Dylan J. Temples: Solution Set Two

Northwestern University, Electrodynamics II
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Contents

1 Complex Polarization Amplitude

In class, we have described the polarization of the EM field in terms of the x and y components of the electric field (E_x or E_y) or their equivalents in the magnetic field. Another way to describe the polarization is as a complex quantity: $E_x + iE_y$. Starting from the discussion of polarization in class, show how you would represent a linearly polarized wave; right- and left-circularly polarized waves; and right- and left-elliptically polarized waves with these new complex amplitudes.

The components of the electric field of an electromagnetic wave can be expressed as

$$E_x = a_1 \cos(\mathbf{k}\cdot\mathbf{x} - \omega t + \delta_1) \tag{1}$$

$$E_y = a_2 \cos(\mathbf{k}\cdot\mathbf{x} - \omega t + \delta_2) \ , \tag{2}$$

where $a_{1,2}$ are the amplitudes of the field, and $\delta_{1,2}$ are the phases. In this representation, we have $\{a_1, a_2\} \in \mathbb{R}$. From this, we can represent the electric field as

$$\mathbf{E}(\mathbf{x}, t) = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} \ , \tag{3}$$

which is a completely real quantity, however we may represent the electric field as a complex quantity using complex exponentials:

$$\mathbf{E}(\mathbf{x}, t) = (a_1 e^{i\delta_1}\hat{\mathbf{x}} + a_2 e^{i\delta_2}\hat{\mathbf{x}})e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \ , \tag{4}$$

which leads to the definition of the complex quantities

$$\alpha_1 = a_1 e^{i\delta_1} \tag{5}$$

$$\alpha_2 = a_2 e^{i\delta_2} \ . \tag{6}$$

We can gather an overall phase, and ignore it, without losing generality, and consider only the relative phase δ, then we have:

$$\mathbf{E}(\mathbf{x}, t) = (\alpha_1\hat{\mathbf{x}} + \alpha_2\hat{\mathbf{y}})e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} = (a_1\hat{\mathbf{x}} + a_2 e^{i\delta}\hat{\mathbf{x}})e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \ , \tag{7}$$

which is the general representation of an arbitrary polarization determined by a_1, a_2, δ. Let us define $\tau = \mathbf{k}\cdot\mathbf{x} - \omega t$.

1.1 Linear Polarization.

In the case of linear polarization, there is no condition on the amplitudes, but the phase must be an integral multiple of π:

$$\mathbf{E}^{(lin)}(\mathbf{x}, t) = (a_1\hat{\mathbf{x}} + a_2 e^{i(n\pi)}\hat{\mathbf{x}})e^{i\tau} = (a_1\hat{\mathbf{x}} \pm a_2\hat{\mathbf{y}})e^{i\tau} \ . \tag{8}$$

1.2 Circular Polarization.

Circular polarization requires that the amplitudes of both components are equal and the phases are $\delta_r = \frac{\pi}{2} + 2n\pi$ for right-handed, and $\delta_l = -\frac{\pi}{2} + 2n\pi$ for left, where $n \in \mathbb{Z}$:

$$\text{Right} : \mathbf{E}^{(cr)}(\mathbf{x}, t) = a(\hat{\mathbf{x}} + e^{i\left(\frac{\pi}{2} + 2n\pi\right)}\hat{\mathbf{y}})e^{i\tau} = a(\hat{\mathbf{x}} + \hat{\mathbf{y}})e^{i\tau} \tag{9}$$

$$\text{Left} : \mathbf{E}^{(cl)}(\mathbf{x}, t) = a(\hat{\mathbf{x}} + e^{i\left(-\frac{\pi}{2} + 2n\pi\right)}\hat{\mathbf{y}})e^{i\tau} = a(\hat{\mathbf{x}} - \hat{\mathbf{y}})e^{i\tau} \tag{10}$$

1.3 Elliptical Polarization.

Elliptical polarization is the general polarization, and has no conditions on the phase or amplitudes, see Equation 7.

