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1 Hertz Vector - Partial Field Expansion.

In class, we derived the expansion for the fields corresponding to the $n = 0$ partial field expansion of the Hertz vector, $\mathbf{\Pi}^0_\omega$. Suppose the current distribution is such that only the magnetic dipole moment $\mathbf{m}$ is important.

1.1 Field Expansion for $n = 1$.

Using the expressions derived in class for $\mathbf{\Pi}^1_\omega$, show that the electric and magnetic fields are given by

$$E^1_{\varphi\omega} = \frac{k^2}{4\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\left(\frac{1}{r} + \frac{i}{kr^2}\right)e^{ikr} m \sin\theta \ , \tag{1}$$

and

$$H^1_{r\omega} = \frac{1}{2\pi}\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr} m\cos\theta \tag{2}$$

$$H^1_{\theta\omega} = \frac{1}{4\pi}\left(\frac{1}{r^3} - \frac{ik}{r^2} - \frac{k^2}{r}\right)e^{ikr} m\sin\theta \ . \tag{3}$$

In class, we found the expression for the $n = 1$ mode of the Hertz vector to be

$$\mathbf{\Pi}^1_\omega = \frac{ik}{4\pi\epsilon_0}\left(\frac{1}{r} + \frac{i}{kr^2}\right)e^{ikr}\frac{\mathbf{m}\times\mathbf{r}}{ri\omega} \ , \tag{4}$$

using a spherical coordinate system, and aligning the z axis with the magnetic moment such that $\mathbf{m} = m\hat{\mathbf{z}}$, simplifies this to

$$\mathbf{\Pi}^1_\omega = \frac{mk}{4\pi\epsilon_0\omega}\left(\frac{1}{r} + \frac{i}{kr^2}\right)e^{ikr}\frac{\mathbf{z}\times\mathbf{r}}{r} \ . \tag{5}$$

The vector product is given by

$$\mathbf{z}\times\mathbf{r} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & 1 \\ r\sin\theta\cos\phi & r\sin\theta\sin\phi & r\cos\theta \end{vmatrix} = \hat{\mathbf{x}}(-r\sin\theta\sin\phi) + \hat{\mathbf{y}}(r\sin\theta\cos\phi) \tag{6}$$

$$= r\sin\theta\left(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}\right) = r\sin\theta\hat{\boldsymbol{\phi}} \ , \tag{7}$$

so the Hertz vector for this mode is

$$\mathbf{\Pi}^1_\omega = \Pi^1_{\phi\omega}\hat{\boldsymbol{\phi}} = \frac{mk}{4\pi\epsilon_0\omega}\left(\frac{1}{r} + \frac{i}{kr^2}\right)e^{ikr}\sin\theta\hat{\boldsymbol{\phi}} \ . \tag{8}$$

This mode of the electric field is given by

$$\mathbf{E}^1_\omega = -\frac{1}{c^2}\frac{\partial^2\mathbf{\Pi}^1_\omega}{\partial t^2} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{\Pi}^1_\omega) \ , \tag{9}$$

let's first examine the second term:

$$\boldsymbol{\nabla}\cdot\mathbf{\Pi}^1_\omega = \boldsymbol{\nabla}\cdot\Pi^1_{\phi\omega}\hat{\boldsymbol{\phi}} = \boldsymbol{\nabla}\cdot\Pi^1_{\phi\omega}\hat{\boldsymbol{\phi}} \propto \frac{\partial}{\partial\phi}\Pi^1_{\phi\omega} = 0 \ , \tag{10}$$

because $\Pi^1_{\phi\omega}$ has no dependence on the azimuthal coordinate. If we assume a harmonic time dependence for the fields, Equation 9 can be written

$$\mathbf{E}^1_\omega = -\frac{1}{c^2}(-\omega^2)\mathbf{\Pi}^1_\omega = \frac{\omega^2}{c^2}\frac{mk}{4\pi\epsilon_0\omega}\left(\frac{1}{r}+\frac{i}{kr^2}\right)e^{ikr}\sin\theta\hat{\boldsymbol{\phi}} \ , \tag{11}$$

then using the dispersion relation $\omega = kc$, this becomes

$$\mathbf{E}^1_\omega = \frac{mk^2}{4\pi\epsilon_0 c}\left(\frac{1}{r}+\frac{i}{kr^2}\right)e^{ikr}\sin\theta\hat{\boldsymbol{\phi}} = \frac{mk^2\sqrt{\mu_0\epsilon_0}}{4\pi\epsilon_0}\left(\frac{1}{r}+\frac{i}{kr^2}\right)e^{ikr}\sin\theta\hat{\boldsymbol{\phi}} \ , \tag{12}$$

noting the speed of light is $c = 1/\sqrt{\mu_0\epsilon_0}$. Final simplification gives the expected result

$$E^1_\omega = \frac{k^2}{4\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\left(\frac{1}{r}+\frac{i}{kr^2}\right)e^{ikr}m\sin\theta \ . \tag{13}$$

The magnetic field is related to the electric field by

$$\mathbf{B}^1_\omega = \frac{1}{c^2}\boldsymbol{\nabla}\times\frac{\partial\mathbf{\Pi}^1_\omega}{\partial t} = -\frac{i\omega}{c^2}\boldsymbol{\nabla}\times\mathbf{\Pi}^1_\omega = -\frac{i\omega}{c^2}\boldsymbol{\nabla}\times\Pi^1_{\phi\omega}\hat{\boldsymbol{\phi}} \ . \tag{14}$$

We have that $\Pi^1_{\theta\omega} = \Pi^1_{r\omega} = 0$, so carrying out the curl yields

$$\mathbf{B}^1_\omega = -\frac{i\omega}{c^2}\left\{\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\Pi^1_{\phi\omega}\sin\theta)\right)\hat{\mathbf{r}} - \left(\frac{1}{r}\frac{\partial}{\partial r}r\Pi^1_{\phi\omega}\right)\hat{\boldsymbol{\theta}}\right\} \ . \tag{15}$$

