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Classical Mechanics and Thermodynamics - Garrod
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## Contents

# 1 Problem #1: Half the Microstates of Most Probable Macrostate.

Consider $N$ coins (or equivalently, non-interacting spin-1/2 particles), where $N$ is very large. Find the ratio of heads to tails (or spin up to spin down) for which the number of microstates is half that of the most probable macrostate ($N/2$ heads, $N/2$ tails).

The most probable macrostate (50-50) has some amplitude (number of microstates) associated with it, so the macrostate (ratio of heads to tails) of interest is when the ratio of the amplitudes (probabilities) is a half,

$$R = \frac{1}{2} = \frac{P(fN)}{P(N/2)} \ , \tag{1}$$

where $P(\eta)$ is the probability of getting $\eta$ heads (or spin up particles). In class it was shown that this function is

$$P(\eta) = \frac{1}{2^N}\binom{N}{n} = \frac{1}{2^N}\frac{N!}{n!(N-n)!} \ , \tag{2}$$

so the ratio is

$$R = \frac{(\frac{N}{2})!(\frac{N}{2})!}{(fN)!(N-fN)!} \ . \tag{3}$$

For large $N$ the distribution is sharply peaked at the most probable macrostate, in this case $\frac{1}{2}$ (ratio of heads to $N$), so the macrostate with half as many microstates will be very close to $\frac{1}{2}$ as well. We can therefore introduce a small parameter $\delta$ such that

$$f = \frac{1}{2} - \delta \ , \tag{4}$$

which makes Equation 3 into

$$\frac{1}{2} = \frac{(\frac{N}{2}!)^2}{(\frac{N}{2}-\delta N)!(\frac{N}{2}+\delta N)!} \ , \tag{5}$$

of which we can take the natural logarithm,

$$\ln\frac{1}{2} = 2\ln\left(\frac{N}{2}!\right) - \ln\left\{(\frac{N}{2}-\delta N)!\right\} - \ln\left\{(\frac{N}{2}+\delta N)!\right\} \ , \tag{6}$$

which allows us to use Stirling's approximation, $\ln N \simeq N\ln N - N$, to get

$$\ln\frac{1}{2} = 2\left\{\frac{N}{2}\ln\frac{N}{2} - \frac{N}{2}\right\} - (\frac{N}{2}-\delta N)\ln\left\{\frac{N}{2}-\delta N\right\} + (\frac{N}{2}-\delta N) - (\frac{N}{2}+\delta N)\ln\left\{\frac{N}{2}+\delta N\right\} + (\frac{N}{2}+\delta N) \ ,$$

note the two $\delta N$ terms cancel and the $-N$ cancels with the two $N/2$ terms, yielding

$$\ln\frac{1}{2} = N\ln\frac{N}{2} - \frac{N}{2}\ln\left\{\left(\frac{N}{2}-\delta N\right)\left(\frac{N}{2}+\delta N\right)\right\} + \delta N\ln\left\{\frac{\frac{N}{2}-\delta N}{\frac{N}{2}+\delta N}\right\} \ , \tag{7}$$

which simplifies to

$$\ln\frac{1}{2} = N\ln\frac{N}{2} - \frac{N}{2}\ln\left\{N^2\left(\frac{1}{4}-\delta^2\right)\right\} + \delta N\ln\left\{\frac{1-2\delta}{1+2\delta}\right\} \ . \tag{8}$$

Using Mathematica to expand the logs to second order in the small parameter $\delta$, we get

$$\ln\frac{1}{2} = N\ln\frac{N}{2} - \frac{N}{2}\ln\frac{N^2}{4} - 2N\delta^2 + \mathcal{O}(\delta^3) = -2N\delta^2 + \mathcal{O}(\delta^3) \ , \tag{9}$$

because the power of two inside the second log can be pulled down to be a coefficient. Therefore the value of the small parameter is

$$\delta(N) = \pm\sqrt{\frac{\ln 2}{2N}} \ , \tag{10}$$

and the macrostate with half as many microstates as the most probable state has a heads to total ratio of $f = \frac{1}{2} \pm (\ln 2/2N)^{1/2}$. So the ratio of heads to tails is

$$\frac{N_H}{N_T} = \frac{N_H}{N - N_H} = \frac{fN}{(1-f)N} = \frac{\frac{1}{2} \pm (\ln 2/2N)^{1/2}}{\frac{1}{2} \mp (\ln 2/2N)^{1/2}} \ . \tag{11}$$

# 2 Garrod 2.23: Combinatorics of Multi-Particle Systems.

Consider a quantum-mechanical system of four particles in a one-dimensional harmonic oscillator potential. The one-particle energy eigenvalues are $(n + \frac{1}{2})\hbar\omega$ and are nondegenerate. For Bose-Einstein particles and Fermi-Dirac particles, determine the number of four-particle quantum states with an energy of $8\hbar\omega$. Assume that the Fermi-Dirac particles have zero spin, although that can be shown to be impossible.

