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Contents

1 Complex Dielectric Constant: Drude Approximation.

Plot the frequency dependence of the real and imaginary part of the dielectric constant as well as the reflectivity of a metal in the Drude approximation, taking into account the frequency dependence of the conductivity, but ignoring any contributions other than that from conduction electrons in the metal. Make sure the frequency range covers all the relevant regions of interest.

The Drude approximation is one model for electrical conduction of metals in which the conduction electrons are viewed as a classical electron gas. It uses a frequency dependent, complex relative permittivity given by

$$\epsilon_r(\omega) = 1 + \omega_p^2 \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \ , \tag{1}$$

where $\omega_p^2 = Ne^2/m_e\epsilon_0$ is the plasma frequency and ω_0 is the resonant frequency of the metal, and $\gamma = 1/\tau$ is the inverse time constant of the metal

$$\tau = \frac{m_e\sigma}{Ne^2} = \frac{1}{\omega_p^2\epsilon_0}\sigma \ , \tag{2}$$

where σ is the conductivity. The relative permittivity has real and imaginary components given by

$$\Re\{\epsilon_r(\omega)\} = 1 - \omega_p^2 \frac{\omega^2 - \omega_0^2}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2} \tag{3}$$

$$\Im\{\epsilon_r(\omega)\} = \omega_p^2 \frac{\gamma\omega}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2} \ . \tag{4}$$

The behavior of these components is shown in Figure 1 with $\omega_p = 2$, $\omega_0 = 4$, and $\gamma = 1.4$.

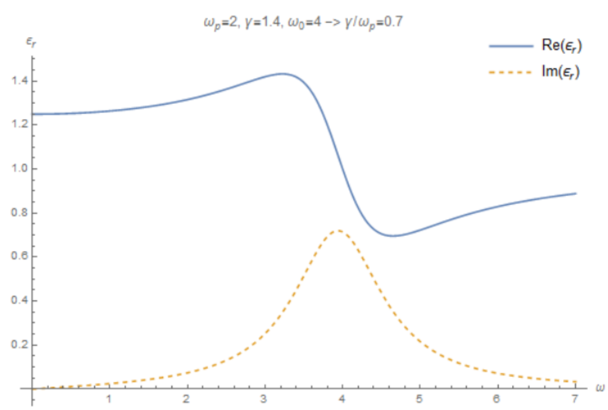
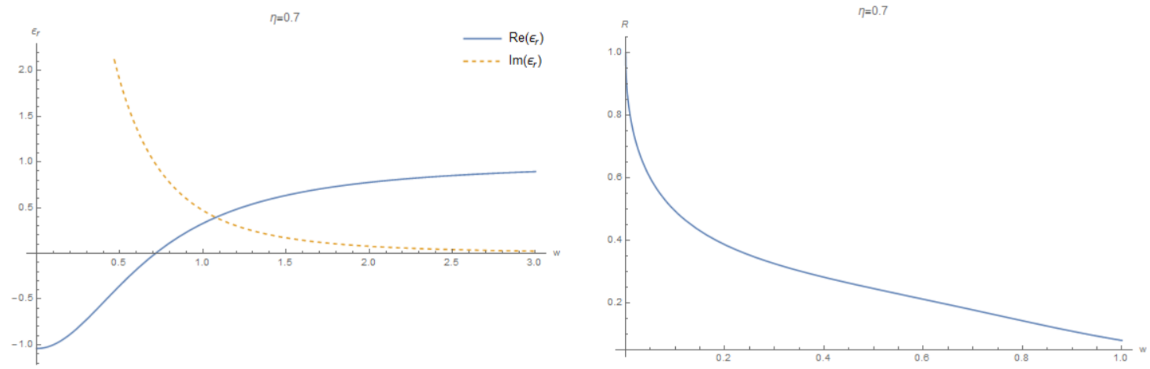


Figure 1: Relative permittivity of metals in the Drude model: real (blue) and imaginary (yellow, dashed) components, with $\omega_0 = 4$, $\omega_p = 2$, and $\gamma = 1.4$.