Let us examine the radial component:

$$B^1_{r\omega} = -\frac{i\omega}{c^2}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\Pi^1_{\phi\omega}\sin\theta)\right) = -\frac{i\omega}{c^2}\left(\frac{1}{r\sin\theta}\frac{mk}{4\pi\epsilon_0\omega}\left(\frac{1}{r}+\frac{i}{kr^2}\right)e^{ikr}\frac{\partial}{\partial\theta}\sin^2\theta\right) \tag{16}$$

$$= -\frac{i}{c^2}\frac{mk}{4\pi\epsilon_0}\frac{1}{r\sin\theta}\left(\frac{1}{r}+\frac{i}{kr^2}\right)e^{ikr}\left(2\sin\theta\cos\theta\right) = \frac{1}{2\pi\epsilon_0 c^2}\left(\frac{-ik}{r^2}+\frac{1}{r^3}\right)e^{ikr}m\cos\theta \ , \tag{17}$$

which simplifies to

$$B^1_{r\omega} = \frac{\mu_0}{2\pi}\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}m\cos\theta \quad \Rightarrow \quad H^1_{r\omega} = \frac{1}{2\pi}\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}m\cos\theta \ . \tag{18}$$

The polar coordinate of the magnetic field is

$$B^1_{\theta\omega} = \frac{i\omega}{c^2}\left(\frac{1}{r}\frac{\partial}{\partial r}r\Pi^1_{\phi\omega}\right) = \frac{i\omega}{c^2}\frac{k}{4\pi\epsilon_0\omega}m\sin\theta\left(\frac{1}{r}\frac{\partial}{\partial r}\left(1+\frac{i}{kr}\right)e^{ikr}\right) \tag{19}$$

$$= \frac{i\omega}{c^2}\frac{k}{4\pi\epsilon_0\omega}m\sin\theta\frac{1}{r}\left(ike^{ikr}+\frac{i}{k}\left(\frac{ike^{ikr}}{r}-\frac{e^{ikr}}{r^2}\right)\right) \tag{20}$$

$$= \frac{1}{4\pi\epsilon_0 c^2}\left(-\frac{k^2}{r}-\frac{ik}{r^2}+\frac{1}{r^3}\right)e^{ikr}m\sin\theta \ , \tag{21}$$

using the derivative:

$$\frac{\partial}{\partial r}\frac{e^{ikr}}{r} = \frac{1}{r}ike^{ikr} - \frac{1}{r^2}e^{ikr} \ . \tag{22}$$

This simplifies to

$$B^1_{\theta\omega} = \frac{\mu_0}{4\pi}\left(\frac{1}{r^3}-\frac{ik}{r^2}-\frac{k^2}{r}\right)e^{ikr}m\sin\theta \quad \Rightarrow \quad H^1_{\theta\omega} = \frac{1}{4\pi}\left(\frac{1}{r^3}-\frac{ik}{r^2}-\frac{k^2}{r}\right)e^{ikr}m\sin\theta \ . \tag{23}$$

1.2 Far-field Time-averaged Flux.

Show that the time-averaged flux of far field radiation from this magnetic dipole is given by

$$-\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{k^4}{12\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}m^2 \ . \tag{24}$$

The Poynting vector for the $n = 1$ partial wave is given by

$$\mathbf{S}^1_\omega = \frac{1}{2}\mathbf{E}^1_\omega \times \mathbf{H}^{1*}_\omega = \frac{1}{2}\left(-E^1_{\varphi\omega}H^{1*}_{\theta\omega}\hat{\mathbf{r}} + E^1_{\varphi\omega}H^{1*}_{r\omega}\hat{\boldsymbol{\theta}}\right) \ , \tag{25}$$

the flux of this vector (the power) through a sphere of radius r is

$$P = \int \mathbf{S}^1_\omega \cdot \mathrm{d}\mathbf{A} = \int \mathbf{S}^1_\omega \cdot \hat{\mathbf{r}}(2\pi r^2 \sin\theta \mathrm{d}\theta) = 2\pi r^2 \int_0^\pi S^1_{r\omega}\sin\theta \mathrm{d}\theta \ , \tag{26}$$

where

$$S^1_{r\omega} \equiv -\frac{1}{2}E^1_{\varphi\omega}H^{1*}_{\theta\omega} \ . \tag{27}$$

If we are only concerned with the far field radiation, we can approximate the fields by retaining only the terms with the highest power of r:

$$E^1_{\varphi\omega} \sim \frac{k^2}{4\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{e^{ikr}}{r}m\sin\theta \tag{28}$$

$$H^{1*}_{\theta\omega} \sim -\frac{1}{4\pi}\frac{k^2 e^{-ikr}}{r}m\sin\theta \ , \tag{29}$$

and so the negative of their product is

$$S^1_{r\omega} = \frac{1}{2}\frac{k^4 m^2 \sin^2\theta}{16\pi^2 r^2}\sqrt{\frac{\mu_0}{\epsilon_0}} \ . \tag{30}$$

The far field power radiated from the magnetic dipole is then

$$-\frac{\mathrm{d}W}{\mathrm{d}t} = P = 2\pi r^2 \frac{1}{2}\frac{k^4\langle m^2\rangle}{16\pi^2 r^2}\sqrt{\frac{\mu_0}{\epsilon_0}}\int_0^\pi \sin^3\theta \mathrm{d}\theta = \frac{k^4 m^2}{16\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\left(\frac{4}{3}\right) = \frac{k^4\langle m^2\rangle}{12\pi}\sqrt{\frac{\mu_0}{\epsilon_0}} \ , \tag{31}$$

where $\langle m^2\rangle$ is the time-averaged magnitude of the magnetic moment.

1.3 Power Radiated from Conducting Loop.

Consider now a conducting loop of radius r_0 with an oscillating current of amplitude I_0 flowing through it. Such a loop has negligible electric dipole moment. Find an expression for the power radiated at large distances from the resulting magnetic dipole in terms of I_0, r_0 and the wavelength λ.