2 Complex Wave Vector.

The electric field $\mathbf{E}$ of an electromagnetic wave in which the real and imaginary components of the complex wave vector $\mathbf{k}$ are in different directions, is linearly polarized. Determine the mutual disposition of the vectors $\mathbf{E}_0$ (the amplitude of the electric field), $\boldsymbol{\mathcal{H}}_1$, $\boldsymbol{\mathcal{H}}_2$, $\mathbf{k}'$, and $\mathbf{k}''$, where $\boldsymbol{\mathcal{H}}_{1,2}$ are the real and imaginary parts of the complex amplitude $\mathbf{H}_0$ of the magnetic field, and $\mathbf{k}'$ and $\mathbf{k}''$ are the real and imaginary parts of the vector $\mathbf{k}$. Find the locus of the end point of the vector $\mathbf{H}$ at a given point in space.

The electric field of an electromagnetic wave propagating with complex wave vector $\mathbf{k} = \mathbf{k}' + i\mathbf{k}''$ can be expressed as

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = \left(\mathbf{E}_0 e^{-\mathbf{k}''\cdot\mathbf{x}}\right) e^{i(\mathbf{k}'\cdot\mathbf{x}-\omega t)} \ , \tag{11}$$

where $\mathbf{E}_0$ describes the polarization of the wave. Since the polarization is linear, we enforce that $\mathbf{E}_0$ is completely real. This wave propagates in the $\mathbf{k}'$ direction, and is damped in the $\mathbf{k}''$ direction. The direction of the amplitude of the electric field (which, due to the linear polarization, is real) is by definition perpendicular to $\mathbf{k}'$. Furthermore, we know the magnitude of the wave vector k is a real number[1], so we enforce $\mathbf{k}\cdot\mathbf{k} = k^2$:

$$k^2 = (\mathbf{k}' + i\mathbf{k}'')\cdot(\mathbf{k}' + i\mathbf{k}'') = |\mathbf{k}'|^2 - |\mathbf{k}''|^2 + 2i(\mathbf{k}'\cdot\mathbf{k}'') \ , \tag{12}$$

which yields the conditions $\mathbf{k}'\cdot\mathbf{k}'' = 0$ and $k^2 = |\mathbf{k}'|^2 - |\mathbf{k}''|^2$, the first of which implies that $\mathbf{k}'$ and $\mathbf{k}''$ are orthogonal[2]. In addition to this we can use Maxwell's equations to write

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 0 = i\mathbf{k}\cdot\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \quad \Rightarrow \quad \mathbf{k}\cdot\mathbf{E}_0 = 0 \ , \tag{13}$$

(after dividing out i and the exponential) so the polarization is in the plane perpendicular to the wave vector (as it should be). If we use the complex representation of the wave vector, we see

$$\mathbf{k}'\cdot\mathbf{E}_0 + i\mathbf{k}''\cdot\mathbf{E}_0 = 0 \ , \tag{14}$$

and since we enforced that $\mathbf{E}_0$ is real, each term must vanish independently. We therefore see that the electric field and wave vector components form a mutually orthogonal set of vectors. Using another of Maxwell's equations, we have

$$\boldsymbol{\nabla}\times\mathbf{E} = -\mu\frac{\partial\mathbf{H}}{\partial t} \quad \Rightarrow \quad i\mathbf{k}\times\mathbf{E} = i\mu\omega\mathbf{H} \ , \tag{15}$$

where we have the magnetic field defined by

$$\mathbf{H} = \mathbf{H}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = (\boldsymbol{\mathcal{H}}_1 + i\boldsymbol{\mathcal{H}}_2) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \ . \tag{16}$$

Inserting the complex wave vector into Equation 15, we see

$$(\mathbf{k}'\times\mathbf{E}_0) + i(\mathbf{k}''\times\mathbf{E}_0) = \mu\omega(\boldsymbol{\mathcal{H}}_1 + i\boldsymbol{\mathcal{H}}_2) \ , \tag{17}$$

after dividing out the exponential factor common to both fields. Comparing real and imaginary parts we see

$$\boldsymbol{\mathcal{H}}_1 = \frac{1}{\mu\omega}\mathbf{k}'\times\mathbf{E}_0 = \frac{k'E_0}{\mu\omega}\left(\frac{\mathbf{k}''}{k''}\right) \tag{18}$$

$$\boldsymbol{\mathcal{H}}_2 = \frac{1}{\mu\omega}\mathbf{k}''\times\mathbf{E}_0 = -\frac{k''E_0}{\mu\omega}\left(\frac{\mathbf{k}'}{k'}\right) \ , \tag{19}$$

[1] Since $k^2 = \mu\epsilon\omega^2$, we are assuming that $\{\mu, \epsilon, \omega\} \in \mathbb{R}$.