This derivation of the dielectric constant using a harmonic model for atomic polarizability of the conductor. If we ignore the resonance (equivalent to setting the restoring force of the harmonic equation of motion in this model) we recover the expressions

$$\Re\{\epsilon_r(\omega)\} = 1 - \omega_p^2 \frac{\omega^2}{\omega^4 + \gamma^2\omega^2} = 1 - \omega_p^2 \frac{\tau^2}{\omega^2\tau^2 + 1} \tag{5}$$

$$\Im\{\epsilon_r(\omega)\} = \omega_p^2 \frac{\gamma\omega}{\omega^4 + \gamma^2\omega^2} = \omega_p^2 \frac{\gamma\omega}{\omega^4 + \gamma^2\omega^2} \frac{\tau^3/\omega}{\tau^3/\omega} = \omega_p^2 \frac{\tau^2}{\omega^3\tau^3 + \omega\tau} \ . \tag{6}$$



(a) The real (blue) and imaginary (yellow, dashed) components of relative permittivity, with $1/(\omega\tau) = 0.7$, for $\omega/\omega_p \in [0, 3]$.

(b) Reflectivity as a function of dimensionless frequency $\omega/\omega_p \in [0, 3]$. Note this is normalized and has a maximum of one, as expected physically.

Figure 2: The simplified Drude model for conductivity in metals, with no resonances, for $1/(\omega\tau) = 0.7$.

Let us divide the numerator and denominator of the second term of the real component by $\omega_p^2\tau^2$, which results in

$$\Re\{\epsilon_r(\omega)\} = 1 - \frac{1}{(\omega/\omega_p)^2 + 1/(\omega_p\tau)^2} \ . \tag{7}$$

While dividing the numerator and denominator of the imaginary component by $\omega_p^3\tau^3$ yields

$$\Im\{\epsilon_r(\omega)\} = \frac{1}{\omega_p\tau}\frac{1}{(\omega/\omega_p)^3 + (\omega/\omega_p)/(\omega_p\tau)^2} \ , \tag{8}$$

note the same quantities appear in both, which leads to the definition of the dimensionless frequency $w = \omega/\omega_p$ and a dimensionless constant $\eta = (\omega_p\tau)^{-1}$. Finally, we can express the real and imaginary components of the dielectric constant in terms of a single parameter and a dimensionless variable:

$$\Re\{\epsilon_r(w)\} = 1 - \frac{1}{w^2 + \eta^2} \tag{9}$$

$$\Im\{\epsilon_r(w)\} = \eta\frac{1}{w^3 + \eta^2 w} \ , \tag{10}$$

which are shown in Figure 2a for $\eta = 0.7$ on the interval of w from zero to three.

The reflectivity is given by

$$\mathcal{R} = \left|\frac{1-n}{1+n}\right|^2 \ , \tag{11}$$

but for metals, the index of refraction is $n \in \mathbb{C}$, and is given by the square root of the relative permittivity,

$$n = \sqrt{\Re\{\epsilon_r(w)\} + i\Im\{\epsilon_r(w)\}} = \sqrt{1 - \frac{1}{w^2+\eta^2} + i\eta\frac{1}{w^3+\eta^2 w}} \ , \tag{12}$$

shown in Figure 2b, for $\eta = 0.7$.

2 Sokhotski–Plemelj Theorem.

In many advanced physics texts, one might see the strange equation

$$\frac{1}{x \pm i\epsilon} = \mathcal{P}\frac{1}{x} \mp i\pi\delta(x) \ . \tag{13}$$

Justify this equation, and in doing so, describe under what conditions it is valid.

The given equation, Sokhotski's formula, is only valid in the limit that $\epsilon \to 0$, but we most approach from the positive side because we have $\epsilon > 0$ (account for both possibilities with the $\pm$). Here, $\mathcal{P}$ represents the Cauchy principal value, which is a distribution, the same goes for the delta function. In light of this, we may view the expression above as an operator expression treating the left and right sides as operators. Now, consider the action of the left-hand side operator on a test function $f(x)$ (where $f(x)$ is complex valued, and continuous on the real axis), and integrate over a region of the distribution:

$$\lim_{\epsilon\to 0^+} \int_a^b \frac{1}{x \pm i\epsilon} f(x) \mathrm{d}x \ , \tag{14}$$

which has poles at $x = \mp i\epsilon$. Multiplying and dividing by the complex conjugagte of the denominator yields the real and imaginary parts:

$$\lim_{\epsilon\to 0^+} \int_a^b f(x)\mathrm{d}x \frac{1}{x \pm i\epsilon}\frac{x \mp i\epsilon}{x \mp i\epsilon} = \lim_{\epsilon\to 0^+} \int_a^b f(x)\mathrm{d}x \frac{x \mp i\epsilon}{x^2 - (i\epsilon)^2} \ , \tag{15}$$

