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Contents

1 The Metric Tensor.

From the fact that $\mathrm{d}s^2 = g_{jk}\mathrm{d}x^j\mathrm{d}x^k$ is an invariant, show that g_{jk} is a symmetric covariant tensor of rank 2.

Consider transforming the differentials to a new coordinate frame:

$$\mathrm{d}\bar{x}^j = \frac{\partial \bar{x}^j}{\partial x^p}\mathrm{d}x^p \tag{1}$$

$$\mathrm{d}\bar{x}^k = \frac{\partial \bar{x}^k}{\partial x^q}\mathrm{d}x^q \ , \tag{2}$$

which allows us to write the line element in these coordinates

$$\mathrm{d}s^2 = \bar{g}_{jk}\mathrm{d}\bar{x}^j\mathrm{d}\bar{x}^k = \bar{g}_{jk}\frac{\partial \bar{x}^j}{\partial x^p}\mathrm{d}x^p\frac{\partial \bar{x}^k}{\partial x^q}\mathrm{d}x^q \ . \tag{3}$$

However, the line element is invariant under coordinate transform, so we may set the quantity from both frames equal:

$$g_{jk}\mathrm{d}x^j\mathrm{d}x^k = \bar{g}_{jk}\frac{\partial \bar{x}^j}{\partial x^p}\mathrm{d}x^p\frac{\partial \bar{x}^k}{\partial x^q}\mathrm{d}x^q \ . \tag{4}$$

Let us relabel the dummy indices on the right-had side, and slightly rearrange the right:

$$g_{pq}\mathrm{d}x^p\mathrm{d}x^q = \bar{g}_{jk}\frac{\partial \bar{x}^j}{\partial x^p}\frac{\partial \bar{x}^k}{\partial x^q}\mathrm{d}x^p\mathrm{d}x^q \ , \tag{5}$$

gathering terms yields

$$\left(g_{pq} - \bar{g}_{jk}\frac{\partial \bar{x}^j}{\partial x^p}\frac{\partial \bar{x}^k}{\partial x^q}\right)\mathrm{d}x^p\mathrm{d}x^q = 0 \ . \tag{6}$$

For this to be valid for all $\mathrm{d}x^p$ and $\mathrm{d}x^q$, we must have that

$$g_{pq} = \bar{g}_{jk}\frac{\partial \bar{x}^j}{\partial x^p}\frac{\partial \bar{x}^k}{\partial x^q} \ . \tag{7}$$

This form is exactly that of a transformation of a covariant, rank two tensor, and thus g_{ij} is a covariant, rank two tensor.

Furthermore, the metric tensor can be expressed as

$$g_{jk} = \frac{1}{2}(g_{jk} + g_{kjj}) + \frac{1}{2}(g_{jk} - g_{kj}) \equiv A_{jk} + B_{jk} \ , \tag{8}$$

where the first term is defined to be A_{jk}, a symmetric rank two tensor, and the second term is defined to be B_{jk}, an anti-symmetric rank two tensor. This allows us to write

$$g_{jk}\mathrm{d}x^j\mathrm{d}x^k = (A_{jk} + B_{jk})\mathrm{d}x^j\mathrm{d}x^k \ , \tag{9}$$

which, after some rearranging yields

$$(g_{jk} - A_{jk})\mathrm{d}x^j\mathrm{d}x^k = B_{jk}\mathrm{d}x^j\mathrm{d}x^k \ . \tag{10}$$

Let us examine the right-hand side:

$$B_{jk}\mathrm{d}x^j\mathrm{d}x^k = B_{kj}\mathrm{d}x^k\mathrm{d}x^j \ , \tag{11}$$

by swapping the dummy indices. Using the fact that B_{jk} is anti-symmetric gives us

$$B_{jk}\mathrm{d}x^j\mathrm{d}x^k = -B_{jk}\mathrm{d}x^k\mathrm{d}x^j \quad \Rightarrow \quad 2B_{jk}\mathrm{d}x^j\mathrm{d}x^k = 0 \ . \tag{12}$$

Using this result, Equation 10 reduces to

$$(g_{jk} - A_{jk})\mathrm{d}x^j\mathrm{d}x^k = 0 \ , \tag{13}$$

so that $g_{jk} = A_{jk}$. By definition we have $A_{jk} = A_{kj}$ so it is symmetric, and therefore so is the metric, g_{jk}. Combining this with the result from Equation 7 we see that g_{jk} is a symmetric tensor of rank two.

