DYLAN J. TEMPLES: SOLUTION SET ONE

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Contents

1 Transfer Function.

Determine the transfer function $H(s)$ ($V_{\rm in} \rightarrow V_{\rm out}$) for the circuit shown in Figure 1 below, and describe the expected $V_{\rm out}$ for a pulse input based on the properties of $H(s)$ in the complex s plane. You may assume that the capacitor is initially uncharged.

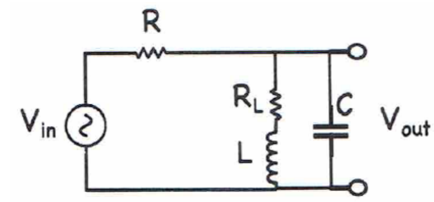

Figure 1: Circuit diagram for problem #1.

The voltage across a component due to a time-dependent current $I(s)$ in s-space is

$$V(s) = RI(s) \ ; \quad V(s) = \frac{1}{sC} I(s) + \frac{V(0)}{s} \ ; \quad V(s) = LsI(s) \ , \tag{1}$$

for a resistor of resistance R, a capacitor of capacitance C, and an inductor of inductance L, respectively. The initial voltage (and therefore charge) on the capacitor is $V(0)$ which we will take to be zero. We will define the current through resistor R as $I_1(s)$, which then splits into two currents: $I_2(s)$ and $I_3(s)$, which are the currents through the inductor and capacitor, respectively. At this node, Kirchoff's laws tell us that

$$I_1(s) = I_2(s) + I_3(s) \ . \tag{2}$$

Consider the path from $V_{\rm in}(s)$ to ground through the inductor, from Kirchoff's laws, it follows that

$$V_{\rm in}(s) = RI_1(s) + R_L I_2(s) + sLI_2(s) \ , \tag{3}$$

similarly for the path to ground through the capacitor:

$$V_{\rm in}(s) = RI_1(s) + \frac{1}{sC} I_3(s) \ . \tag{4}$$

Applying the same reasoning to the paths from $V_{\rm out}$ to ground, we obtain the equations

$$V_{\rm out} = \left[R_L + sL\right] I_2(s) \tag{5}$$

$$V_{\rm out} = \frac{1}{sC} I_3(s) \ . \tag{6}$$

We now have a system of five equations, for which we want to solve for the four unknowns $I_1(s), I_2(s), I_3(s), V_{\rm out}(s)$ in terms of $V_{\rm in}(s)$ and the constants $R, L,$ and C. This over constrains the problem, so we select four equations to solve (using MATHEMATICA). To determine the transfer function $H(s)$ (which takes $V_{\rm in}$ to $V_{\rm out}$) we are only interested in the result for the output voltage:

$$V_{\rm out} = \left[\frac{Ls + R_L}{CLRs^2 + (CRR_L + L)s + (R + R_L)} \right] V_{\rm in} \ , \tag{7}$$

(this result was obtained by using Equations 2 through 5, but was verified using Equation 6 instead of 6, with consistent results). The transfer function is

$$H(s) = \frac{Ls + R_L}{(Ls + R_L) + (CLRs^2 + CRR_L s + R)} \ . \tag{8}$$

This transfer function has roots at

$$s = \frac{-(CRR_L + L) \pm \sqrt{(CRR_L + L)^2 - 4CLR(R + R_L)}}{2CLR} \ , \tag{9}$$

so the system is critically damped if

$$(CRR_L + L)^2 = 4CLR(R + R_L) \ , \tag{10}$$

because both roots have the same value. Alternatively, if

$$(CRR_L + L)^2 < 4CLR(R + R_L) \ , \tag{11}$$

the poles will have some imaginary component, so the output will oscillate given an input pulse. If the opposite is true, the poles are real, but there is damping, so the system is under-damped.

2 Control Circuit.

Following up on the example done in class last quarter on Laplace transforms in feedback circuits, consider the control circuit in Figure 2 below, which for definiteness we can take to be the circuit controlling the temperature in a common kitchen oven. In usual operation, we would like to set the temperature to a certain set-point $u(t)$, and we would like to design the control circuit so that the response $r(t)$, which in this case is the temperature of the oven, comes to the set point as quickly as possible. The error signal $e(t) = u(t) - r(t)$ is the input to the control circuit, which in turn outputs a control signal $y(t)$, which is fed to the "plant", which includes the heating element, but also the other components of the oven that determine how fast the oven warms up or cools down. The time domain response of the plant is quite often difficult to model, particularly for complicated systems, but in this case we will model it as a low pass filter with a time constant of τ.