which we can separate:

$$\lim_{\epsilon\to 0^+} \int_a^b \frac{1}{x \pm i\epsilon} f(x)\mathrm{d}x = \lim_{\epsilon\to 0^+} \int_a^b f(x)\mathrm{d}x \frac{x}{x^2+\epsilon^2} \mp \lim_{\epsilon\to 0^+} \int_a^b f(x)\mathrm{d}x \frac{i\epsilon}{x^2+\epsilon^2} \ , \tag{16}$$

let us multiply and divide the first term on the right-hand side by x, and the do the same with the second, but with π:

$$\lim_{\epsilon\to 0^+} \int_a^b \frac{f(x)\mathrm{d}x}{x \pm i\epsilon} = \lim_{\epsilon\to 0^+} \int_a^b \frac{f(x)\mathrm{d}x}{x}\frac{x^2}{x^2+\epsilon^2} \mp i\pi \int_a^b f(x)\mathrm{d}x \lim_{\epsilon\to 0^+} \frac{\epsilon}{\pi(x^2+\epsilon^2)} \ . \tag{17}$$

The imaginary term represents a nascent delta function of the form

$$\eta_\epsilon(x) = \frac{1}{\epsilon}\eta\left(\frac{x}{\epsilon}\right) \quad \text{with} \quad \eta(\xi) = \frac{1}{\pi(\xi^2+1)} \ , \tag{18}$$

which in the limit $\epsilon \to 0^+$, this approaches a Dirac delta function $\delta(x)$. In the following regions the factor in the real term behaves as

$$\lim_{x\gg|\epsilon|} \frac{x^2}{x^2+\epsilon^2} = 1 \tag{19}$$

$$\lim_{x\ll|\epsilon|} \frac{x^2}{x^2+\epsilon^2} = 0 \ , \tag{20}$$

so the integral:

$$\int \frac{f(x)\mathrm{d}x}{x}\frac{x^2}{x^2+\epsilon^2} = \int^{-\epsilon} \frac{f(x)\mathrm{d}x}{x}\frac{x^2}{x^2+\epsilon^2} + \int_{-\epsilon}^{\epsilon} \frac{f(x)\mathrm{d}x}{x}\frac{x^2}{x^2+\epsilon^2} + \int_{\epsilon} \frac{f(x)\mathrm{d}x}{x}\frac{x^2}{x^2+\epsilon^2} \ , \tag{21}$$

will reduce to

$$\int^{-\epsilon} \frac{f(x)}{x} \mathrm{d}x + \int_{\epsilon} \frac{f(x)}{x} \mathrm{d}x \ , \tag{22}$$

which in the limit $\epsilon \to 0^+$ is simply the definition of the Cauchy principal value. Now the integral in Equation 14 can be written

$$\int_a^b \frac{f(x)}{x \pm i\epsilon} \mathrm{d}x = \int_a^b \left\{ \mathcal{P}\frac{1}{x} \mp i\pi\delta(x) \right\} f(x) \mathrm{d}x \ , \tag{23}$$

in the limit $\epsilon \to 0$. We can then identify the operator

$$\frac{1}{x \pm i\epsilon} = \mathcal{P}\frac{1}{x} \mp i\pi\delta(x) \ , \tag{24}$$

as the Sokhotski formula.

3 Kramers-Kronig Relations.

Under the assumption that "cause" and "effect" are real quantities, derive the Kramers-Kronig relations for the frequency dependent real and imaginary parts of the relative dielectric constant $\epsilon(\omega) = \epsilon'(\omega) + i\epsilon''(\omega)$:

$$\epsilon'(\omega) - 1 = \frac{2}{\pi}\mathcal{P}\int_0^\infty \mathrm{d}x \frac{x\epsilon''(x)}{x^2 - \omega^2} \tag{25}$$

$$\epsilon''(\omega) = -\frac{2}{\pi}\omega\mathcal{P}\int_0^\infty \mathrm{d}x \frac{\epsilon'(x) - 1}{x^2 - \omega^2} \ . \tag{26}$$