Given the total energy, we have that

$$8\hbar\omega = \hbar\omega[(n_1 + \frac{1}{2}) + (n_2 + \frac{1}{2}) + (n_3 + \frac{1}{2}) + (n_4 + \frac{1}{2})] \quad \Rightarrow \quad 6 = n_1 + n_2 + n_3 + n_4 \ , \tag{12}$$

where $n_i$ is the energy state of the $i$th particle. Note that the particles are indistinguishable, so exchanging particles does not add microstates. For example $\{n_i\} = (0, 1, 2, 3)$ is the same microstate as $(1, 3, 0, 2)$.

## 2.1 Bose-Einstein Particles.

The number of microstates available to this system is all the ways you can add four positive integers to equal six. Due to the fact the particles are indistinguishable, we can have the following microstates

$$\begin{aligned} \{n_i\} &= (6,0,0,0) && (13)\\ &= (5,1,0,0) && (14)\\ &= (4,2,0,0) && (15)\\ &= (4,1,1,0) && (16)\\ &= (3,3,0,0) && (17)\\ &= (3,2,1,0) && (18)\\ &= (3,1,1,1) && (19)\\ &= (2,2,2,0) && (20)\\ &= (2,2,1,1)\ , && (21) \end{aligned}$$

which totals nine microstates. While this method works for a system with $N = 4$, it soon becomes intractable for larger $N$.

## 2.2 Fermi-Dirac Particles.

Due to the Pauli principle, no two particles can occupy the same quantum state, in this case, no two $n_i$ can be equal (spin is ignored). Therefore there is only one microstate, $\{n_i\} = (0, 1, 2, 3)$.

## 3 Garrod 2.26: Number of Particles in Harmonic Potential.

Consider a system of spinless one-dimensional particles in a harmonic oscillator potential of angular frequency $\omega$. The energy spectrum is then $\varepsilon_n = (n + \frac{1}{2})\hbar\omega$. Using Equations (2.63) and (2.64), evaluate the number of particles in the system as a function of the affinity $\alpha$ and the temperature $T$ for the case $\alpha \gg 1$ for both fermions and bosons.

We begin by noting the distribution functions for both types of particles

$$f_{FD}(\varepsilon_n) = \frac{1}{\exp[\alpha + \beta\varepsilon_n] + 1} \tag{22}$$

$$f_{BE}(\varepsilon_n) = \frac{1}{\exp[\alpha + \beta\varepsilon_n] - 1} \ , \tag{23}$$

where $\beta = 1/(kT)$ with $k$ being Boltzmann's constant. If we take the limit that $\alpha \gg 1$, the $\pm 1$ term in the denominator is dwarfed by the exponential, so

$$f(\varepsilon_n) = f_{FD}(\varepsilon_n) = f_{BE}(\varepsilon_n) = e^{-\alpha} \exp\left[-\beta\hbar\omega\left(n + \frac{1}{2}\right)\right] = e^{-\alpha} e^{-\beta\hbar\omega n} e^{--\beta\hbar\omega/2} \ . \tag{24}$$

The number of particles of the system is the sum over all discrete energy states,

$$N = \sum_{n=0}^{\infty} e^{-\alpha} e^{-\beta\hbar\omega n} e^{-\beta\hbar\omega/2} = e^{-\alpha} e^{-\beta\hbar\omega/2} \sum_{n=0}^{\infty} (e^{-\beta\hbar\omega})^n \ , \tag{25}$$

now let us define $\gamma = \hbar\omega/k$ so that

$$N = e^{-\alpha} e^{-\gamma/(2T)} \sum_{n=0}^{\infty} (e^{-\gamma/T})^n = \frac{1}{\exp\left[\alpha + \frac{\gamma}{2T}\right]} \left(\frac{1}{1 - e^{-\gamma/T}}\right) \ . \tag{26}$$

# 4 Garrod 3.8: Particles in One-Dimensional Box.

A system of $N$ noninteracting one-dimensional particles are constrained to an interval of length $L$ and have a total energy $E = N\varepsilon$.

## 4.1 Probability Density.

Write an explicit formula for the uniform ensemble probability density $P(x_1, \ldots, x_N; p_1, \ldots, pN)$.