2 Covariant Derivative of Scalar.

Show that $\frac{\partial \varphi}{\partial x^i}$ transforms as a covariant tensor of rank one, where ϕ is a scalar.

Let us define $\varphi_i = \frac{\partial \varphi}{\partial x^i}$, and consider this quantity in a transformed coordinate system:

$$\bar{\varphi}_k = \frac{\partial \varphi}{\partial \bar{x}^k} \ , \tag{14}$$

using the chain rule for differentiation, we have

$$\bar{\varphi}_k = \frac{\partial \varphi}{\partial x^i}\frac{\partial x^i}{\partial \bar{x}^k} = \frac{\partial x^i}{\partial \bar{x}^k}\varphi_i \ . \tag{15}$$

Thus we see that $\phi_i = \frac{\partial \varphi}{\partial x^i}$ transforms as a covariant tensor of tank one.

3 Covariant Derivative of a Covector.

Show that $\frac{\partial t_i}{\partial x^j}$ is not a tensor, even if t_i is a tensor of rank one.

Consider the transformation of the tensor t_i to another coordinate frame:

$$\bar{t}_k = \frac{\partial x^i}{\partial \bar{x}^k} t_i \ , \tag{16}$$

which is a covariant transformation. We can now take the derivative with respect to $\bar{x}^l$:

$$\frac{\partial \bar{t}_k}{\partial \bar{x}^l} = \frac{\partial}{\partial \bar{x}^l}\left(\frac{\partial x^i}{\partial \bar{x}^k} t_i\right) = \frac{\partial^2 x^i}{\partial \bar{x}^l \partial \bar{x}^k} t_i + \frac{\partial x^i}{\partial \bar{x}^k}\frac{\partial t_i}{\partial \bar{x}^l} \ , \tag{17}$$

using the product rule for differentiation. Applying the chain rule to the second term, we obtain

$$\frac{\partial \bar{t}_k}{\partial \bar{x}^l} = \frac{\partial^2 x^i}{\partial \bar{x}^l \partial \bar{x}^k} t_i + \frac{\partial x^i}{\partial \bar{x}^k}\frac{\partial t_i}{\partial x^j}\frac{\partial x^j}{\partial \bar{x}^l} \ , \tag{18}$$

thus we see

$$\frac{\partial \bar{t}_k}{\partial \bar{x}^l} = \frac{\partial x^i}{\partial \bar{x}^k}\frac{\partial x^j}{\partial \bar{x}^l}\frac{\partial t_i}{\partial x^j} + \frac{\partial^2 x^i}{\partial \bar{x}^l \partial \bar{x}^k} t_i \ . \tag{19}$$

A tensor transforms in the following ways:

$$\text{covariant} : \bar{A}_{ij} = \frac{\partial x^l}{\partial \bar{x}^i}\frac{\partial x^m}{\partial \bar{x}^j} A_{lm} \tag{20}$$

$$\text{contravariant} : \bar{A}^{ij} = \frac{\partial \bar{x}^i}{\partial x^l}\frac{\partial \bar{x}^j}{\partial x^m} A^{lm} \tag{21}$$

$$\text{mixed} : \bar{A}^i{}_j = \frac{\partial \bar{x}^i}{\partial x^l}\frac{\partial x^m}{\partial \bar{x}^j} A^l{}_m \ , \tag{22}$$

so if the second term in Equation 19 were identically zero, the quantity $\frac{\partial t_i}{\partial x^j}$ would transform as a tensor. Since this term is not guaranteed to be zero, we conclude that $\frac{\partial t_i}{\partial x^j}$ is not a tensor.