If we treat the electric displacement field as the effect caused by the electric field in the dielectric, we may write this field[1] as

$$\mathbf{D}(\mathbf{x}, t) = \epsilon_0 \left\{ \mathbf{E}(\mathbf{x}, t) + \int_0^\infty G(\tau)\mathbf{E}(\mathbf{x}, t - \tau)\mathrm{d}\tau \right\} \ , \tag{27}$$

where $G(\tau)$ is the Fourier transform of the electric susceptibility $\chi_e = \epsilon(\omega) - 1$ (note we have defined our ϵ such that it is the relative permittivity: $\boldsymbol{\epsilon} = \epsilon_0\epsilon_r(\omega)$ where $\boldsymbol{\epsilon}$ is the dielectric constant) given by

$$G(\tau) = \frac{1}{2\pi}\int_{-\infty}^\infty [\epsilon(\omega) - 1] \, e^{-i\omega t}\mathrm{d}\omega \ , \tag{28}$$

which can be inverted to write

$$\epsilon(\omega) = 1 + \int_0^\infty G(\tau)e^{i\omega\tau}\mathrm{d}\tau \ . \tag{29}$$

This model (Equation 27) is valid regardless of the model of the frequency dependent dielectric constant, which in this case is a complex quantity. Given that the displacement (effect) and electric (cause) fields are real, so must be $G(\tau)$, we use the form of the previous equation to write

$$\epsilon(-\omega) = 1 + \int_0^\infty G(\tau)e^{i(-\omega)\tau}\mathrm{d}\tau \tag{30}$$

$$\epsilon^*(\omega) = 1 + \int_0^\infty G(\tau)e^{(-i)\omega\tau}\mathrm{d}\tau \ , \tag{31}$$

so $\epsilon(-\omega) = \epsilon^*(\omega^*)$ (we have $\omega \in \mathbb{R} \to \omega = \omega^*$), and is therefore analytic in the complex ϵ upper half plane, given that $G(\tau)$ is well-behaved. Consider a complex-valued function $f(z)$ which is analytic in the complex upper half plane; using Cauchy's integral formula, we can take the integral along the closed contour C in the complex plane to obtain

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\omega)}{\omega - z}\mathrm{d}\omega \ . \tag{32}$$

We can identify the function $f(\omega') = \epsilon(\omega') - 1$, and similarly $f(z) = \epsilon(z) - 1$, so using Cauchy's integral formula we write

$$\epsilon(z) - 1 = \frac{1}{2\pi i}\oint_C \frac{\epsilon(\omega') - 1}{\omega' - z}\mathrm{d}\omega' \ , \tag{33}$$

[1] Jackson, Classical Electrodynamics, 3ed. Equation 7.111.

we can now use a contour C enclosing the upper half plane in a semicircle of infinite radius. This contour consists of the real axis, an infinitesimal semicircular arc of radius ϵ below the origin (to allow for the possibility of a singularity at the origin), and semicircular arc with $R \to \infty$. The integrand vanishes along the arc at infinity and does not contribute to the contour integral, leaving only the integral along the real axis from negative to positive infinity and the infinitesimal arc below the origin:

$$\epsilon(z) - 1 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\epsilon(\omega') - 1}{\omega' - z} \mathrm{d}\omega' \ , \tag{34}$$

where z is any point in the upper half complex plane. Consider the points z an infinitesimal displacement δ away from the real axis: $z = \omega + i\delta$, and then take the limit $\delta \to 0^+$:

$$\lim_{\delta \to 0^*} \epsilon(w + i\delta) - 1 = \epsilon(w) - 1 = \lim_{\delta \to 0^*} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\epsilon(\omega') - 1}{\omega' - \omega - i\delta} \mathrm{d}\omega' \ . \tag{35}$$

Rewriting this in terms of the function $f(\omega) = \epsilon(w) - 1$, we see

$$f(\omega) = \lim_{\delta \to 0^*} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(\omega')}{\omega' - \omega - i\delta} \mathrm{d}\omega' \ . \tag{36}$$