We can begin by writing the Hamiltonian of the particles, which is just that of $N$ free particles,

$$\mathcal{H}(x_1, \ldots, x_N; p_1, \ldots, p_N) = \sum_{i=1}^{N} \frac{p_i^2}{2m} = \sum_{i=1}^{N} q_i^2 \ , \tag{27}$$

using a change of variables $q_i = p_i/\sqrt{2m}$. Using this, and Garrod Equation 3.16, we can write the normalization constant, which is the total number of states

$$C(E) = \int_\Omega \Theta \left( N\varepsilon - \sum_{i=1}^{N} q_i^2 \right) \ d^N x_i \ d^N[(2m)^{1/2} q_i] \ , \tag{28}$$

we can immediately integrate over the spatial variables that do not appear in the integrand, so we pull out $N$ factors of the spatial scale,

$$C(E) = (L\sqrt{2m})^N \int_\Omega \Theta \left( N\varepsilon - \sum_{i=1}^{N} q_i^2 \right) \ d^N q_i \ . \tag{29}$$

This is the equation for the surface of a sphere in $N$ dimensional phase-space, as shown in Garrod Equation A.30. Using the result given in the appendix, we find that

$$C(E) = (L\sqrt{2m})^N \frac{\pi^{N/2}}{(N/2)!} \left( \sqrt{N\varepsilon} \right)^N = \frac{(L\sqrt{2\pi m N\varepsilon})^N}{(N/2)!} \ . \tag{30}$$

Therefore the expression for the probability distribution is given by Garrod Equation 3.14,

$$P(x_i, p_i) = \frac{1}{C(E)} \Theta \left( N\varepsilon - \sum_{i=1}^{N} \frac{p_i^2}{2m} \right) = \frac{(N/2)!}{(L\sqrt{2\pi m N\varepsilon})^N} \Theta \left( N\varepsilon - \sum_{i=1}^{N} \frac{p_i^2}{2m} \right) \ . \tag{31}$$

## 4.2 Momentum of $N$th Particle.

By integrating over all coordinates and momenta except $p_N$, determine the probability distribution for the momentum of the $N$th particle.

Starting from Equation 31, we can integrate this distribution over all position space and all momenta except that of the last particle, which will give the probability distribution for the momentum of the last ($N$th) particle. We have an integral of the form of Equation 28, just with one less integration over momentum, so we can write

$$P(p_M) = \frac{(L\sqrt{2m})^{N-1}}{C(E)} \int_\Omega \Theta \left( N\varepsilon - \frac{p_N^2}{2m} - \sum_{i=1}^{N-1} q_i^2 \right) \ d^{N-1} q_i \ , \tag{32}$$

which again is a sphere in phase space with radius $R = \sqrt{N\varepsilon - p_N^2/2m}$. Therefore, using the Appendix again, we can write

$$P(p_N) = \frac{(N/2)!}{(L\sqrt{2\pi m N\varepsilon})^N}(L^N)(\sqrt{2m})^{N-1}\frac{\pi^{(N-1)/2}}{((N-1)/2)!}(N\varepsilon - p_N^2/2m)^{(N-1)/2} \ . \tag{33}$$

Note that we can write $x^{(a-1)/2} = x^{a/2}/\sqrt{x}$, so the above expression becomes

$$P(p_N) = \frac{\frac{N}{2}!}{\frac{N-1}{2}!}\frac{1}{(N\varepsilon)^{N/2}}\frac{1}{\sqrt{2\pi m}}\left[(N\varepsilon)\left(1 - \frac{p_N^2}{2mN\varepsilon}\right)\right]^{(N-1)/2} \tag{34}$$

$$= \frac{\frac{N}{2}!}{\frac{N-1}{2}!}(2\pi m N\varepsilon)^{-1/2}\left(1 - \frac{p_N^2}{2mN\varepsilon}\right)^{(N-1)/2} \ , \tag{35}$$

which gives the probability distribution for the momentum of the $N$th particle in the ensemble.

### 4.3 Relation to Maxwell Distribution.

Take the logarithm of the result obtained in (b), assuming that $N \gg 1$, use Stirling's approximation to rewrite it, and then exponentiate the result to show that the momentum distribution associated with the uniform ensemble is just the Maxwell distribution.