The magnitude of the magnetic dipole moment of a loop of radius r_0 with current $I_0 \cos\omega t$ is given by

$$m = \pi I_0 \cos(\omega t) r_0^2 \quad \Rightarrow \quad \langle m^2\rangle = \pi^2 r_0^4 I_0^2 \langle\cos^2\omega t\rangle = \frac{\pi^2}{2}r_0^4 I_0^2 \ . \tag{32}$$

The wavenumber k is related to the wavelength by $k = 2\pi/\lambda$, and so the far field power is

$$P = \frac{\left(\frac{2\pi}{\lambda}\right)^4 \frac{1}{2}\left(\pi I_0 r_0^2\right)^2}{12\pi}\sqrt{\frac{\mu_0}{\epsilon_0}} = \frac{1}{2}\frac{4\pi}{3} I_0^2 \left(\frac{\pi r_0}{\lambda}\right)^4 \sqrt{\frac{\mu_0}{\epsilon_0}} \ , \tag{33}$$

so the average power is

$$P = \frac{2\pi}{3} I_0^2 \left(\frac{\pi r_0}{\lambda}\right)^4 \sqrt{\frac{\mu_0}{\epsilon_0}} \ , \tag{34}$$

1.4 Power Through Resistor.

If the oscillating current of maximum amplitude I_0 were to flow through a resistance $\mathcal{R}$, calculate the value this resistance must have in order to dissipate the same amount of power that you found in part 1.3.

The power through a resistor is

$$P = I_0^2 \mathcal{R} \quad \Rightarrow \quad \langle P \rangle = \langle I_0^2 \rangle \mathcal{R} = \frac{1}{2} I_0^2 \mathcal{R} \ , \tag{35}$$

so to dissipate the power found previously the value of $\mathcal{R}$ must be

$$\mathcal{R} = \frac{2P}{I_0^2} = \frac{4\pi}{3}\left(\frac{\pi r_0}{\lambda}\right)^4 \sqrt{\frac{\mu_0}{\epsilon_0}} \ . \tag{36}$$

2 Alternative Solutions to the Wave Equation.

Let ψ be a solution of the wave equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \nabla^2\psi + k^2\psi = 0 \ . \tag{37}$$

2.1 Solution: Derivatives of ψ.

Show that the derivatives of ψ are also a solution to the wave equation.

Let us define the derivatives of the solution as

$$\boldsymbol{\phi} = \boldsymbol{\nabla}\psi \ , \tag{38}$$

which if we insert into the wave equation yields

$$\nabla^2\boldsymbol{\phi} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{\phi}}{\partial t^2} = \nabla^2(\boldsymbol{\nabla}\psi) - \frac{1}{c^2}\frac{\partial^2\boldsymbol{\nabla}\psi}{\partial t^2} \ , \tag{39}$$

which can be separated into components

$$\nabla^2\boldsymbol{\phi} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{\phi}}{\partial t^2} = \sum_{i=1}^{3}\left[\nabla^2\left(\frac{\partial\psi}{\partial x_i}\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{\partial\psi}{\partial x_i}\right]\hat{\mathbf{x}}_i = \sum_{i=1}^{3}\left[\frac{\partial}{\partial x_i}\left(\nabla^2\psi\right) - \frac{1}{c^2}\frac{\partial}{\partial x_i}\frac{\partial^2\psi}{\partial t^2}\right]\hat{\mathbf{x}}_i \ . \tag{40}$$

The time derivative and Laplacian both commute with the derivative with respect to a single coordinate. If we pull the derivative outside the sum

$$\nabla^2\boldsymbol{\phi} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{\phi}}{\partial t^2} = \frac{\partial}{\partial x_i}\sum_{i=1}^{3}\left[\left(\nabla^2\psi\right) - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}\right]\hat{\mathbf{x}}_i \ , \tag{41}$$

we see the summand is the wave equation for ψ which is satisfied. Therefore,

$$\nabla^2\boldsymbol{\phi} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{\phi}}{\partial t^2} = 0 \ , \tag{42}$$

and so the derivatives of ψ, $\boldsymbol{\phi} = \boldsymbol{\nabla}\psi$ satisfy the wave equation. Note the derivative with respect to time commutes in a similar way with both the Laplacian and second time derivative. This can also be shown using the form of the wave equation with the time derivative carried out:

$$0 = \nabla^2(\boldsymbol{\nabla}\psi) + k^2(\boldsymbol{\nabla}\psi) \tag{43}$$

$$= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\psi) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\psi) + k^2(\boldsymbol{\nabla}\psi) \ , \tag{44}$$

using vector identity (1)[1]. Noting that $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\psi = 0$, yields the condition which is satisfied if $\boldsymbol{\nabla}\psi$ is a solution to the wave equation:

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\psi) = -k^2(\boldsymbol{\nabla}\psi) \ . \tag{45}$$

Let us examine the left-hand side:

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\psi) = \boldsymbol{\nabla}(\nabla^2\psi) \ , \tag{46}$$

using the definition of the Laplacian. The wave equation for ψ tells us $\nabla^2\psi = -k^2\psi$, so

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\psi) = \boldsymbol{\nabla}(-k^2\psi) \ , \tag{47}$$

and we may factor out the $-k^2$ because it is a scalar, and see that the condition in Equation 45 is satisfied and therefore $\boldsymbol{\nabla}\psi$ is a solution to the wave equation.

[1] See section 7, page 15.

2.2 Solution: $\mathbf{r} \times \boldsymbol{\nabla}\psi$.

Show that $\mathbf{r} \times \boldsymbol{\nabla}\psi$ is a solution to the wave equation.