4 Euler - Lagrange Equations.

Prove that a necessary condition that

$$I = \int_{t_1}^{t_2} F(t, x, \dot{x}) \mathrm{d}t \ , \tag{23}$$

be an extremum (maximum or minimum is that F satisfies the Euler-Lagrange equation

$$\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial F}{\partial \dot{x}} = 0 \ . \tag{24}$$

Consider a differential of this quantity:

$$\mathrm{d}I = \mathrm{d} \int_{t_1}^{t_2} F(t, x, \dot{x}) \mathrm{d}t = \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial q} \mathrm{d}q + \frac{\partial F}{\partial \dot{q}} \mathrm{d}\dot{q} \right) \mathrm{d}t \ . \tag{25}$$

Using the definition $\dot{q} = \frac{\mathrm{d}q}{\mathrm{d}t}$, we have

$$\mathrm{d}I = \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial q} \mathrm{d}q + \frac{\partial F}{\partial \dot{q}} \frac{\mathrm{d}(\mathrm{d}q)}{\mathrm{d}t} \right) \mathrm{d}t \ . \tag{26}$$

Consider integrating the second term by parts with

$$u = \frac{\partial L}{\partial \dot{q}} \quad \Rightarrow \quad \mathrm{d}u = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} \mathrm{d}t \tag{27}$$

$$\mathrm{d}v = \mathrm{d}(\mathrm{d}q) \quad \Rightarrow \quad v = \mathrm{d}q \ , \tag{28}$$

such that

$$\int_{t_i}^{t_2} \frac{\partial F}{\partial \dot{q}} \frac{\mathrm{d}(\mathrm{d}q)}{\mathrm{d}t} = \left[\frac{\partial L}{\partial \dot{q}} (\mathrm{d}q) \right] \Big|_{t_i}^{t_2} - \int_{t_i}^{t_2} (\mathrm{d}q) \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} \mathrm{d}t \ . \tag{29}$$

The differential of I is then

$$\mathrm{d}I = \left[\frac{\partial L}{\partial \dot{q}} (\mathrm{d}q) \right] \Big|_{t_i}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial q} \mathrm{d}q - (\mathrm{d}q) \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} \right) \mathrm{d}t \ . \tag{30}$$

The endpoints $q|_{t_1}$ and $q|_{t_2}$ are fixed while the path between them varies, therefore $\mathrm{d}q|_{t_1} = \mathrm{d}q|_{t_2} = 0$, so the boundary term from integration by parts vanishes. We are left with

$$\mathrm{d}I = \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} \right) \mathrm{d}q \mathrm{d}t \ . \tag{31}$$

For I to be an extremum (stationary value of I), we have that $\mathrm{d}I = 0$ for all $\mathrm{d}q$:

$$\frac{\mathrm{d}I}{\mathrm{d}q} = 0 = \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} \right) \mathrm{d}t \ , \tag{32}$$

which is valid for all choices of t_1 and t_2 if

$$0 = \frac{\partial F}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} \ , \tag{33}$$

which is the Euler-Lagrange equation.

5 Special Relativity.

In special relativity, as discussed in class the metric in Minkowski space-time is given by

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \ , \tag{34}$$

with $\mathrm{d}s^2 \geq 0$ for all velocities less than c, and has the same value in all Galilean coordinate systems. This defines an improper Euclidean space V_4 with a signature $(+,-,-,-)$. The Galilean coordinates define an orthogonal rectilinear coordinate system for this space, with an orthonormal frame in V_4 defined by the reduced Galilean coordinates x^α

$$x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^0 = ct, \tag{35}$$

which enables the metric to be rewritten in the form

$$\mathrm{d}s^2 = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2 \tag{36}$$

or

$$\mathrm{d}s^2 = \eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \ , \tag{37}$$

where all $\eta_{\alpha\beta}$ are zero except:

$$\eta_{11} = \eta_{22} = \eta_{33} = -\eta_{00} = -1. \tag{38}$$

5.1 Unit Velocity Vector.

Consider a mass at a point P which describes a trajectory C in V_4. C can be defined by giving the coordinates of the mass x^α as a function of some parameter s measure along C. Define the unit velocity vector by

$$u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}s}, \ (\alpha = 0,1,2,3). \tag{39}$$

Show that

$$u^i = \frac{v^i}{c\sqrt{1-\beta^2}}, \ (i=1,2,3); \quad u^0 = \frac{1}{\sqrt{1-\beta^2}} \ , \tag{40}$$

where $v = \beta c$ is the velocity of the mass and $v^i = \mathrm{d}x^i/\mathrm{d}t$.