[2] Compare this to the argument given by Jackson, Classical Electrodynamics, 3 ed. Chapter 7, equation 7.15.

it is arbitrary which component gets the negative sign, but one must be negative to maintain a right-handed coordinate system spanned by $\{\mathbf{E}_0, \mathbf{k}', \mathbf{k}''\}$, see Figure 1. If we align a standard Cartesian coordinate system with $\hat{\mathbf{z}}$ aligned with the electric field, we can write the the magnetic field as

$$\mathbf{H} = \frac{E_0}{\mu\omega}(k'\hat{\mathbf{x}} - ik''\hat{\mathbf{y}})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \ . \tag{20}$$

If we expand the complex exponential and take the real part, we see

$$\Re(\mathbf{H}) = \frac{E_0}{\mu\omega}\left(k'\cos(\mathbf{k}\cdot\mathbf{x}-\omega t)\hat{\mathbf{x}} + k''\sin(\mathbf{k}\cdot\mathbf{x}-\omega t)\hat{\mathbf{y}}\right) \ \Rightarrow \ \begin{cases} H_x &= \frac{E_0}{\mu\omega}k'\cos(\mathbf{k}\cdot\mathbf{x}-\omega t) \\ H_y &= \frac{E_0}{\mu\omega}k''\sin(\mathbf{k}\cdot\mathbf{x}-\omega t) \ , \end{cases} \tag{21}$$

which is elliptically polarized. At a time t, the locus of the end-point of a magnetic field line located at a point $\mathbf{x}_0$ is

$$\mathbf{x}_0 + \frac{E_0}{\mu\omega}k'\cos(\mathbf{k}\cdot\mathbf{x}_0-\omega t)\hat{\mathbf{x}} + \frac{E_0}{\mu\omega}k''\sin(\mathbf{k}\cdot\mathbf{x}_0-\omega t)\hat{\mathbf{y}} \tag{22}$$

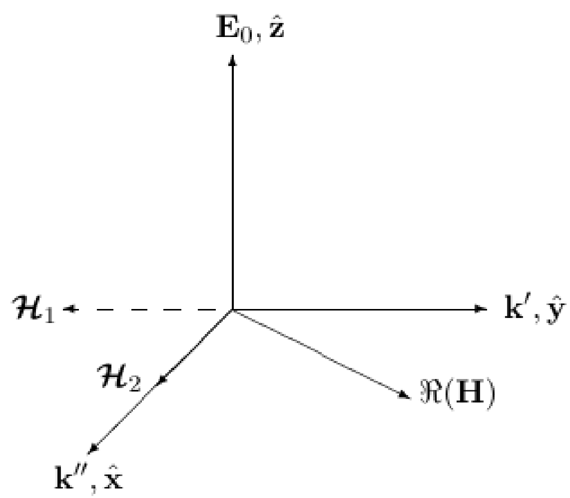

Figure 1: Relative positions of the electric field, real and imaginary parts of both the wave vector and magnetic field.

3 Relative Strength of Electromagnetic Wave Components in the Vacuum.

Calculate the numerical value of the ratio of the E and H fields of an electromagnetic wave in a vacuum, with appropriate units (use the simplest units possible).