Let us define

$$\boldsymbol{\phi} = \mathbf{r} \times \boldsymbol{\nabla}\psi \ , \tag{48}$$

which if we insert into the wave equation yields

$$\nabla^2 \boldsymbol{\phi} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{\phi}}{\partial t^2} = \nabla^2(\mathbf{r} \times \boldsymbol{\nabla}\psi) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\mathbf{r} \times \boldsymbol{\nabla}\psi) = \nabla^2(\mathbf{r} \times \boldsymbol{\nabla}\psi) + k^2(\mathbf{r} \times \boldsymbol{\nabla}\psi) = 0 \ . \tag{49}$$

Let us expand the first term using vector identity (1):

$$\nabla^2(\mathbf{r} \times \boldsymbol{\nabla}\psi) = \boldsymbol{\nabla}\left\{\boldsymbol{\nabla} \cdot (\mathbf{r} \times \boldsymbol{\nabla}\psi)\right\} - \boldsymbol{\nabla} \times \left\{\boldsymbol{\nabla} \times (\mathbf{r} \times \boldsymbol{\nabla}\psi)\right\} \ , \tag{50}$$

which we can also examine term-by-term. The first of which is, by vector identity (3):

$$\boldsymbol{\nabla} \cdot (\mathbf{r} \times \boldsymbol{\nabla}\psi) = \boldsymbol{\nabla}\psi \cdot (\boldsymbol{\nabla} \times \mathbf{r}) - \mathbf{r} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi) = \boldsymbol{\nabla}\psi \cdot (\boldsymbol{\nabla} \times \mathbf{r}) \ , \tag{51}$$

using $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi = 0$. If we permute the order of the vectors in the remaining term, we see

$$\boldsymbol{\nabla}\psi \cdot (\boldsymbol{\nabla} \times \mathbf{r}) = \mathbf{r} \cdot (\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}) = -\mathbf{r} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi) = 0 \ , \tag{52}$$

and thus the first term on the right-hand side of Equation 50 is zero. We therefore have, using vector identity (2):

$$\begin{aligned} \nabla^2(\mathbf{r} \times \boldsymbol{\nabla}\psi) &= -\boldsymbol{\nabla} \times \left\{\boldsymbol{\nabla} \times (\mathbf{r} \times \boldsymbol{\nabla}\psi)\right\} &(53)\\ &= -\boldsymbol{\nabla} \times \left\{\mathbf{r}(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\psi) - \boldsymbol{\nabla}\psi(\boldsymbol{\nabla} \cdot \mathbf{r}) + (\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla})\mathbf{r} - (\mathbf{r} \cdot \boldsymbol{\nabla})\boldsymbol{\nabla}\psi\right\} \ . &(54) \end{aligned}$$

The vector $\mathbf{r}$ is the coordinate of a point, and as such the vector relations $\boldsymbol{\nabla} \cdot \mathbf{r} = 3$ and $\boldsymbol{\nabla} \times \mathbf{r} = 0$ hold (the second identity would have made Equation 51 vanish without much work). The expression above then simplifies to

$$\begin{aligned} \nabla^2(\mathbf{r} \times \boldsymbol{\nabla}\psi) &= -\boldsymbol{\nabla} \times \left\{\mathbf{r}(\nabla^2\psi) - 3\boldsymbol{\nabla}\psi + (\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla})\mathbf{r} - (\mathbf{r} \cdot \boldsymbol{\nabla})\boldsymbol{\nabla}\psi\right\} &(55)\\ &= -\boldsymbol{\nabla} \times \mathbf{r}(-k^2\psi) - 3\boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi + \boldsymbol{\nabla} \times (\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla})\mathbf{r} - \boldsymbol{\nabla} \times (\mathbf{r} \cdot \boldsymbol{\nabla})\boldsymbol{\nabla}\psi &(56)\\ &= \left\{k^2\boldsymbol{\nabla} \times (\mathbf{r}\psi)\right\} + \left\{\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla})\mathbf{r}\right\} - \left\{\boldsymbol{\nabla} \times (\mathbf{r} \cdot \boldsymbol{\nabla})\boldsymbol{\nabla}\psi\right\} \ . &(57) \end{aligned}$$

Let us examine this term-by-term, the first of which is

$$k^2\boldsymbol{\nabla} \times (\mathbf{r}\psi) = k^2\left(\boldsymbol{\nabla}\psi \times \mathbf{r} + \psi\boldsymbol{\nabla} \times \mathbf{r}\right) = -k^2(\mathbf{r} \times \boldsymbol{\nabla}\psi) \ , \tag{58}$$

which is the desired result. If the other two terms are both zero, or cancel each other identically, then we have

$$\nabla^2(\mathbf{r} \times \boldsymbol{\nabla}\psi) = -k^2(\mathbf{r} \times \boldsymbol{\nabla}\psi) \ , \tag{59}$$

which satisfies Equation 49, and thus $\mathbf{r} \times \boldsymbol{\nabla}\psi$ is a solution to the wave equation. Let us return to the remaining two terms. In the first remaining term, the vector having its curl taken can be expanded in Cartesian coordinates:

$$(\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla})\mathbf{r} = \left((\boldsymbol{\nabla}\psi)_x \frac{\partial}{\partial x} + (\boldsymbol{\nabla}\psi)_y \frac{\partial}{\partial y} + (\boldsymbol{\nabla}\psi)_z \frac{\partial}{\partial z} \right) \mathbf{r} \tag{60}$$

$$= \left((\boldsymbol{\nabla}\psi)_x \frac{\partial x}{\partial x} + (\boldsymbol{\nabla}\psi)_y \frac{\partial x}{\partial y} + (\boldsymbol{\nabla}\psi)_z \frac{\partial x}{\partial z} \right) \hat{\mathbf{x}} \tag{61}$$

$$+ \left((\boldsymbol{\nabla}\psi)_x \frac{\partial y}{\partial x} + (\boldsymbol{\nabla}\psi)_y \frac{\partial y}{\partial y} + (\boldsymbol{\nabla}\psi)_z \frac{\partial y}{\partial z} \right) \hat{\mathbf{y}} \tag{62}$$