The magnitude of the velocity is given by

$$v = \sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2} = \sqrt{\left(\frac{\mathrm{d}x^1}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x^2}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x^3}{\mathrm{d}t}\right)^2} \ . \tag{41}$$

The metric can be rewritten as

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = c^2 - \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 = c^2 - v^2 = c^2\left(1 - \frac{v^2}{c^2}\right) \ , \tag{42}$$

but since $x^0 = ct$, we have that $\mathrm{d}t = \mathrm{d}x^0/c$ and so

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = c^2\left(\frac{\mathrm{d}s}{\mathrm{d}x^0}\right)^2 = c^2\left(1 - \frac{v^2}{c^2}\right) \ , \tag{43}$$

and as such

$$\frac{\mathrm{d}x^0}{\mathrm{d}s} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}} \tag{44}$$

Using the chain rule, the unit velocity components are given by

$$u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}x^0}\frac{\mathrm{d}x^0}{\mathrm{d}s} = \frac{1}{c}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{1}{\sqrt{1-\beta^2}} \tag{45}$$

Using the definitions from Equation 35, the Cartesian unit velocities ($i = 1, 2, 3$) are

$$u^i = \frac{1}{c\sqrt{1-\beta^2}}\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{v^i}{c\sqrt{1-\beta^2}} \ . \tag{46}$$

The time-like component is given by

$$u^0 = \frac{1}{c}\frac{\mathrm{d}x^0}{\mathrm{d}t}\frac{1}{\sqrt{1-\beta^2}} = \frac{1}{c\sqrt{1-\beta^2}}c\frac{\mathrm{d}t}{\mathrm{d}t} = \frac{1}{\sqrt{1-\beta^2}} \ . \tag{47}$$

5.2 Minkowski Equations of Motion.

Minkowski's reformulation of Newton's equations of motion in the relativistic case take the form

$$m_0 c^2 \frac{\mathrm{d}u^\alpha}{\mathrm{d}s} = \Phi^\alpha \ . \tag{48}$$

Here m_0 is a parameter that has the dimensions of a mass, and Φ^α is a generalization of the Newtonian force vector. Using this equation and the results above, show that

$$m_0 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v^i}{\sqrt{1-\beta^2}}\right) = f^i, \quad (i = 1, 2, 3) \tag{49}$$

$$m_0 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{c}{\sqrt{1-\beta^2}}\right) = f^0 \ , \tag{50}$$

where

$$f^\alpha = \Phi^\alpha\sqrt{1-\beta^2} \ , \tag{51}$$

and consequently, the usual Newtonian motions are replaced by

$$m_0 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{v}}{\sqrt{1-\beta^2}}\right) = \mathbf{f} \tag{52}$$

in ordinary vector notation.

The equations of motion can be written

$$m_0 c^2 \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = m_0 c^2 \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}\left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}\right) = \Phi^\alpha \ , \tag{53}$$

using Equation 42 this becomes

$$m_0 c^2 \left(\frac{1}{c\sqrt{1-\beta^2}} \right) \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} \frac{1}{c\sqrt{1-\beta^2}} \right) = \Phi^\alpha \ . \tag{54}$$

Simplification yields the expression

$$m_0 \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} \frac{1}{\sqrt{1-\beta^2}} \right) = \Phi^\alpha \sqrt{1-\beta^2} \ . \tag{55}$$

For the space-like coordinates ($i = 1, 2, 3$), we have

$$m_0 \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{1}{\sqrt{1-\beta^2}} \right) = \Phi^i \sqrt{1-\beta^2} \ , \tag{56}$$

and therefore we obtain

$$m_0 \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{v^i}{\sqrt{1-\beta^2}} \right) = \Phi^i \sqrt{1-\beta^2} = f^i \ . \tag{57}$$

For the time-like component, we have

$$m_0 \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathrm{d}x^0}{\mathrm{d}t} \frac{1}{\sqrt{1-\beta^2}} \right) = m_0 \frac{\mathrm{d}}{\mathrm{d}t} \left(c\frac{\mathrm{d}t}{\mathrm{d}t} \frac{1}{\sqrt{1-\beta^2}} \right) = m_0 \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{c}{\sqrt{1-\beta^2}} \right) = \Phi^0 \sqrt{1-\beta^2} = f^0 \ . \tag{58}$$