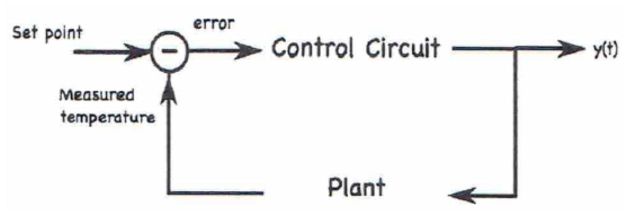

Figure 2: Control circuit diagram for problem #2.

For the control circuit, we consider a traditional "proportional-integral" or PI controller whose response to the error $e(t)$ is given by

$$y(t) = k_p e(t) + k_i \int_0^t e(t)\mathrm{d}t \ . \tag{12}$$

Given these parameters, design the control circuit, *i.e.*, determine the values of k_p and k_i such that the temperature of the oven approaches the set-point as quickly as possible after a step change in the set-point.

Let us consider this circuit in complex s space by Laplace transforming all the functions:

$$\mathcal{L}[u(t)] = U(s) \ ; \quad \mathcal{L}[r(t)] = R(s) \ ; \quad \mathcal{L}[e(t)] = E(s) = U(s) - R(s) \ , \tag{13}$$

and

$$Y(s) = \mathcal{L}[y(t)] = k_p E(s) + k_i \frac{E(s)}{s} = \left[k_p + \frac{k_i}{s} \right] E(s) \ , \tag{14}$$

using the properties of the Laplace transform. Let us define two transfer functions $G(s)$ and $H(s)$, which are the actions of the control circuit and the plant respectively. If we model the plant as a passive low-pass filter (RC circuit), with time constant $\tau = RC$, the transfer function is then

$$H(s) = \frac{1}{1 + \tau s} \ , \tag{15}$$

and from Equation 14, the transfer function for the control circuit is

$$G(s) = k_p + \frac{k_i}{s} \ . \tag{16}$$

The action of these transfer functions are as follows:

$$Y(s) = G(s)E(s) \tag{17}$$
$$R(s) = H(s)Y(s) \ , \tag{18}$$

so the function $G(s)$ takes the error and returns the control signal, while the function $H(s)$ takes the control signal and returns the response (oven temperature). We can find a function that describes how the error of the system responds to a step change in the set-point. Upon substitution of Equation 18 into the error signal, we obtain

$$E(S) = U(s) - H(s)G(s)E(s) \quad \Rightarrow \quad E(s) = \frac{1}{1 + H(s)G(s)}U(s) = \left[1 + \frac{k_p + \frac{k_i}{s}}{1 + \tau s}\right]^{-1} U(s) \ , \tag{19}$$

so the transfer function that takes the set-point to the error signal is

$$K(s) = \left[1 + \frac{k_p + \frac{k_i}{s}}{1 + \tau s}\right]^{-1} \tag{20}$$

which has poles at

$$-1 = \frac{k_p + \frac{k_i}{s}}{1 + \tau s} \quad \Rightarrow \quad s = -\frac{1}{2\tau}\left[(k_p + 1) \pm \sqrt{-4\tau k_i + k_p^2 + 2k_p + 1}\right] \ . \tag{21}$$

We require the system be critically damped, so that the error signal asymptotes to zero as quickly as possible; for this to be the case, there must only be one value for the roots of the quadratic (damping occurs when the poles lie along the real axis, and oscillation if they have an imaginary component). Thus, we enforce the sum in the radical is equal to zero:

$$-4\tau k_i + k_p^2 + 2k_p + 1 = 0 \quad \Rightarrow \quad k_p = -1 \pm 2\sqrt{\tau k_i} \ . \tag{22}$$

Using this result, the transfer function $K(s)$ can be written in terms of k_i only

$$K(s) = \frac{s(1 + \tau s)}{k_i + s\left(s\tau \pm 2\sqrt{\tau k_i}\right)} \ . \tag{23}$$

We can put this result into the control signal:

$$y(t) = (-1 \pm 2\sqrt{\tau k_i})e(t) + k_i \int_0^t e(t)\mathrm{d}t \ , \tag{24}$$

where k_i can be chosen to affect the damping rate, larger k_i results in a faster damping.