$$+ \left((\boldsymbol{\nabla}\psi)_x \frac{\partial z}{\partial x} + (\boldsymbol{\nabla}\psi)_y \frac{\partial z}{\partial y} + (\boldsymbol{\nabla}\psi)_z \frac{\partial z}{\partial z} \right) \hat{\mathbf{z}} \tag{63}$$

$$= (\boldsymbol{\nabla}\psi)_x \hat{\mathbf{x}} + (\boldsymbol{\nabla}\psi)_y \hat{\mathbf{y}} + (\boldsymbol{\nabla}\psi)_z \hat{\mathbf{z}} = \boldsymbol{\nabla}\psi \ , \tag{64}$$

and as such, is zero when we take it's curl. The final remaining term can be evaluated in Cartesian coordinates:

$$\boldsymbol{\nabla} \times (\mathbf{r} \cdot \boldsymbol{\nabla})\boldsymbol{\nabla}\psi = \boldsymbol{\nabla} \times \left(\sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i} \right) \boldsymbol{\nabla}\psi = \sum_{i=1}^{3} \left(\boldsymbol{\nabla} \times x_i \frac{\partial \boldsymbol{\nabla}\psi}{\partial x_i} \right) = \sum_{i=1}^{3} \left(\boldsymbol{\nabla} \times x_i \boldsymbol{\nabla} \frac{\partial \psi}{\partial x_i} \right) \ . \tag{65}$$

Let us now define ψ'_i as the derivative with respect to the ith coordinate, so the above is equivalent to

$$\sum_{i=1}^{3} \left(\boldsymbol{\nabla} \times x_i \boldsymbol{\nabla} \psi'_i \right) = \sum_{i=1}^{3} \left(\boldsymbol{\nabla} \times x_i \left\{ \psi'_{ix} \hat{\mathbf{x}} + \psi'_{iy} \hat{\mathbf{y}} + \psi'_{iz} \hat{\mathbf{z}} \right\} \right) \ . \tag{66}$$

Consider the x component ($i = 1$)

$$\begin{aligned} &\boldsymbol{\nabla} \times x \left\{ \psi'_{xx} \hat{\mathbf{x}} + \psi'_{xy} \hat{\mathbf{y}} + \psi'_{xz} \hat{\mathbf{z}} \right\} \\ &\quad = \hat{\mathbf{x}} \left(x\psi'_{xzy} - x\psi'_{xyz} \right) + \hat{\mathbf{y}} \left(x\psi'_{xxz} - \left[\psi'_{xz} + x\psi'_{xzx} \right] \right) + \hat{\mathbf{z}} \left(\left[\psi'_{xy} + x\psi'_{xyx} \right] - x\psi'_{xxy} \right) \ , \end{aligned} \tag{67}$$

which simplifies to

$$\boldsymbol{\nabla} \times x \boldsymbol{\nabla} \psi'_x = -\psi'_{xz} \hat{\mathbf{y}} + \psi'_{xy} \hat{\mathbf{z}} \ , \tag{68}$$

and through permutation, we find similarly

$$\boldsymbol{\nabla} \times y \boldsymbol{\nabla} \psi'_y = -\psi'_{yx} \hat{\mathbf{z}} + \psi'_{yz} \hat{\mathbf{x}} \tag{69}$$

$$\boldsymbol{\nabla} \times z \boldsymbol{\nabla} \psi'_z = -\psi'_{zy} \hat{\mathbf{x}} + \psi'_{zx} \hat{\mathbf{y}} \ . \tag{70}$$

Summing these results gives

$$\boldsymbol{\nabla} \times (\mathbf{r} \cdot \boldsymbol{\nabla})\boldsymbol{\nabla}\psi = \sum_{i=1}^{3} \left(\boldsymbol{\nabla} \times x_i \boldsymbol{\nabla} \psi'_i \right) = 0 \ , \tag{71}$$

because each component cancels exactly. We now return to Equation 57, and see that the only the first term is nonzero, which was shown to give the desired result. Therefore $\mathbf{r} \times \boldsymbol{\nabla}\psi$ is a solution to the wave equation.

3 Vector Solutions to the Wave Equation.

As we have see, the time dependent electric and magnetic fields are solutions to the wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{72}$$

$$\nabla^2 \mathbf{H} - \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \ . \tag{73}$$

For scalar functions satisfying the wave equation, we know in principle how to obtain the solutions, but for vector functions, things are not so straightforward. In Cartesian coordinates, we can split up each vector field into its three component scalar fields, and solve for them independently, but it is not clear how to do this for a general curvilinear coordinate system, where the unit vectors themselves may be functions of position. This is the case for spherical coordinates, which we shall look into in this problem.

One way to get around this problem in the case when the divergence of the vector field vanishes (as is the case for source-free regions) is to break up the vector fields into the sum of two partial fields.

3.1 Divergence-less Fields.

From the result of the problem above, we see that if ψ is a scalar function, then $\mathbf{r} \times \boldsymbol{\nabla}\psi$ is a solution to the wave equation. Show that if we express the electric field as $\mathbf{E} = \mathbf{r} \times \boldsymbol{\nabla}\psi$, then the divergence of the resulting electric and magnetic fields vanish. (Conversely, if $\mathbf{H} = \mathbf{r} \times \boldsymbol{\nabla}\psi$, the divergences also vanish.)

The divergence of the electric field, by vector identity (3), is

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \boldsymbol{\nabla} \cdot (\mathbf{r} \times \boldsymbol{\nabla}\psi) = \boldsymbol{\nabla}\psi(\boldsymbol{\nabla} \times \mathbf{r}) - \mathbf{r} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi) \ . \tag{74}$$

Using vector identity (5), and noting that $\mathbf{r}$ is the coordinate of a point so $\boldsymbol{\nabla} \times \mathbf{r} = 0$, we have that

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0 \ . \tag{75}$$

Using Maxwell's equations, we see

$$\boldsymbol{\nabla} \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} = -i\omega\mu\mathbf{H} \ , \tag{76}$$

for harmonic fields. Taking the divergence of both sides yields

$$\boldsymbol{\nabla} \cdot \mathbf{H} = \frac{1}{i\omega\mu}\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) = 0 \ , \tag{77}$$

because the divergence of the curl of a vector is zero. Therefore both the electric and magnetic fields are divergence-less.