5.3 Force and Velocity.

The derivative $\mathrm{d}u^\alpha/\mathrm{d}s$ is perpendicular to u^α, so that $\Phi^\alpha u_\alpha = 0$. Using this fact, show that $f^0 = (\mathbf{f} \cdot \mathbf{v})/c$ and consequently that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{m_0 c^2}{\sqrt{1-\beta^2}} \right) = \mathbf{f} \cdot \mathbf{v} \ . \tag{59}$$

Using the fact $\Phi^\alpha u_\alpha = 0$, we have

$$\Phi^0 u_0 - \Phi^1 u_1 - \Phi^2 u_2 - \Phi^3 u_3 = 0 = \Phi^0 u_0 - \mathbf{\Phi} \cdot \mathbf{u} \ , \tag{60}$$

where

$$u_0 = g_{0\alpha} u^\alpha = g_{00} u^0 = u^0 \ , \tag{61}$$

since the Minkowski metric is diagonal. Inserting the results from the previous two sections yields

$$0 = \sqrt{1-\beta^2} f^0 \left(\frac{1}{\sqrt{1-\beta^2}} \right) - \sqrt{1-\beta^2} \mathbf{f} \cdot \left(\frac{\mathbf{v}}{c\sqrt{1-\beta^2}} \right) \ , \tag{62}$$

yielding the result

$$\frac{\mathbf{f} \cdot \mathbf{v}}{c} = f^0 \ . \tag{63}$$

With this result, Equation 50 becomes

$$m_0 \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{c}{\sqrt{1-\beta^2}} \right) = \frac{\mathbf{f} \cdot \mathbf{v}}{c} \quad \Rightarrow \quad \mathbf{f} \cdot \mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{m_0 c^2}{\sqrt{1-\beta^2}} \right) \ . \tag{64}$$

5.4 Energy-Momentum Vector.

In Newtonian mechanics, the time dependence of the energy E of a mass is given by

$$\frac{\partial E}{\partial t} = \mathbf{f} \cdot \mathbf{v}, \tag{65}$$

which suggests that the term in the parenthesis in part c should be identified with the relativistic energy of the mass

$$E = \frac{m_0 c^2}{\sqrt{1-\beta^2}} \ , \tag{66}$$

which as you know, does not go to zero as $v \to 0$, but tends to the rest energy $E_0 = m_0 c^2$. Defining the momentum-energy vector by the equation

$$p^\alpha = m_0 c u^\alpha, \tag{67}$$

determine the space-like and time-like components of p in terms of m_0, v_i and E.

The space-like component is simply

$$p^0 = m_0 c u^0 = m_0 c \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{c}\frac{m_0 c^2}{\sqrt{1-\beta^2}} = E/c \ , \tag{68}$$

and the space-like components are given by

$$p^i = m_0 c u^i = m_0 c \frac{v^i}{c\sqrt{1-\beta^2}} = \frac{m_0 v^i}{\sqrt{1-\beta^2}} = \frac{1}{c^2}\frac{m_0 c^2}{\sqrt{1-\beta^2}} v^i = E\frac{v^i}{c^2} \ . \tag{69}$$

6 The Potential Tensor.

6.1 Sourceless Field Tensor.

Consider the differential equation for the field tensor in the absence of any sources ($J^\lambda = 0$)

$$\partial_\mu F^{\lambda\mu} = 0 \ . \tag{70}$$

Show that this can be written in the form

$$\partial^\mu F_{\nu\mu} = 0 \ , \tag{71}$$

and consequently the potential A_μ obeys the equation

$$(\partial_\mu \partial^\mu) A_\nu - \partial_\nu (\partial^\mu A_\mu) = 0 \ . \tag{72}$$

Multiplying the metric on the right of Equation 70 yields

$$0 = \partial_\mu F^{\mu\lambda} g_{\lambda\nu} \ , \tag{73}$$

inserting the identity,

$$0 = (\partial_\mu g^{\mu\gamma}) \left(g_{\mu\gamma} F^{\mu\lambda} g_{\lambda\nu} \right) = (\partial^\gamma) \left(g_{\mu\gamma} F^\mu_\nu \right) = \partial^\gamma F_{\gamma\nu} \ , \tag{74}$$

relabeling the dummy indices produces Equation 71. The potential is related to the field tensor by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \tag{75}$$

and differentiation gives

$$\partial^\mu F_{\mu\nu} = \partial^\mu (\partial_\mu A_\nu) - \partial^\mu (\partial_\nu A_\mu) = 0 \ . \tag{76}$$