We may then apply the Sokhotski formula (Equation 23 with $x \to \omega' - \omega$) to this integral, yielding

$$f(\omega) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left\{ \mathcal{P} \frac{1}{\omega' - \omega} + i\pi\delta(\omega' - \omega) \right\} f(\omega') \mathrm{d}\omega' \tag{37}$$

$$= \frac{1}{2} \left\{ \frac{1}{\pi i} \mathcal{P} \int_{-\infty}^{\infty} \frac{1}{\omega' - \omega} f(\omega') \mathrm{d}\omega' + \int_{-\infty}^{\infty} \delta(\omega' - \omega) f(\omega') \mathrm{d}\omega' \right\} \tag{38}$$

$$= \frac{1}{2} \frac{1}{\pi i} \mathcal{P} \int_{-\infty}^{\infty} \frac{1}{\omega' - \omega} f(\omega') \mathrm{d}\omega' + \frac{1}{2} f(\omega) \tag{39}$$

$$f(\omega) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \mathcal{P} \frac{f(\omega')}{\omega' - \omega} \mathrm{d}\omega' \quad \Rightarrow \quad \epsilon(w) = 1 + \frac{1}{\pi i} \mathcal{P} \int_{-\infty}^{\infty} \frac{\epsilon(\omega') - 1}{\omega' - \omega} \mathrm{d}\omega' \ . \tag{40}$$

Let us expand this in real and imaginary components

$$\epsilon(w) = 1 + \frac{1}{\pi i} \mathcal{P} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{\omega' - \omega} \big(\epsilon(\omega') - 1 \big) \tag{41}$$

$$= 1 + \frac{1}{\pi i} \mathcal{P} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{\omega' - \omega} \left\{ \Re \big(\epsilon(\omega') - 1 \big) + i \Im \big(\epsilon(\omega') \big) \right\} \tag{42}$$

$$= 1 + \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{\omega' - \omega} \Im \big(\epsilon(\omega') \big) - i \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{\omega' - \omega} \Re \big(\epsilon(\omega') - 1 \big) \right\} \ . \tag{43}$$

We can now easily identify the real and imaginary components $\epsilon'(\omega)$ and $\epsilon''(\omega)$ to be

$$\Re[\epsilon(\omega)] = \epsilon'(\omega) = 1 + \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\Im[\epsilon(\omega')]}{\omega' - \omega} \mathrm{d}\omega' \tag{44}$$

$$\Im[\epsilon(\omega)] = \epsilon''(\omega) = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\Re[\epsilon(\omega') - 1]}{\omega' - \omega} \mathrm{d}\omega' \ . \tag{45}$$

We may cast these in a form such that the integral does not range over negative frequency, first by multiplying and dividing both integrals by the conjugate of the denominator. Consider the integrals:

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{\Im\left[\epsilon(\omega')\right]}{\omega'-\omega}\frac{\omega'+\omega}{\omega'+\omega}\mathrm{d}\omega' = \omega\mathcal{P}\int_{-\infty}^{\infty}\frac{\Im\left[\epsilon(\omega')\right]}{\omega'^2-\omega^2}\mathrm{d}\omega' + \mathcal{P}\int_{-\infty}^{\infty}\omega'\frac{\Im\left[\epsilon(\omega')\right]}{\omega'^2-\omega^2} \tag{46}$$

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{\Re\left[\epsilon(\omega')-1\right]}{\omega'-\omega}\frac{\omega'+\omega}{\omega'+\omega}\mathrm{d}\omega' = \omega\mathcal{P}\int_{-\infty}^{\infty}\frac{\Re\left[\epsilon(\omega')-1\right]}{\omega'^2-\omega^2}\mathrm{d}\omega' + \mathcal{P}\int_{-\infty}^{\infty}\omega'\frac{\Re\left[\epsilon(\omega')-1\right]}{\omega'^2-\omega^2}\ . \tag{47}$$

We previously stated $\epsilon(-\omega) = \epsilon^*(\omega)$, so let us investigate the repercussions of this with our definition of the complex dielectric constant $\epsilon(\omega) = \epsilon'(\omega) + i\epsilon''(\omega)$:

$$\epsilon(-\omega) = \epsilon'(-\omega) + i\epsilon''(-\omega) \tag{48}$$

$$\epsilon^*(\omega) = \epsilon'(\omega) - i\epsilon''(-\omega)\ , \tag{49}$$

and since these expressions are equivalent, we must have $\epsilon'(-\omega) = \epsilon'(\omega)$ so $\Re[\epsilon(\omega)-1]$ is even in ω and $\epsilon''(-\omega) = -\epsilon''(\omega)$ so $\Im[\epsilon(\omega)]$ is odd in ω. Using this fact the first integral in Equation 46, and the second in Equation 47, are zero by symmetry. Therefore the real and imaginary parts of the dielectric constant can be written