3.2 Analog to Waveguides

It can be seen that $\mathbf{E}_1 = \mathbf{r} \times \boldsymbol{\nabla}\psi$ cannot be the full general solution of the wave equation, since this electric field has no component parallel to $\mathbf{r}$. However, a second, linearly independent solution $\mathbf{E}_2$ can be obtained by defining the magnetic field as $\mathbf{H}_2 = \mathbf{r} \times \boldsymbol{\nabla}\psi$, and obtaining the electric field $-i\omega\epsilon\mathbf{E}_2 = \boldsymbol{\nabla} \times \mathbf{H}_2$. Since $\mathbf{E}_1$ has no component in the radial direction, it can be thought of as being the analog of a TE wave, while $\mathbf{H}_2$ can be thought of as being the analog of a TM wave. Consider now the solutions of the wave equations inside a conducting spherical shell of radius r_0.

3.2.1 TE Mode Allowed Frequencies.

Show that for the "TE" modes, the allowed frequencies are given by

$$\omega_{ln} = c\frac{\alpha_{ln}}{r_0} \ , \tag{78}$$

where α_{ln} is the nth zero of the spherical Bessel function $j_l(x)$.

In this analog to waveguides, the transverse component is the radial component in spherical coordinates. Therefore, for a "TE" mode, we have $E_r = 0$ and the condition that $\mathbf{H}_t|_S = 0$. The transverse component of the magnetic field satisfies the Helmholtz equation

$$(\nabla_t^2 + k^2)\psi = 0 \ , \tag{79}$$

where $H_r = \psi e^{i(kz-\omega t)}$ and $k = \omega/c$. The solutions to this differential equation are of the form

$$\psi(r,\theta,\phi) = R_l(kr)Y_l^m(\theta,\phi) \ , \tag{80}$$

where $Y_l^m(\theta,\phi)$ are the spherical harmonics. The radial function is a linear combination of spherical Bessel functions of the first and second kind. The boundary conditions are

$$0 = \psi(r_0,\theta,\phi) = R_l(kr_0)Y_l^m(\theta,\phi) \ , \tag{81}$$

for all θ,ϕ. The volume of interest contains the origin, so we may exclude the spherical Bessel functions of the second kind. The boundary conditions then give

$$j_l(kr_0) = 0 \quad \Rightarrow \quad k = \frac{\alpha_{ln}}{r_0} \ , \tag{82}$$

where α_{ln} is the nth zero of the lth spherical Bessel function. Using $k = \omega/c$ we get the result

$$\omega = c\frac{\alpha_{ln}}{r_0} \ . \tag{83}$$

3.2.2 TM Mode Allowed Frequencies.

Find out the condition that determines the frequencies of the "TM" modes.

Similarly, for a "TM" mode, we have $H_r = 0$ and the condition that $\partial_{\hat{\mathbf{n}}}\mathbf{E}_t|_S = 0$, where $\partial_{\hat{\mathbf{n}}}$ denotes the normal derivative. The solution to the Helmholtz equation ψ has the same functional form, but represents the radial electric field $E_r = \psi e^{i(kz-\omega t)}$. The boundary condition gives

$$0 = \frac{\partial}{\partial r} j_l(kr)\bigg|_{r=r_0} = j_l'(kr_0) \ , \tag{84}$$

so we have

$$k = \frac{\alpha_{ln}'}{r_0} \ , \tag{85}$$

where α_{ln}' is the nth zero of the first derivative of the lth spherical Bessel function. Again, we use$k = \omega/c$ we get the result

$$\omega = c\frac{\alpha_{ln}'}{r_0} \ . \tag{86}$$

4 Covariant Tensor - Change of Coordinate Basis.

A covariant tensor has components $xy, 2y - z^2, xz$ in rectangular coordinates. Find its covariant components in spherical coordinates.

Under a change of coordinates, a covariant tensor transforms as

$$\bar{A}_j = \frac{\partial x^i}{\partial \bar{x}^j} A_i \ , \tag{87}$$

where the A_i are the coefficients of the basis vectors in the source coordinates. Consider a tensor with covariant components A_i, in the basis $x^i \in \{x, y, z\}$, with components $\bar{A}_j$ in spherical coordinates $\bar{x}^j \in \{r, \theta, \phi\}$. The transformed components are thus

$$\bar{A}_j = \frac{\partial x^1}{\partial \bar{x}^j} A_1 + \frac{\partial x^2}{\partial \bar{x}^j} A_2 + \frac{\partial x^3}{\partial \bar{x}^j} A_3 \ , \tag{88}$$

where the untransformed components are

$$A_1 = x^1 x^2 \qquad A_2 = 2x^2 - (x^3)^2 \qquad A_3 = x^1 x^3 \ . \tag{89}$$

In terms of the destination coordinates, the source coordinates are

$$x = x^1 = r \cos\phi \sin\theta \quad \Rightarrow \quad x^1 = \bar{x}^1 \cos\bar{x}^3 \sin\bar{x}^2 \tag{90}$$

$$y = x^2 = r \sin\phi \sin\theta \quad \Rightarrow \quad x^2 = \bar{x}^1 \sin\bar{x}^3 \sin\bar{x}^2 \tag{91}$$

$$z = x^3 = r \cos\theta \quad \Rightarrow \quad x^3 = \bar{x}^1 \cos\bar{x}^2 \ , \tag{92}$$

so the components in the source basis can be expressed

$$A_1 = x^1 x^2 = (r\sin\theta)^2 \cos\phi \sin\phi = (\bar{x}^1)^2 (\sin\bar{x}^2)^2 \cos\bar{x}^3 \sin\bar{x}^3 \tag{93}$$

$$A_2 = 2x^2 - (x^3)^2 = 2(r\cos\phi\sin\theta)^2 - (r\cos\theta)^2 = 2(\bar{x}^1)^2(\cos\bar{x}^3)^2(\sin\bar{x}^2)^2 - (\bar{x}^1)^2(\cos\bar{x}^2)^2 \tag{94}$$

$$A_3 = x^1 x^3 = r^2 \cos\phi \sin\theta \cos\theta = (\bar{x}^1)^2 \cos\bar{x}^3 \sin\bar{x}^2 \cos\bar{x}^2 \ . \tag{95}$$