An electromagnetic wave propagating in the $\hat{\mathbf{n}}$ direction has electric and magnetic fields at a location $\mathbf{x}$, at time t, given by

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}_0 e^{i(k\hat{\mathbf{n}}\cdot\mathbf{x}-\omega t)} \tag{23}$$

$$\mathbf{B}(\mathbf{x},t) = \mathbf{E}_0 e^{i(k\hat{\mathbf{n}}\cdot\mathbf{x}-\omega t)} \ , \tag{24}$$

which satisfy the Helmholtz wave equation[3] given that $k^2\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = \mu\epsilon\omega^2$, where μ and ϵ are the permeability and permittivity of the substance the electromagnetic waves permeate, respectively. Under this condition, we have[4],

$$H = \sqrt{\frac{\epsilon_0}{\mu_0}} E \sin\theta_{\mathbf{EB}} \ , \tag{25}$$

where $\theta_{\mathbf{EB}}$ is the angle between the electric and magnetic field. Since these fields are perpendicular, $\theta_{\mathbf{EB}} = \pm\pi/2 + 2\pi m$ with $m \in \mathbb{Z}$, and the ratio is

$$\frac{E}{H} = \sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{\frac{1.257\times10^{-6}}{8.854\times10^{-12}}}\left(\frac{\mathrm{N}\cdot\mathrm{A}^{-2}}{\mathrm{F}\cdot\mathrm{m}^{-1}}\right)^{1/2} \ . \tag{26}$$

Let us now investigate the units[5]:

$$\frac{1}{A^2}\frac{\mathrm{N}\cdot\mathrm{m}}{\mathrm{F}} = \frac{1}{\mathrm{A}^2}\frac{\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-2}}{(\mathrm{J}/\mathrm{V}^2)} = \frac{\mathrm{V}^2}{\mathrm{A}^2} \ , \tag{27}$$

so the ratio of the electric and magnetic field magnitudes is

$$\frac{E}{H} = 376.6\ \Omega \ , \tag{28}$$

which is a resistance.

[3] See Jackson, Classical Electrodynamics, 3 ed. Chapter 7, equations 7.3 and 7.9.
[4] See Jackson, Classical Electrodynamics, 3 ed. Chapter 7, equation 7.11.
[5] N=newton, A=ampere, F=farad, J=joule, V=volt, Ω=ohm.

4 Complex Components of Refractive Index for a Metal.

A linearly polarized wave is incident at an angle θ_i on the surface of a metal. The direction of the electric field vector is at an angle $\pi/4$ to the plane of incidence. The experimentally determined ratio of the perpendicular and parallel (to the plane of incidence) components of the reflected wave is found to be $R_{||}/R_{\perp} = \tan\rho$, and the phase difference δ between the components is such that

$$\frac{R_{||}}{R_{\perp}} = \tan\rho e^{i\delta} \ . \tag{29}$$

Show that the real and imaginary parts of the refractive index $n = n_2/n_1 = n' + in''$ under the condition $|(n')^2 - (n'')^2| \gg \sin^2\theta_i$ are given by

$$n' = \frac{\sin\theta_i \tan\theta_i \cos 2\rho}{1 + \sin 2\rho \cos\delta} \tag{30}$$

$$n'' = -\frac{\sin\theta_i \tan\theta_i \sin 2\rho \sin\delta}{1 + \sin 2\rho\cos\delta} \ . \tag{31}$$

Fresnel formulae give the amplitude of the parallel and perpendicular components of the reflected waves

$$R_{||} = A_{||}\frac{n\cos\theta_i - \cos\theta_t}{n\cos\theta_i + \cos\theta_t} \qquad R_{\perp} = A_{\perp}\frac{\cos\theta_i - n\cos\theta_t}{\cos\theta_i + n\cos\theta_t} \ . \tag{32}$$

Using Snell's law, we have

$$n_1 \sin\theta_i = n_2 \sin\theta_t \quad \Rightarrow \quad n = \frac{\sin\theta_i}{\sin\theta_t} \ , \tag{33}$$

with $n = n_2/n_1$. We can insert this into the Fresnel equations and take their ratio, which after some simplification yields

$$\frac{R_{||}}{R_{\perp}} = \frac{A_{||}}{A_{\perp}}(-1)\frac{\cos(\theta_i + \theta_t)}{\cos(\theta_i - \theta_t)} \ . \tag{34}$$

