physicists, 7 ed. Page 928. The exact form that follows is from Kurt Nalty: http://www.kurtnalty.com/Helmholtz.pdf, and verified against the result presented by Bouwkamp and Casimir in 1948: On the mutual inductance of two parallel coaxial circles of circular cross-section.

If we retain only terms to zeroth order in the small parameter (reasonable to do because to return to our original problem we need $\delta = 0$) we see the integral can be written

$$K(\nu) = \frac{1}{2} \int_{1-\nu^2}^{1} \frac{\mathrm{d}y}{y\sqrt{(1-y)}} + \mathcal{O}(\delta^1) \ , \tag{74}$$

which is the first term in the Taylor expansion (as says MATHEMATICA). In the limit $y \to 0$ ($\nu \to 1$) the term in the radical becomes unity and the integral is trivial to integrate,

$$K(\nu) \sim -\frac{1}{2} \log(1-\nu^2) + \mathcal{O}(\delta^1), \ \nu \to 1 \ , \tag{75}$$

so the integral diverges logarithmically. The fact this integral diverges makes sense, because if $d \to 0$, the two wire rings are occupying the exact same space. In reality, this is not possible, but when the rings are a finite distance apart, and brought closer, the inductance increase logarithmically.

6 Problem #6: Inductance of Coaxial Cable.

Calculate the inductance per unit length of a coaxial cable that consists of an inner conductor of radius a and permeability μ_1 surrounded by a thin conducting shell of radius b, with the space between the two conductors being filled with a medium of permeability μ_2.

Consider a cylindrical coordinate system with the z direction aligned along the axis of the cable, with radial coordinate r and axial coordinate ϑ. Let there be a current $+I$ on the inner conductor and a current $-I$ on the outer conductor. Now consider an Amperian contour C_1 of $r < a$ (assume for simplicity, in the $z = 0$ plane), the current enclosed by this contour is $I(r^2/a^2)$, so by Ampere's law, we have

$$\int_{C_1} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = B(2\pi r) = \mu_1 I \frac{r^2}{a^2} \quad \Rightarrow \quad \mathbf{B} = \frac{\mu_1 I r}{2\pi a^2} \hat{\boldsymbol{\vartheta}} \ . \tag{76}$$

The energy density of the magnetic field created is given by Jackson equation 5.148:

$$u = \frac{1}{2} \mathbf{H} \cdot \mathbf{B} = \frac{1}{2\mu} |\mathbf{B}|^2 \ , \tag{77}$$

so in the region $r < a$, the energy density is

$$u = \frac{1}{2\mu_1} \left(\frac{\mu_1 I r}{2\pi a^2} \right)^2 = \frac{1}{2} \mu_1 \left(\frac{I}{2\pi} \right)^2 \frac{r^2}{a^4} \ , \tag{78}$$

which we can integrate over this entire region to determine the total energy in this region. We are interested in quantites per unit length, so we do not need to integrate over the z coordinate (we have no dependence on z, so integrating over the total length of the solenoid just adds a factor of total length, which we divide by to get per unit length quantities, so it is equivalent to multiplying by one). The total energy per unit length in this region is

$$E = \frac{1}{2} \mu_1 \left(\frac{I}{2\pi} \right)^2 \int_{\vartheta=0}^{2\pi} \int_{r=0}^{a} \frac{r^2}{a^4} r \mathrm{d}r \mathrm{d}\vartheta = \frac{1}{2} \mu_1 \frac{I^2}{2\pi a^4} \left(\frac{1}{4} a^4 \right) = \frac{I^2}{2} \frac{1}{2\pi} \frac{\mu_1}{4} \ . \tag{79}$$

Now consider an Amperian contour C_2 with $a < r < b$, using Ampere's law, we see

$$\int_{C_2} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = B(2\pi r) = \mu_2(-I) \quad \Rightarrow \quad \mathbf{B} = -\frac{\mu_2 I}{2\pi r} \hat{\boldsymbol{\vartheta}} \ . \tag{80}$$

In this region, the energy density is given by

$$u = \frac{1}{2\mu_2} \left(\frac{\mu_2 I}{2\pi r} \right)^2 = \frac{1}{2} \mu_2 \left(\frac{I}{2\pi} \right)^2 \frac{1}{r^2} \ , \tag{81}$$

and thus, the total energy per unit length is

$$E = \frac{1}{2} \mu_2 \left(\frac{I}{2\pi} \right)^2 \int_{\vartheta=0}^{2\pi} \int_{r=a}^{b} \frac{1}{r^2} r \mathrm{d}r \mathrm{d}\vartheta = \frac{1}{2} \mu_2 \frac{I^2}{2\pi} \log(b/a) = \frac{I^2}{2} \frac{1}{2\pi} \mu_2 \log(b/a) \ . \tag{82}$$

Adding this energy to the energy per unit length in Equation 79, the total energy per unit length in the region $r \leq b$ is

$$E = \frac{I^2}{2} \frac{1}{2\pi} \left[\frac{\mu_1}{4} + \mu_2 \log \left(\frac{b}{a} \right) \right] \ . \tag{83}$$

The energy stored in an inductor of inductance L when a current I applied is given by $E = \frac{1}{2}LI^2$, which can be solved for the inductance:

$$L = 2\frac{E}{I^2} \ , \tag{84}$$

and for the coaxial cable, we see the inductance per unit length is

$$L = \frac{1}{2\pi}\left[\frac{\mu_1}{4} + \mu_2 \log\left(\frac{b}{a}\right)\right] \ . \tag{85}$$