Contracting the first term, and using the fact that the derivatives commute, this may be written as

$$(\partial_\mu \partial^\mu) A_\nu - \partial_\nu (\partial^\mu A_\mu) = 0 \ . \tag{77}$$

6.2 Potential Tensor.

Assuming a solution of the form

$$A_\mu(x) = \epsilon_\mu e^{ip_\nu x^\nu} \ , \tag{78}$$

where the vector ϵ_μ is the polarization vector, show that

$$(p_\mu p^\mu) \epsilon_\nu = (p^\mu \epsilon_\mu) p_\nu \ . \tag{79}$$

First, let us note $\partial_\nu = g_{\mu\nu} \partial^\nu$, so

$$\partial^\mu = \partial^0 e_0 + \partial^1 e_1 + \partial^2 e_2 + \partial^3 e_3 \tag{80}$$

$$\partial_\nu = \partial_0 e^0 - \partial_1 e^1 - \partial_1 e^2 - \partial_1 e^3 \ , \tag{81}$$

and thus

$$\partial_\mu \partial^\mu = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2 \ . \tag{82}$$

Inserting the ansatz into the first term in Equation 77 yields

$$(\partial_\mu \partial^\mu)\epsilon_\nu e^{ip_\lambda x^\lambda} = (\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2)\epsilon_\nu e^{ip_0 x^0} e^{ip_1 x^1} e^{ip_2 x^2} e^{ip_3 x^3} \tag{83}$$

$$= ((ip_0)^2 - (ip_1)^2 - (ip_2)^2 - (ip_3)^2)\epsilon_\nu e^{ip_\lambda x^\lambda} = -(p_0^2 - p_1^2 - p_2^2 - ip_3^2)\epsilon_\nu e^{ip_\lambda x^\lambda} \tag{84}$$

$$= -(p_\mu p^\mu)\epsilon_\nu e^{ip_\lambda x^\lambda} \ . \tag{85}$$

The second term in Equation 77 is

$$\partial_\nu(\partial^\mu \epsilon_\mu e^{ip_\nu x^\nu}) = \partial_\nu \left(\left(\partial^0 \epsilon_0 + \partial^1 \epsilon_1 + \partial^2 \epsilon_2 + \partial^3 \epsilon_3 \right) e^{ip_\nu x^\nu} \right) \tag{86}$$

$$= \partial_\nu \left(\left(ip^0 \epsilon_0 + ip^1 \epsilon_1 + ip^2 \epsilon_2 + ip^3 \epsilon_3 \right) e^{ip_\nu x^\nu} \right) = i\partial_\nu \left(p^\mu \epsilon_\mu e^{ip_\nu x^\nu} \right) \ , \tag{87}$$

now the remaining derivative commutes, so

$$\partial_\nu(\partial^\mu \epsilon_\mu e^{ip_\nu x^\nu}) = ip^\mu \epsilon_\mu \partial_\nu \left(e^{ip_\nu x^\nu} \right) = ip^\mu \epsilon_\mu \left(ip_0 e^0 - ip_1 e^1 - ip_2 e^2 - ip_3 e^3 \right) e^{ip_\nu x^\nu} \tag{88}$$

$$= -p^\mu \epsilon_\mu \left(p_0 e^0 - p_1 e^1 - p_2 e^2 - p_3 e^3 \right) e^{ip_\nu x^\nu} = -p^\mu \epsilon_\mu p_\nu e^{ip_\nu x^\nu} \ . \tag{89}$$

Combining these as in Equation 77 yields

$$0 = \left(-(p_\mu p^\mu)\epsilon_\nu e^{ip_\lambda x^\lambda} \right) - \left(-p^\mu \epsilon_\mu p_\nu e^{ip_\nu x^\nu} \right) \tag{90}$$

$$0 = -(p_\mu p^\mu)\epsilon_\nu + p^\mu \epsilon_\mu p_\nu \tag{91}$$

$$(p_\mu p^\mu)\epsilon_\nu = (p^\mu \epsilon_\mu)\, p_\nu \ , \tag{92}$$

proving Equation 79.