If we use the fact that the electric field makes an angle $\pi/4$ with the plane of incidence the components $A_{||}$ and $A_{\perp}$ must be equal, so the ratio is one[6]. The ratio of parallel to perpendicular components of the reflected wave can now be represented as

$$\frac{R_{||}}{R_{\perp}} = -\frac{\cos\theta_i\cos\theta_t - \sin\theta_i\sin\theta_t}{\cos\theta_i\cos\theta_t + \sin\theta_i\sin\theta_t} \ . \tag{35}$$

Once again, using Snell's law we have

$$\sin\theta_t = \frac{1}{n}\sin\theta_i \quad \Rightarrow \quad \frac{1}{n^2}\sin^2\theta_i + \cos^2\theta_t = 1 \ , \tag{36}$$

and so $\cos\theta_t = \frac{1}{n}\sqrt{n^2 - \sin^2\theta_i}$, let us now introduce the complex refractive index $n = m + ik$ (note we have switched notation $n' \to m$ and $n'' \to k$). Snell's law becomes

$$\cos\theta_t = \frac{1}{n}\sqrt{m^2 - k^2 + 2imk - \sin^2\theta_i} \sim \frac{1}{n}\sqrt{m^2 - k^2 + 2imk} = \frac{1}{n}\sqrt{n^2} = 1 \ , \tag{37}$$

[6] The parallel and perpendicular components are with respect to the plane of incidence.

therefore using the condition $|(n')^2 - (n'')^2| \gg \sin^2\theta_i$ we have that $\cos\theta_t \sim 1$. If we introduce this, and the condition given on the left of the arrow in Equation 36, back into the ratio in Equation 35, we obtain the expression

$$\frac{R_{||}}{R_\perp} = -\frac{\cos\theta_i - \frac{1}{n}\sin^2\theta_i}{\cos\theta_i + \frac{1}{n}\sin^2\theta_i} = -\frac{n\cos\theta_i - \sin^2\theta_i}{n\cos\theta_i + \sin^2\theta_i} \ , \tag{38}$$

after multiplying and dividing the refractive index. Inserting the complex refractive index to this expression yields

$$\frac{R_{||}}{R_\perp} = -\frac{(m+ik)\cos\theta_i - \sin^2\theta_i}{(m+ik)\cos\theta_i + \sin^2\theta_i} = -\frac{(m\cos\theta - \sin^2\theta_i) + ik\cos\theta_i}{(m\cos\theta + \sin^2\theta_i) + ik\cos\theta_i} \ , \tag{39}$$

which we can multiply and divide by the complex conjugate of the denominator to ensure the denominator is real. This gives the real and imaginary components of the ratio to be

$$\Re\left(\frac{R_{||}}{R_\perp}\right) = \frac{\sin^4\theta_i - k^2\cos^2\theta_i - m^2\cos^2\theta_i}{\sin^4\theta_i + k^2\cos^2\theta_i + m^2\cos^2\theta_i + 2m\sin^2\theta_i\cos\theta_i} \tag{40}$$

$$\Im\left(\frac{R_{||}}{R_\perp}\right) = -\frac{2k\sin^2\theta_i\cos\theta_i}{\sin^4\theta_i + k^2\cos^2\theta_i + m^2\cos^2\theta_i + 2m\sin^2\theta_i\cos\theta_i} \ . \tag{41}$$

Using the measured ratio given in the problem, and equating the real parts and imaginary parts with the expressions above yield the two equations

$$\tan\rho\cos\delta = \frac{\sin^4\theta_i - k^2\cos^2\theta_i - m^2\cos^2\theta_i}{\sin^4\theta_i + k^2\cos^2\theta_i + m^2\cos^2\theta_i + 2m\sin^2\theta_i\cos\theta_i} \tag{42}$$

$$\tan\rho\sin\delta = -\frac{2k\sin^2\theta_i\cos\theta_i}{\sin^4\theta_i + k^2\cos^2\theta_i + m^2\cos^2\theta_i + 2m\sin^2\theta_i\cos\theta_i} \ . \tag{43}$$