The transformed coordinates are

$$\bar{A}_1 = \frac{\partial x}{\partial r} A_1 + \frac{\partial y}{\partial r} A_2 + \frac{\partial z}{\partial r} A_3 = \cos\phi\sin\theta A_1 + \sin\phi\sin\theta A_2 + \cos\theta A_3 \tag{96}$$

$$= r^2 \sin\theta \left\{ \cos^2\phi \sin\phi \sin^2\theta + \sin\phi \left(2(\cos\phi\sin\theta)^2 - \cos^2\theta \right) + \cos\phi \cos^2\theta \right\} \tag{97}$$

$$= r^2 \sin\theta \left\{ (\sin\phi + 2\sin\phi) \cos^2\phi \sin^2\theta + (-\sin\phi + \cos\phi) \cos^2\theta \right\} \tag{98}$$

$$\bar{A}_2 = \frac{\partial x}{\partial \theta} A_1 + \frac{\partial y}{\partial \theta} A_2 + \frac{\partial z}{\partial \theta} A_3 = r\cos\phi\cos\theta A_1 + r\sin\phi\cos\theta A_2 - r\sin\theta A_3 \tag{99}$$

$$= r^3 \cos\theta \left\{ \cos^2\phi \sin\phi \sin^2\theta + \sin\phi \left(2(\cos\phi\sin\theta)^2 - \cos^2\theta \right) - \sin\theta\cos\phi\sin\theta \right\} \tag{100}$$

$$= r^3 \cos\theta \left\{ (3\cos\phi\sin\phi - 1) \cos\phi \sin^2\theta - \sin\phi \cos^2\theta \right\} \tag{101}$$

$$\tag{102}$$

$$\bar{A}_3 = \frac{\partial x}{\partial \phi} A_1 + \frac{\partial y}{\partial \phi} A_2 + \frac{\partial z}{\partial \phi} A_3 = -r\sin\phi\sin\theta A_1 + r\cos\phi\sin\theta A_2 \tag{103}$$

$$= r^3 \sin\theta \left\{ -\cos\phi \sin^2\phi \sin^2\theta + \cos\phi \left(2(\cos\phi\sin\theta)^2 - \cos^2\theta \right) \right\} \tag{104}$$

$$= r^3 \sin\theta \cos\phi \left\{ \left(2\cos^2\phi - \sin^2\phi \right) \sin^2\theta - \cos^2\theta \right\} \tag{105}$$

5 Contraction of Tensor.

Prove that the contraction of the tensor T^p_q is a scalar or invariant.

A tensor with rank greater than or equal to rank 2 is contracted by setting two indices equal, and carrying out the implied sum over the repeated index. Thus, the contraction of the tensor T^p_q is T^q_q. This operation reduces the rank of a tensor by two, and therefore the contraction of T^p_q, a rank 2 tensor, results in a rank zero tensor, which is a scalar. Consider this tensor under a coordinate transformation:

$$\bar{T}^i_j = \frac{\partial \bar{x}_i}{\partial x_p} \frac{\partial x_q}{\partial \bar{x}_j} T^p_q \ , \tag{106}$$

which we may contract by setting $i = j$:

$$\bar{T}^i_i = \frac{\partial \bar{x}_i}{\partial x_p} \frac{\partial x_q}{\partial \bar{x}_i} T^p_q = \frac{\partial x_q}{\partial x_p} T^p_q \ . \tag{107}$$

The remaining derivative is a Kronecker delta, so we are left with

$$\bar{T}^i_i = \delta_{qp} T^p_q = T^q_q \ , \tag{108}$$

which is the contraction of the tensor in the original coordinate basis. Therefore the contraction of a rank 2 tensor is invariant under change of coordinates, and is therefore a scalar, confirming the claim above.

6 Metric Tensors.

Determine the metric tensor g_{ij} in cylindrical and spherical coordinates.

A metric tensor is defined as

$$g_{ij} = \sum_{k=1}^{3} \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_j} \ , \tag{109}$$

with $i, j \in \{1, 2, 3\}$. The coordinates x_i are the Cartesian coordinates x, y, z and the q_j are the coordinates of the destination system (*e.g.*, cylindrical, spherical). Note that since scalar multiplication is commutative, we have

$$g_{ij} = g_{ji} \ , \tag{110}$$

so the metric tensor is symmetric. Additionally, the diagonal elements are given by

$$g_{ii} = \sum_{k=1}^{3} \left(\frac{\partial x_k}{\partial q_i} \right)^2 = h_i^2 \ , \tag{111}$$

where h_i are the scale factors.