6.3 Gauge Transformation.

If $p_\mu p^\mu \neq 0$, then the result of part 6.2 gives

$$\epsilon_\nu = \frac{p^\mu \epsilon_\mu}{p_\mu p^\mu} p_\nu \ , \tag{93}$$

so that $\epsilon_\nu \propto p_\nu$. Using this fact, show that with a choice of a suitable gauge transformation, one can make a potential A_μ vanish, so that $p_\mu p^\mu \neq 0$ corresponds to a trivial solution of the wave equation.

We may select a scalar gauge $\psi(x)$ which makes the potential

$$A_\lambda \to A'_\lambda = A_\lambda + \partial_\lambda \psi \ , \tag{94}$$

such that the potential is

$$A'_\mu(x) = \frac{p^\lambda \epsilon_\lambda}{p_\lambda p^\lambda} p_\mu e^{ip_\nu x^\nu} + \partial_\mu \psi(x) \ , \tag{95}$$

which vanishes if

$$\partial_\mu \psi(x) = -\frac{p^\lambda \epsilon_\lambda}{p_\lambda p^\lambda} p_\mu e^{ip_\nu x^\nu} = -\frac{p^\lambda \epsilon_\lambda}{p_\lambda p^\lambda} \frac{1}{i} \partial_\mu e^{ip_\nu x^\nu} \ , \tag{96}$$

so by inspection we see

$$\psi(x) = i\frac{p^\lambda \epsilon_\lambda}{p_\lambda p^\lambda} e^{ip_\nu x^\nu} \ . \tag{97}$$

Let us insert this into the expression for the gauged potential

$$A'_\lambda = A_\lambda + \partial_\lambda \left(i\frac{p^\mu \epsilon_\mu}{p_\mu p^\mu} e^{ip_\nu x^\nu} \right) = A_\lambda + i\frac{p^\mu \epsilon_\mu}{p_\mu p^\mu} \partial_\lambda \left(e^{ip_\nu x^\nu} \right) \tag{98}$$

$$= A_\lambda + i\frac{p^\mu \epsilon_\mu}{p_\mu p^\mu} (ip_\lambda) \left(e^{ip_\nu x^\nu} \right) = A_\lambda - \frac{p^\mu \epsilon_\mu}{p_\mu p^\mu} p_\lambda \left(e^{ip_\nu x^\nu} \right) = A_\lambda - A_\lambda = 0 \ , \tag{99}$$

and we see that it vanishes. We see that in the case $p_\mu p^\mu \neq 0$, the potential vanishes and the wave equation, Equation 77, is simply

$$0 = (\partial_\mu \partial^\mu) A'_\nu - \partial_\nu (\partial^\mu A'_\mu) \tag{100}$$

$$0 = 0 \ , \tag{101}$$

which is a trivial solution.

6.4 Mass of Electromagnetic Field.

If we do not have $p_\mu p^\mu \neq 0$, then $p_\mu p^\mu = 0$. Using the results of Problem 5, show that this condition implies that the electromagnetic field is massless.

The non-trivial solutions to the wave equation, require

$$p_\mu p^\mu = 0 = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 \ , \tag{102}$$

which if we insert the results from Equations 68 and 69 becomes

$$0 = (E/c)^2 - (\gamma m_0)^2 \left((v^1)^2 + (v^2)^2 + (v^3)^2 \right) = \frac{E^2}{c^2} - \gamma^2 m_0^2 v^2 \ , \tag{103}$$

where $v^2 = (v^1)^2 + (v^2)^2 + (v^3)^2$ is the velocity and $\gamma = (1 - \beta^2)^{-1/2}$. Rearranging and using the definition of momentum $p = \gamma m_0 v$, we have

$$E^2 = p^2 c^2 \ . \tag{104}$$

However, in general, the relativistic energy includes the kinetic term and the rest mass term

$$E^2 = p^2 c^2 + (m_0 c^2)^2 \ , \tag{105}$$

but since the speed of light is a constant, we have that $m_0 = 0$, and thus the rest mass of the electromagnetic field is zero.

7 Transverse Electromagnetic Waves.

Consider the non-trivial solution to the potential of Problem 6 above, with $p_\mu p^\mu = 0$, and consider a wave propagating in the z direction, so that $p_\mu = (0, 0, k, \omega/c)$. Using an appropriate gauge transformation, show that non-transverse components of A_μ can be made to vanish so that the resulting wave is purely transverse.