Pleading Mathematica to solve these equations for m and k, we obtain the results

$$m = \frac{\sin(\theta_i)\tan(\theta_i)\cos(2\rho)\sec^2(\rho)}{2\cos(\delta)\tan(\rho) + \sec^2(\rho)} = \frac{\sin\theta_i\tan\theta_i\cos 2\rho}{1 + \frac{2\cos\delta\tan\rho}{\sec^2\rho}} \tag{44}$$

$$k = -\frac{2\sin(\delta)\sin(\theta_i)\tan(\theta_i)\tan(\rho)}{2\cos(\delta)\tan(\rho) + \sec^2(\rho)} = -\frac{2\sin\theta_i\tan\theta_i\sin\delta\frac{\tan\rho}{\sec^2\rho}}{1 + \frac{2\cos\delta\tan\rho}{\sec^2\rho}} \ . \tag{45}$$

Let us investigate the denominator:

$$1 + \frac{2\cos\delta\tan\rho}{\sec^2\rho} = 1 + 2\cos\delta\sin\rho\cos\rho = 1 + \cos\delta\sin 2\rho \ , \tag{46}$$

which is as desired. The last factor in the expression for k (including the prefactor of 2) is

$$2\frac{\tan\rho}{\sec^2\rho} = 2\sin\rho\cos\rho = \sin 2\rho \ , \tag{47}$$

which yields the final result:

$$m = n' = \frac{\sin\theta_i\tan\theta_i\cos 2\rho}{1 + \cos\delta\sin 2\rho} \tag{48}$$

$$k = n'' = -\frac{\sin\theta_i\tan\theta_i\sin 2\rho\sin\delta}{1 + \cos\delta\sin 2\rho} \ . \tag{49}$$

5 Electron Density of Interstellar Medium.

A pulsar emits a pulse with frequencies ω_1, ω_2 considerably larger than the plasma frequency ω_p of the interstellar medium. The times of arrival of the pulses with frequencies ω_1, ω_2 are measured. Show that this permits the determination of the electron density of the medium integrated over the distance L to the pulsar, *i.e.*, $\int_0^L n(l)\mathrm{d}l$ is the distance dependent electron density.

The wave number is related to the frequency of the wave by

$$k^2 = \epsilon\mu\omega^2 \sim \epsilon_0\mu_0\epsilon_r(\omega)\omega^2 \ , \tag{50}$$

which we can insert the definition[7] of the relative permittivity of a plasma,

$$\epsilon_r = 1 - \frac{\omega_p^2}{\omega^2} \ , \tag{51}$$

where $\omega_p(z)$ is the distance dependent plasma frequency:

$$\omega_p^2(z) = \frac{e^2}{m\epsilon_0}N(z) \ , \tag{52}$$

with e and m as the charge and effective mass of the charge carriers (in this case electrons) in the plasma, respectively, and $N(z)$ is the distance dependent density of charge carriers. Using the permittivity of a plasma, we see

$$\omega^2 - \omega_p^2 = c^2k^2 \ , \tag{53}$$

where $c = 1/\sqrt{\epsilon_0\mu_0}$. The group velocity of the wave is then

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \left(\frac{\mathrm{d}k}{\mathrm{d}\omega}\right)^{-1} = \left(\frac{\mathrm{d}}{\mathrm{d}\omega}\frac{1}{c}\sqrt{\omega^2-\omega_p^2}\right)^{-1} = \left(\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}\omega}\frac{1}{2}\left(\omega^2-\omega_p^2\right)^{-1/2}(2\omega)\right)^{-1} \ , \tag{54}$$

which we can invert to obtain the group velocity:

$$v_g = c\sqrt{\omega^2-\omega_p^2}\frac{1}{\omega} = c\sqrt{1-\frac{\omega_p^2}{\omega^2}} \ . \tag{55}$$