$$\epsilon'(\omega) = 1 + \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\omega'\epsilon''(\omega')}{\omega'^2-\omega^2}\mathrm{d}\omega' = 1 + \frac{2}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{x\epsilon''(x)}{x^2-\omega^2}\mathrm{d}x \tag{50}$$

$$\epsilon''(\omega) = -\frac{\omega}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\epsilon'(\omega')-1}{\omega'^2-\omega^2}\mathrm{d}\omega' = -\frac{2\omega}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\epsilon'(x)-1}{x^2-\omega^2}\mathrm{d}x\ , \tag{51}$$

after writing $\Re[\epsilon(\omega)-1] = \Re[\epsilon(\omega)]-1 = \epsilon'(\omega)-1$ and $\Im[\epsilon(\omega)] = \epsilon''(\omega)$. Note the change in integration limits, because the integrands are even functions. These are the Kramers-Kronig relations, in the desired form.

4 Position-Dependent Permittivity.

In deriving the wave equation satisfied by **E** and **H** fields, we have assumed that the permittivity and the permeability are constants independent of position. Show that if the relative permittivity is a function of position, the wave equation for the **H** field is modified to

$$\nabla^2 \mathbf{H} + \epsilon \frac{\omega^2}{c^2} \mathbf{H} + \frac{1}{\epsilon} (\boldsymbol{\nabla} \epsilon) \times (\boldsymbol{\nabla} \times \mathbf{H}) = 0 \ , \tag{52}$$

where $\mu \sim \mu_0$ and $c^2 = 1/(\mu_0 \epsilon_0)$.

Let us begin from Maxwell's equations

$$0 = \boldsymbol{\nabla} \times \mathbf{E} + \mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{53}$$

$$0 = \boldsymbol{\nabla} \times \mathbf{H} - \epsilon \frac{\partial \mathbf{E}}{\partial t} \ , \tag{54}$$

where the permittivity is $\epsilon = \epsilon_0 \epsilon_r(\mathbf{x})$. Let us take the curl of the second, and the time derivative of the first, which yields

$$0 = \boldsymbol{\nabla} \times \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \frac{\partial^2 \mathbf{H}}{\partial t^2} \quad \Rightarrow \quad \boldsymbol{\nabla} \times \frac{\partial \mathbf{E}}{\partial t} = -\mu_0 \frac{\partial^2 \mathbf{H}}{\partial t^2} \tag{55}$$

$$0 = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{H} - \boldsymbol{\nabla} \times \left(\epsilon \frac{\partial \mathbf{E}}{\partial t} \right) \quad \Rightarrow \quad \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{H}) - \nabla^2 \mathbf{H} = \left(\boldsymbol{\nabla} \epsilon \times \frac{\partial \mathbf{E}}{\partial t} + \epsilon \boldsymbol{\nabla} \times \frac{\partial \mathbf{E}}{\partial t} \right) \ , \tag{56}$$

given that there are no magnetic monopoles the first term in the second equation is zero. We are left with the equation

$$0 = \nabla^2 \mathbf{H} + \left(\boldsymbol{\nabla} \epsilon \times \frac{\partial \mathbf{E}}{\partial t} + \epsilon \boldsymbol{\nabla} \times \frac{\partial \mathbf{E}}{\partial t} \right) = \nabla^2 \mathbf{H} + \boldsymbol{\nabla} \epsilon \times \frac{\partial \mathbf{E}}{\partial t} - \mu_0 \epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} \ , \tag{57}$$

after inserting Equation 55. We can then insert Equation 54, and write the wave equation as

$$0 = \nabla^2 \mathbf{H} + \boldsymbol{\nabla} \epsilon \times \frac{1}{\epsilon} (\boldsymbol{\nabla} \times \mathbf{H}) - \mu_0 \epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} \ , \tag{58}$$

and since we assume the magnetic field has a harmonic form,

$$\mathbf{H}(\mathbf{x}, t) = \mathbf{H}_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \ , \tag{59}$$

the second time derivative contributes a factor of $(-i\omega)^2$, so

$$0 = \nabla^2 \mathbf{H} + \mu_0 \epsilon \omega^2 \mathbf{H} + \frac{1}{\epsilon} (\boldsymbol{\nabla} \epsilon) \times (\boldsymbol{\nabla} \times \mathbf{H}) \ . \tag{60}$$