The non-trivial solution to the wave equation gives us the condition

$$p_\mu p^\mu = 0 = p_\mu (g^{\mu\nu} p_\nu) = (\omega/c)^2 - k^2 = 0 \ , \tag{106}$$

which yields the dispersion relation $\omega = ck$ and thus $p_0 = p_3$. Furthermore, using the form of the wave equation in Equation 79, we see

$$0 = (p^\mu \epsilon_\mu)\, p_\nu \ , \tag{107}$$

if we use the potential of the form

$$A_\mu(x) = \epsilon_\mu e^{ip_\nu x^\nu} \ . \tag{108}$$

In general, $p_\nu \neq 0$, so we have

$$0 = p^\mu \epsilon_\mu = p^0 \epsilon_0 + p^3 \epsilon_3 = p_0 \epsilon_0 - p_3 \epsilon_3 = \frac{\omega}{c} \epsilon_0 - k \epsilon_3 \ , \tag{109}$$

and using the dispersion relation yields

$$\epsilon_0 = \epsilon_3 \ . \tag{110}$$

For the given momentum, we see

$$p_\nu x^\nu = p_0 x^0 + p_1 x^1 + p_2 x^2 + p_3 x^3 = \frac{\omega}{c} x^0 + k x^3 \ , \tag{111}$$

so the potential has the form

$$A_\mu = \epsilon_\mu e^{ip_0 x^0} e^{ip_3 x^3} = \epsilon_\mu e^{i \frac{\omega}{c} x^0} e^{ikx^3} \ . \tag{112}$$

Choosing a gauge ψ, sets the new potential to be

$$A'_\mu = \epsilon_\mu e^{i \frac{\omega}{c} x^0} e^{ikx^3} + \partial_\mu \psi \ , \tag{113}$$

but we are interested in a potential with only transverse components ($\mu = 1, 2$), so $A'_0 = A'_3 = 0$, and as such we have the conditions

$$0 = A_0 + \partial_0 \psi \quad \Rightarrow \quad -\frac{\partial \psi}{\partial x^0} = \epsilon_0 e^{ip_0 x^0} e^{ip_3 x^3} \tag{114}$$

$$0 = A_3 + \partial_3 \psi \quad \Rightarrow \quad -\frac{\partial \psi}{\partial x^3} = \epsilon_3 e^{ip_0 x^0} e^{ip_3 x^3} = \epsilon_0 e^{ip_0 x^0} e^{ip_0 x^3} \ . \tag{115}$$

The solution to these differential equations is

$$\psi = \epsilon_0 e^{ip_0 x^3} \frac{i}{p_0} e^{ip_0 x^0} = \frac{i\epsilon_0}{\omega/c} e^{ip_\nu x^\nu} = \frac{i\epsilon_0}{\omega/c} e^{i(\frac{\omega}{c} x^0 + kx^3)} \ , \tag{116}$$

to verify, the derivatives are

$$\partial_0 \psi = \frac{\partial \psi}{\partial x^0} = i\frac{\omega}{c}\frac{i\epsilon_0}{\omega/c}e^{i(\frac{\omega}{c}x^0 + kx^3)} = -\epsilon_0 e^{ip_\nu x^\nu} = -A_0 \tag{117}$$

$$\partial_1 \psi = \frac{\partial \psi}{\partial x^1} = 0 \tag{118}$$

$$\partial_2 \psi = \frac{\partial \psi}{\partial x^2} = 0 \tag{119}$$

$$\partial_3 \psi = \frac{\partial \psi}{\partial x^3} = k\frac{i\epsilon_0}{\omega/c}e^{i(\frac{\omega}{c}x^0 + kx^3)} = -\frac{k\epsilon_0}{k}e^{ip_\nu x^\nu} = -\epsilon_3 e^{ip_\nu x^\nu} = -A_3 \ . \tag{120}$$

Therefore the components of the potential with the gauge ψ is

$$A_0' = A_0 + (-A_0) = 0 \tag{121}$$

$$A_1' = A_1 + 0 = A_1 \tag{122}$$

$$A_2' = A_2 + 0 = A_2 \tag{123}$$

$$A_3' = A_3 + (-A_3) = 0 \ , \tag{124}$$

which has only transverse components, as expected.