Using these relations, if a signal of frequency ω_1 was emitted from a source at $t = 0$, at a distance L away, the time to detect the signal is given by

$$t = \int_0^L \frac{\mathrm{d}z}{v_g(z)} = \int_0^L \frac{\mathrm{d}z}{c\sqrt{\epsilon_r(z)}} = \int_0^L \frac{\mathrm{d}z}{c}\left[1 - \frac{e^2}{m\epsilon_0}\frac{N(z)}{\omega_1^2}\right]^{-1/2} \simeq \int_0^L \frac{\mathrm{d}z}{c}\left[1 + \frac{e^2}{2m\epsilon_0}\frac{N(z)}{\omega_1^2}\right] \ , \tag{56}$$

if we assume the frequency $\omega_1 \gg \omega_p$, which is the regime in which Equation 51 applies. If we now consider two signals of frequencies ω_1 and ω_2, the time difference in their arrivals at a detector a distance L away is

$$\Delta t = t_2 - t_1 \simeq \int_0^L \frac{\mathrm{d}z}{c}\left[1 + \frac{e^2}{2m\epsilon_0}\frac{N(z)}{\omega_2^2} - 1 - \frac{e^2}{2m\epsilon_0}\frac{N(z)}{\omega_1^2}\right] = \frac{e^2}{2mc\epsilon_0}\left(\frac{1}{\omega_2^2} - \frac{1}{\omega_1^2}\right)\int_0^L N(z)\mathrm{d}z \ . \tag{57}$$

In the expression above the first factor is just a product of fundamental constants, so assuming the frequencies of the two signals are known, and the time difference in their arrival can be measured, the electron density of the medium integrated over the distance L can be determined:

$$\int_0^L N(z)\mathrm{d}z = \epsilon_0\frac{2mc(\Delta t)}{e^2}\frac{(\omega_1\omega_2)^2}{\omega_1^2-\omega_2^2} \ . \tag{58}$$

[7] Jackson, Classical Electrodynamics, 3 ed. Chapter 7, equation 7.59.

6 Electromagnetic Waves in a Plasma.

In the absence of absorption, permittivity of a plasma is given by

$$\epsilon = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2} \right) \ , \tag{59}$$

where $\omega_p^2 = Ne^2/m\epsilon_0$ (where e is the magnitude of the electron charge) defines the plasmon frequency. Discuss the propagation of electromagnetic waves in a plasma whose concentration is described by the linear function $N(z) = N_0 z$. Consider the case where a plane monochromatic wave is incident normally on a non-homogeneous layer of plasma. (This is one model for the propagation of radio waves in the ionosphere.)

First, we can note that the index of refraction is

$$n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} = \sqrt{\frac{\mu}{\mu_0}} \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \ , \tag{60}$$

which is purely imaginary if the frequency of the wave is less than the plasma frequency:

$$\omega^2 < \frac{N_0 e^2}{m\epsilon_0} z \ , \tag{61}$$

below this the wave does not propagate. Now consider the Maxwell equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{62}$$

$$\nabla \times \mathbf{B} = \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} \ , \tag{63}$$

of which we can take the curl of the first and the partial derivative with respect to time of the second, yielding

$$\nabla \times \nabla \times \mathbf{E} = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E} = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} \tag{64}$$

$$\nabla \times \frac{\partial \mathbf{B}}{\partial t} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \ , \tag{65}$$

which we can insert into each other yielding the differential equation

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \ , \tag{66}$$

which is the vector wave equation. We can then insert the permittivity of a plasma to the equation:

$$\nabla^2 \mathbf{E} - \mu \left(\epsilon_0 - \frac{N_0 e^2}{m\omega^2} z \right) \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \ , \tag{67}$$

and assume a harmonic solution of the form

$$\mathbf{E} = \mathbf{F}(z) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \ , \tag{68}$$

where $\mathbf{F}(z)$ is an arbitrary function of z. Inserting this to the wave equation yields

$$[\nabla^2\mathbf{F}(z)]e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + \mathbf{F}(z)[\nabla^2 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}] - \mu\left(\epsilon_0 - \frac{N_0 e^2}{m\omega^2}z\right)(-\omega^2)\mathbf{F}(z)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = 0 \ , \tag{69}$$