Using the relative permittivity defined above, we see the equation has the form

$$0 = \nabla^2 \mathbf{H} + \mu_0 \epsilon_0 \epsilon_r(\mathbf{x}) \omega^2 \mathbf{H} + \frac{1}{\epsilon_r(\mathbf{x})} (\boldsymbol{\nabla} \epsilon_r(\mathbf{x})) \times (\boldsymbol{\nabla} \times \mathbf{H}) \ , \tag{61}$$

and since $\mu_0 \epsilon_0 = c^{-2}$, we obtain the result

$$0 = \nabla^2 \mathbf{H} + \epsilon_r(\mathbf{x}) \frac{\omega^2}{c^2} \mathbf{H} + \frac{1}{\epsilon_r(\mathbf{x})} (\boldsymbol{\nabla} \epsilon_r(\mathbf{x})) \times (\boldsymbol{\nabla} \times \mathbf{H}) \ . \tag{62}$$

5 Surface Magnetic Field Wave at a Dielectric Boundary.

Show that a "surface" H wave can be propagated along a plane boundary between two media whose dielectric permittivities ϵ_1 and ϵ_2 are of opposite sign. The wave is damped in both media. Determine the relation between the frequency and wave number.

Let the boundary between the two dielectrics be the $z = 0$ plane, with dielectric constants ϵ_1 for $z > 0$ and ϵ_2 for $z < 0$, which have opposite signs. Let the electromagnetic wave propagate in the y direction, and define a x direction to complete the right-handed Cartesian coordinate system. We have previously shown that complex wave vectors lead to damping in the direction of the imaginary component, which motivates us to define the complex wave vectors

$$\mathbf{k}_1 = k\hat{\mathbf{y}} + i\kappa_1\hat{\mathbf{z}} \tag{63}$$

$$\mathbf{k}_2 = k\hat{\mathbf{y}} - i\kappa_2\hat{\mathbf{z}} \ , \tag{64}$$

with $\{k, \kappa_1, \kappa_2\} \geq 0$, which describe the propagation of the wave in dielectric one ($z > 0$) and two ($z < 0$), respectively. Note we must define k_2 with a negative sign so when $z < 0$, the wave exhibits exponential decay, not growth. We may then assume harmonic forms for the electric field in each region

$$\mathbf{E}_1(\mathbf{r}, t) = E_1\hat{\mathbf{z}}e^{i(\mathbf{k}_1\cdot\mathbf{r}-\omega t)} = E_1\hat{\mathbf{z}}e^{-\kappa_1 z}e^{i(ky-\omega t)} \tag{65}$$

$$\mathbf{E}_2(\mathbf{r}, t) = E_2\hat{\mathbf{z}}e^{i(\mathbf{k}_2\cdot\mathbf{r}-\omega t)} = E_2\hat{\mathbf{z}}e^{\kappa_2 z}e^{i(ky-\omega t)} \ , \tag{66}$$

where we have neglected any phase difference, and we have $\mathbf{E}_i$ is perpendicular to the direction of propagation. We may insert these into the wave equation

$$\nabla^2\mathbf{E}_i = \frac{\epsilon_i}{c^2}\frac{\partial^2\mathbf{E}_i}{\partial t^2} \quad \Rightarrow \quad \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\mathbf{E}_i = \epsilon_i\frac{(-i\omega)^2}{c^2}\mathbf{E}_i \tag{67}$$

where we have assumed $\mu \sim \mu_0$ with $i \in \{1, 2\}$. Note we have written ϵ_i as the *relative* permittivity. Performing the spatial derivatives yields

$$\left((ik)^2 + (\mp\kappa_i)^2\right)\mathbf{E}_i = \epsilon_i\frac{(-i\omega)^2}{c^2}\mathbf{E}_i \quad \Rightarrow \quad k^2 - \kappa_i^2 = \epsilon_i\frac{\omega^2}{c^2} \ , \tag{68}$$

and we can identify the dispersion relation of free space and define $k_0 = \omega/c$. The condition at the boundary $z = 0$ on the normal components of the electric displacement field gives

$$\hat{\mathbf{n}} \cdot (\epsilon_0\epsilon_2\mathbf{E}_2 - \epsilon_0\epsilon_1\mathbf{E}_1)\Big|_{z=0} = 0 \ , \tag{69}$$