3 Polarization of Electromagnetic Waves.

3.1 Linear Polarization.

Show mathematically that a linearly polarized wave can be thought of as a superposition of a right circularly polarized and a left circularly polarized wave.

Let us define a rectilinear coordinate system with the $\hat{\mathbf{k}}$ axis along the direction of propagation of the linearly polarized electromagnetic wave, with two mutually orthogonal axes $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2$. A linearly polarized electromagnetic wave with an arbitrary polarization axis can be represented as

$$\mathbf{E}(\mathbf{x},t) = (E_1\hat{\mathbf{e}}_1 + E_2\hat{\mathbf{e}}_2)e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} \ , \tag{25}$$

where $|\mathbf{E}|^2 = |E_1|^2 + |E_2|^2$ and $\{E_1, E_2\} \in \mathbb{C}$, but with the same phase. However, we are free to align one of the two free axes with the magnetic field (and therefore the polarization), so we can simplify the expression for a linearly polarized wave to

$$\mathbf{E}(\mathbf{x},t) = E\hat{\mathbf{e}}_1 e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} \ , \tag{26}$$

A circularly polarized wave, of indicated handedness, is represented as[1]

$$\text{right}: \ \mathbf{E}^{(r)} = E^{(r)}(\hat{\mathbf{e}}_1 - i\hat{\mathbf{e}}_2)e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} \tag{27}$$

$$\text{left}: \ \mathbf{E}^{(l)} = E^{(l)}(\hat{\mathbf{e}}_1 + i\hat{\mathbf{e}}_2)e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} \ . \tag{28}$$

Now consider a superposition of right and left handed circularly polarized waves of equal magnitude ($E^{(r)} = E^{(l)} \equiv E$), and same overall phase:

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}^{(r)} + \mathbf{E}^{(l)} = 2E\hat{\mathbf{e}}_1 e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} \ , \tag{29}$$

which is clearly a linearly polarized wave (magnetic field points along one rectilinear direction).

3.2 Half-wave Plate.

A half-wave plate is a film made of material that has two mutually perpendicular axes, the so-called fast and slow axes, that are parallel to the film's surface. On exiting the film, linearly polarized waves incident on one side of the film that are parallel to the slow axis are shifted in phase by π with respect to waves polarized parallel to the fast axis. Show that a half-wave plate changes the handedness of a circularly polarized wave, *i.e.*, it convert a right circularly polarized wave into a left circularly polarized wave, and vice versa.

Let us use the same coordinate system as above, with $\hat{\mathbf{e}}_2$ as the fast axis, and $\hat{\mathbf{e}}_1$ as the slow. The half-wave plate transforms the electromagnetic wave with $\mathbf{E}$ to one with $\mathbf{E}'$. Consider a wave polarized along the fast axis (note the polarization axis is the direction which the magnetic field points, so the electric field is perpendicular to both this and the direction of propagation):

$$\mathbf{E} = E\hat{\mathbf{e}}_1 e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} \to \mathbf{E}' = E\hat{\mathbf{e}}_1 e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} \ , \tag{30}$$

while the half-wave plate affects an electromagnetic wave polarized along the slow axis in the following way

$$\mathbf{E} = E\hat{\mathbf{e}}_2 e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} \to \mathbf{E}' = Ee^{i\pi}\hat{\mathbf{e}}_2 e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} = -E\hat{\mathbf{e}}_2 e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} \ . \tag{31}$$

[1] Jackson, Classical Electrodynamics, 3 ed. Section 7.2, page 299.

Now consider a circularly polarized wave with definite handedness:

$$\mathbf{E} = E(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} \ , \tag{32}$$

after it passes through a half-wave plate the component along $\hat{\mathbf{e}}_1$ remains the same, but the component along $\hat{\mathbf{e}}_2$ picks up a relative phase of π, so the transmitted wave is

$$\mathbf{E}' = E(\hat{\mathbf{e}}_1 \pm i(-1)\hat{\mathbf{e}}_2)e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} = E(\hat{\mathbf{e}}_1 \mp i\hat{\mathbf{e}}_2)e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} \ , \tag{33}$$

which is a circularly polarized wave of opposite handedness as the incident wave.