which simplifies, after dividing out the exponential, to

$$\frac{\partial^2\mathbf{F}}{\partial z^2} - k^2\mathbf{F} + \mu\left(\epsilon_0 - \frac{N_0 e^2}{m\omega^2}z\right)\omega^2\mathbf{F} = 0 \ , \tag{70}$$

let us define $q(z) \equiv -k^2 + \mu\epsilon(z)\omega^2 = -k^2 + \mu\epsilon_0\omega^2 - (\mu N_0 e^2/m)z$. If we then assume $\mu \sim \mu_0$ then we can write $k^2 = \mu_0\epsilon_0\omega^2$ and then cancels with the $-k^2$ term[8]. We can express the differential equation as

$$\mathbf{F}'' + q(z)\mathbf{F} = \mathbf{F}'' - \left(\sqrt{\epsilon_0\mu}\omega_p\right)^2 z\mathbf{F} = 0 \ \rightarrow \ \mathbf{F}'' - \left(\frac{\omega_0}{c}\right)^2 z\mathbf{F} = 0 \ , \tag{71}$$

with $\omega_0^2 = \frac{N_0 e^2}{m\epsilon_0}$, which is of the form of the Airy equation[9]. The solutions are then of the form

$$F(z) = C_1 A\left(\left(\frac{\omega_0}{c}\right)^{2/3} z\right) + C_2 B\left(\left(\frac{\omega_0}{c}\right)^{2/3} z\right) \ , \tag{72}$$

where $A(z)$ and $B(z)$ are the Airy functions. We can set $C_2 = 0$ becuase the Airy function of the second kind (B) diverges as $z \to \infty$, while $A \to 0$ and $z \to \infty$, which is the desired behavior. As we said in the beginning, there is some height at which the dielectric constant becomes negative, and the index of refraction will be purely imaginary. Past this point, there is no propagation of a wave, and it must be reflected (allowing for some absorption). This height is

$$z = \frac{m\epsilon_0\omega^2}{N_0 e^2} \ . \tag{73}$$

[9]Weisstein, Eric W. "Airy Differential Equation." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/AiryDifferentialEquation.html

6.1 Solution without Dispersion Relation Assumption.

We can express the differential equation as

$$\mathbf{F}'' + q(z)\mathbf{F} = 0 \ , \tag{74}$$

where the primes denote derivatives with respect to the coordinate z, with

$$q(z) = -k^2 + \mu\epsilon_0\omega^2 - (\mu N_0 e^2/m)z \tag{75}$$

This ODE can be represented as[10]

$$y''(x) + ay'(x) + (bx + c)y = 0 \ , \tag{76}$$

with

$$a = 0 \tag{77}$$

$$b = -\frac{\mu N_0 e^2}{m\omega^2} \tag{78}$$

$$c = \mu\epsilon_0 - k^2 \ , \tag{79}$$

and thus has solutions

$$F(z) = \sqrt{z - \frac{(\mu\epsilon_0 - k^2)m\omega^2}{\mu N_0 e^2}} \Bigg\{ C_1 J_{1/3}\left(\frac{2}{3}\sqrt{-\frac{\mu N_0 e^2}{m\omega^2}}\left[z - \frac{(\mu\epsilon_0 - k^2)m\omega^2}{\mu N_0 e^2}\right]^{3/2}\right) + C_2 Y_{1/3}\left(\frac{2}{3}\sqrt{-\frac{\mu N_0 e^2}{m\omega^2}}\left[z - \frac{(\mu\epsilon_0 - k^2)m\omega^2}{\mu N_0 e^2}\right]^{3/2}\right)\Bigg\} \ , \tag{80}$$

where C_1 and C_2 are arbitrary constants, and $J_{1/3}$ and $Y_{1/3}$ are fractional Bessel functions. If we consider $z > 0$ this is the solution, but for $z \geq 0$ we can set $C_2 = 0$, for the Bessel functions of the second kind diverge at the origin. Both of the Bessel functions decay as $z \to \infty$ so the amplitudes of waves farther into the plasma are much lower than those just near the surface of the plasma.

[10] Andrei D. Polyanin, EqWorld. http://eqworld.ipmnet.ru/en/solutions/ode/ode0204.pdf.