6.1 Cylindrical Coordinates.

In cylindrical coordinates $q_i = \{\rho, \varphi, z\}$, the Cartesian coordinates can be expressed as

$$x_1 = \rho \sin\varphi \qquad x_2 = \rho \sin\varphi \qquad x_3 = z \ , \tag{112}$$

and the relevant derivatives are

$$\frac{\partial x_1}{\partial q_1} = \cos\varphi \tag{113}$$
$$\frac{\partial x_1}{\partial q_2} = -\rho \sin\varphi \tag{114}$$
$$\frac{\partial x_1}{\partial q_3} = 0 \tag{115}$$
$$\frac{\partial x_2}{\partial q_1} = \sin\varphi \tag{116}$$
$$\frac{\partial x_2}{\partial q_2} = \rho \cos\varphi \tag{117}$$
$$\frac{\partial x_2}{\partial q_3} = 0 \tag{118}$$
$$\frac{\partial x_3}{\partial q_1} = 0 \tag{119}$$
$$\frac{\partial x_3}{\partial q_2} = 0 \tag{120}$$
$$\frac{\partial x_3}{\partial q_3} = 1 \tag{121}$$

From this we see that any off-diagonal term $g_{i3} = g_{3i}$ must be zero:

$$g_{i3} = \frac{\partial x_1}{\partial q_i} \frac{\partial x_1}{\partial q_3} + \frac{\partial x_2}{\partial q_i} \frac{\partial x_2}{\partial q_3} + \frac{\partial x_3}{\partial q_i} \frac{\partial x_3}{\partial q_3} \ , \tag{122}$$

since $i \neq 3$, this reduces to

$$g_{i3} = \frac{\partial x_1}{\partial q_i}(0) + \frac{\partial x_2}{\partial q_i}(0) + (0)\frac{\partial x_3}{\partial q_3} = 0 \ . \tag{123}$$

The other off-diagonal terms are

$$g_{12} = \frac{\partial x_1}{\partial q_1} \frac{\partial x_1}{\partial q_2} + \frac{\partial x_2}{\partial q_1} \frac{\partial x_2}{\partial q_2} = -\rho \sin\varphi \cos\varphi + \rho \cos\varphi \sin\varphi = 0 \ , \tag{124}$$

and thus the metric tensor is diagonal. The diagonal terms are

$$g_{11} = (\cos\varphi)^2 + (\sin\varphi)^2 = 1 \tag{125}$$
$$g_{22} = (-\rho \sin\varphi)^2 + (\rho \cos\varphi)^2 = \rho^2 \tag{126}$$
$$g_{33} = 1 \ , \tag{127}$$

which gives the metric tensor in cylindrical coordinates to be

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ . \tag{128}$$

6.2 Cylindrical Coordinates.

In spherical coordinates $q_i = \{r, \theta, \varphi\}$, the Cartesian coordinates can be expressed as

$$x_1 = r\sin\theta\cos\varphi \qquad x_2 = r\sin\theta\sin\varphi \qquad x_3 = r\cos\theta \ , \tag{129}$$

and the relevant derivatives are

$$\frac{\partial x_1}{\partial q_1} = \sin\theta\cos\varphi \quad (130) \qquad \frac{\partial x_1}{\partial q_2} = r\cos\theta\cos\varphi \quad (131) \qquad \frac{\partial x_1}{\partial q_3} = -r\sin\theta\sin\varphi \quad (132)$$

$$\frac{\partial x_2}{\partial q_1} = \sin\theta\sin\varphi \quad (133) \qquad \frac{\partial x_2}{\partial q_2} = r\cos\theta\sin\varphi \quad (134) \qquad \frac{\partial x_2}{\partial q_3} = r\sin\theta\cos\varphi \quad (135)$$

$$\frac{\partial x_3}{\partial q_1} = \cos\theta \quad (136) \qquad \frac{\partial x_3}{\partial q_2} = -r\sin\theta \quad (137) \qquad \frac{\partial x_3}{\partial q_3} = 0 \quad (138)$$

The diagonal terms of the metric are given by

$$g_{11} = (\sin\theta\cos\varphi)^2 + (\sin\theta\sin\varphi)^2 + (\cos\theta)^2 = 1 \tag{139}$$

$$g_{22} = (r\cos\theta\cos\varphi)^2 + (r\cos\theta\sin\varphi)^2 + (-r\sin\theta)^2 = r^2 \tag{140}$$

$$q_{33} = (-r\sin\theta\sin\varphi)^2 + (r\sin\theta\cos\varphi)^2 + 0 = r^2\sin^2\theta \ , \tag{141}$$

and the off-diagonal elements are

$$g_{12} = (\sin\theta\cos\varphi)(r\cos\theta\cos\varphi) + (\sin\theta\sin\varphi)(r\cos\theta\sin\varphi) + (\cos\theta)(-r\sin\theta) \tag{142}$$

$$= r\sin\theta\cos\theta(\cos^2\varphi + \sin^2\varphi - 1) = 0 \tag{143}$$

$$g_{13} = (\sin\theta\cos\varphi)(-r\sin\theta\sin\varphi) + (\sin\theta\sin\varphi)(r\sin\theta\cos\varphi) + (\cos\theta)(0) \tag{144}$$

$$= r\sin^2\theta(-\cos\varphi\sin\varphi + \sin\varphi\cos\varphi) = 0 \tag{145}$$

$$g_{23} = (r\cos\theta\cos\varphi)(-r\sin\theta\sin\varphi) + (r\cos\theta\sin\varphi)(r\sin\theta\cos\varphi) + (-r\sin\theta)(0) \tag{146}$$

$$= r^2(-\cos\theta\cos\varphi\sin\theta\sin\varphi + \cos\theta\sin\varphi\sin\theta\cos\varphi) = 0 \ , \tag{147}$$

and therefore the metric in spherical coordinates is

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix} \ . \tag{148}$$

7 Vector Identities.

Relevant vector identities given by Jackson, Classical Electrodynamics, 3 ed.

1. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{a}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$
2. $\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\boldsymbol{\nabla} \cdot \mathbf{b}) - \mathbf{b}(\boldsymbol{\nabla} \cdot \mathbf{a}) + (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{a} - (\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{b}$
3. $\boldsymbol{\nabla} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\boldsymbol{\nabla} \times \mathbf{a}) - \mathbf{a} \cdot (\boldsymbol{\nabla} \times \mathbf{b})$
4. $\boldsymbol{\nabla} \times (\psi \mathbf{a}) = \boldsymbol{\nabla}\psi \times \mathbf{a} + \psi \boldsymbol{\nabla} \times \mathbf{a}$
5. $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi = 0$