3.3 Intensity of Transmitted Waves from a Linear Polarizer.

Suppose we have a linear polarizer that allows only waves whose electric field vector is parallel to the y axis. Calculate the time average (over one time period of the wave) of the square of the electric field vector (*i.e.*, the time averaged intensity) of the wave that is transmitted through the polarizer, for the case of right circularly polarized light, with $\delta = \pi/2$, and linearly polarized light, with $\delta = 0$.

Consider an arbitrarily polarized electromagnetic wave

$$\mathbf{E} = (E_1\hat{\mathbf{e}}_1 + e^{i\delta}E_2\hat{\mathbf{e}}_2)e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} \ , \tag{34}$$

with $\{E_1, E_2\} \in \mathbb{R}$ the relative phase difference is indicated by the value of δ (we can ignore the overall phase without loss of generality). The transmitted wave now only has the component which was parallel to $\hat{\mathbf{e}}_2$ (the y axis), so the electric field of this wave is

$$\mathbf{E}' = E_2\hat{\mathbf{e}}_2 e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t+\delta)} \ , \tag{35}$$

if we consider the real part of the wave, which is the only part that interacts with the polarizer, we see can say

$$\mathbf{E}' = E_2\hat{\mathbf{e}}_2 \cos(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t+\delta) \quad \Rightarrow \quad |\mathbf{E}'|^2 = \mathbf{E}'\cdot\mathbf{E}' = E_2^2\cos^2(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t+\delta) \ , \tag{36}$$

which we can integrate over one period, and average

$$u = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}|\mathbf{E}'|^2 \mathrm{d}t = \frac{\omega}{2\pi}E_2^2\int_0^{2\pi/\omega}\cos^2(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t+\delta)\mathrm{d}t = \frac{\omega}{2\pi}E_2^2\frac{\pi}{\omega} = \frac{E_2^2}{2} \ , \tag{37}$$

which yields the time average intensity of the electric field over one period.

A circularly polarized electric field with $\delta = \pi/2$ (right handed) is given by

$$\mathbf{E}^{(c)} = E_0(\hat{\mathbf{e}}_1 + i\hat{\mathbf{e}}_2)e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} \ \rightarrow \ |\mathbf{E}^{(c)}|^2 = 2E_0^2 \ , \tag{38}$$

so in regards to Equation 37, $E_2 \rightarrow E_0$, and the intensity of the transmitted wave is

$$u = \frac{E_0^2}{2} = \frac{1}{4}|\mathbf{E}^{(c)}|^2 \ . \tag{39}$$

A linearly polarized electric field (with $\delta = 0$) is given by

$$\mathbf{E}^{(l)} = (E_1\hat{\mathbf{e}}_1 + E_2\hat{\mathbf{e}}_2)e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} \ \rightarrow \ |\mathbf{E}^{(l)}|^2 = E_1^2 + E_2^2 \ , \tag{40}$$

so the time averaged intensity of a linearly polarized wave is

$$u = \frac{E_2^2}{2} = \frac{1}{2}\left(\mathbf{E}^{(l)}\cdot\hat{\mathbf{e}}_2\right)\left(\mathbf{E}^{(l)}\cdot\hat{\mathbf{e}}_2\right)^* \ . \tag{41}$$

4 Superposition of Plane Waves.

Two plane monochromatic linearly polarized waves of the same frequency propagate along the z-axis. The first wave is polarized along the x-axis and has amplitude a, and the second is polarized along the y-axis and has an amplitude b. The phase of the second wave leads the phase of the first wave by χ. Find the polarization of the resulting wave.

Consider the first wave, with electric field polarized along the x-axis, defined by

$$\mathbf{E}_1 = a\hat{\mathbf{x}}e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} , \tag{42}$$

and the second with electric field polarized along the y axis defined by

$$\mathbf{E}_2 = be^{i\chi}\hat{\mathbf{y}}e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} . \tag{43}$$

The superposition of these waves is

$$\mathbf{E} = (a\hat{\mathbf{x}} + be^{i\chi}\hat{\mathbf{y}})e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)} , \tag{44}$$

which is an arbitrarily polarized wave. In the case $a = b$ and $\chi = \pm\pi/2$ (or any odd integer multiple thereof), the wave is circularly polarized[2] (right handed for the minus, and left handed for the plus). In the case $\chi = 0$ (or any integer multiple of pi), the wave is linearly polarized. For any other combination of a, b, χ the wave is elliptically polarized[3].

[2]See Equations 27 and 28.

[3]Jackson, Classical Electrodynamics, 3 ed. Section 7.2, page 299.