because there is no free surface charge on the boundary. Since the electric field is completely in the normal direction, we have $\epsilon_2 E_2 = \epsilon_1 E_1$ because the complex exponentials cancel, and the decaying terms are unity at $z = 0$. The magnetic field is given[2] by the cross-product of the wave vector and electric field:

$$\mathbf{H}_i = \sqrt{\frac{\epsilon_i\epsilon_0}{\mu_0}}\frac{\mathbf{k}_i \times \mathbf{E}_i}{|\mathbf{k}_i|} = \sqrt{\frac{\epsilon_0}{\mu_0}}\sqrt{\epsilon_i}\frac{(k\hat{\mathbf{y}} \pm i\kappa_i\hat{\mathbf{z}}) \times \hat{\mathbf{z}}}{\sqrt{k^2 + \kappa_i^2}}E_i e^{\mp\kappa_i z}e^{i(ky-\omega t)} \ , \tag{70}$$

[2] Jackson, Classical Electrodynamics, 3 ed. Equation 7.18.

note the imaginary term in the wave vector vanishes after carrying out the cross product:

$$\mathbf{H}_i = \sqrt{\frac{\epsilon_0}{\mu_0}} k\sqrt{\epsilon_i} \frac{E_i e^{\mp\kappa_i z} e^{i(ky-\omega t)}}{\sqrt{k^2+\kappa_i^2}} \hat{\mathbf{x}} \ . \tag{71}$$

Now, we may apply the condition on the tangential components of the magnetic field at the boundary $z = 0$:

$$\hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1)\Big|_{z=0} = 0 \ , \tag{72}$$

and since the magnetic field at the boundary has only a tangential component, this reduces to

$$H_2(z=0) = H_1(z=0) \quad \Rightarrow \quad \sqrt{\frac{\epsilon_0}{\mu_0}} k\sqrt{\epsilon_1} \frac{E_1 e^{-\kappa_1(0)} e^{i(ky-\omega t)}}{\sqrt{k^2+\kappa_1^2}} = \sqrt{\frac{\epsilon_0}{\mu_0}} k\sqrt{\epsilon_2} \frac{E_2 e^{\kappa_2(0)} e^{i(ky-\omega t)}}{\sqrt{k^2+\kappa_2^2}} \ , \tag{73}$$

and after canceling the appropriate factors

$$\sqrt{\epsilon_1}\frac{E_1}{\sqrt{k^2+\kappa_1^2}} = \sqrt{\epsilon_2}\frac{E_2}{\sqrt{k^2+\kappa_2^2}} \quad \Rightarrow \quad \frac{E_1}{E_2} = \sqrt{\frac{\epsilon_2}{\epsilon_1}\frac{k^2+\kappa_1^2}{k^2+\kappa_2^2}} \ . \tag{74}$$

Let us collect our results from the dispersion relation (Equation 68), and the boundary conditions (Equations 69 and 74):

$$\kappa_1^2 = k^2 - \epsilon_1 k_0^2 \tag{75}$$
$$\kappa_2^2 = k^2 - \epsilon_2 k_0^2 \tag{76}$$
$$\frac{E_1}{E_2} = \frac{\epsilon_2}{\epsilon_1} \tag{77}$$
$$\frac{E_1^2}{E_2^2} = \frac{\epsilon_2}{\epsilon_1}\frac{k^2+\kappa_1^2}{k^2+\kappa_2^2} \ . \tag{78}$$

If we insert the thrid equation into the last, we obtain

$$\frac{\epsilon_2}{\epsilon_1} = \frac{k^2+\kappa_1^2}{k^2+\kappa_2^2} \ , \tag{79}$$

and then inserting the first two yields

$$\frac{\epsilon_2}{\epsilon_1} = \frac{2k^2-\epsilon_1 k_0^2}{2k^2-\epsilon_2 k_0^2} \quad \Rightarrow \quad (2k^2-\epsilon_2 k_0^2)(\epsilon_2) = (2k^2-\epsilon_1 k_0^2)(\epsilon_1) \ , \tag{80}$$

which has the solution

$$k^2 = \frac{k_0^2}{2}\left(\frac{\epsilon_2^2-\epsilon_1^2}{\epsilon_2-\epsilon_1}\right) \quad \Rightarrow \quad k = \frac{\omega}{c}\sqrt{\frac{1}{2}\frac{\epsilon_2^2-\epsilon_1^2}{\epsilon_2-\epsilon_1}} \ , \tag{81}$$

yielding the dispersion relation.