We can express the logarithm of the probability distribution obtained in section 1.2 as

$$\ln[P(p_N)] = \ln\left[\frac{N}{2}!\right] - \ln\left[\frac{N-1}{2}!\right] - \frac{1}{2}\ln\left[2\pi m N\varepsilon\right] + \frac{N-1}{2}\ln\left[1 - \frac{p_N^2}{2mN\varepsilon}\right] \ . \tag{36}$$

If we assume that $N \gg 1$, we can use Stirling's approximation to rewrite the first two logarithms as

$$\frac{N}{2}\ln\left[\frac{N}{2}\right] - \frac{N}{2} - \frac{N-1}{2}\ln\left[\frac{N-1}{2}\right] + \frac{N-1}{2} \tag{37}$$

$$= -\frac{1}{2}[1 + \ln 2 - N\ln N + N\ln(N-1) - \ln(N-1)] \tag{38}$$

$$= -\frac{1}{2}\left\{1 + \ln 2 - N\ln N + N\ln\left[N\left(1 - \tfrac{1}{N}\right)\right] - \ln\left[N\left(1 - \tfrac{1}{N}\right)\right]\right\} \tag{39}$$

$$= -\frac{1}{2}\left\{1 + \ln 2 - N\ln N + N\ln N + N\ln\left[1 - \tfrac{1}{N}\right] - \ln N - \ln\left[1 - \tfrac{1}{N}\right]\right\} \ . \tag{40}$$

Now we can expand the logarithms in the large $N$ limit, so $1/N \ll 1$ and therefore the logarithm of one plus a small parameter is just the small parameter. After cancelling the $N\ln N$ terms in the above expression, and expanding the logs, we get

$$-\frac{1}{2}\left\{1 + \ln 2 + N\left[-\tfrac{1}{N}\right] - \ln N + \tfrac{1}{N}\right\} = -\frac{1}{2}\left\{\ln 2 - \ln N + \tfrac{1}{N}\right\} \ . \tag{41}$$

Plugging this back in to the probability distribution, Equation 36, factoring out a $-\frac{1}{2}$ out of the remaining terms, and multiplying through by $-2$, gives us

$$-2\ln[P] = \ln 2 - \ln N + \tfrac{1}{N} + \ln\left[2\pi m N\varepsilon\right] - (N-1)\ln\left[1 - \frac{p_N^2}{2mN\varepsilon}\right] \ . \tag{42}$$

Again we expand the right-most logarithm in large $N$ to zeroth order in $1/N$,

$$(N-1)\ln\left[1-\frac{p_N^2}{2mN\varepsilon}\right] = N(1-\tfrac{1}{N})\left[-\frac{p_N^2}{2mN\varepsilon}\right] = -\frac{p_N^2}{2m\varepsilon} + \mathcal{O}(\tfrac{1}{N}) \ . \tag{43}$$

Finally, the expression for the log of the probability distribution is

$$\ln[P(p_N)] = -\frac{1}{2}\left[\ln\left(\frac{(2)2\pi mN\varepsilon}{N}\right) + \frac{1}{N} + \frac{p_N^2}{2m\epsilon}\right] = \ln\left[\frac{1}{\sqrt{4\pi m\varepsilon}}\right] - \frac{p_N^2}{4m\varepsilon} + \mathcal{O}(\tfrac{1}{N}) \tag{44}$$

We can now exponentiate both sides of the equation and neglect terms of order $1/N$, to get

$$P(p_N) = \frac{1}{\sqrt{4\pi m\varepsilon}}\exp\left[-\frac{p_N^2}{4m\varepsilon}\right] \tag{45}$$

which is just the Maxwell-Boltzmann distribution, with $k_B = 1$.

# 5 Garrod 3.16: Particles Confined in Cylindrical Volume.

Consider a system of N noninteracting particles, confined to move within a smooth circular cylinder of radius $R$ and length $\ell$. The axis of the cylinder is the $z$ axis. Suppose that, besides fixing the number of particles in the system, we fix the total energy to be $E$ and the $z$ component of angular momentum to be $L_z$.

## 5.1 Equilibrium Distribution.

By the method used to derive the Maxwell-Boltzmann distribution in the previous chapter, show that the equilibrium distribution function is of the form, $f(\mathbf{r}, \mathbf{v}) = C \exp[-\beta(mv^2/2 - m\Omega(xv_y - yv_x))]$.

We follow the Garrod's treatment of the derivation of the Maxwell-Boltzmann distribution in section 2.3. We assume all microstates are equally probable, and we impose the following conservations must be satisfied:

$$N = \sum N_{kl} \quad E = \sum E_{kl} N_{kl} \quad L_z = \sum \ell_{kl} N_{kl} \ , \tag{46}$$

where $k$ is a momentum space index, and $l$ is a position space index. Let $\ell_{kl}$ be the $z$ projection of the total angular momentum of the particles in the $kl$th phase space bin. Additionally we note the definition of the $z$ component of angular momentum,

$$L_z = m(xp_y - yp_x) \ , \tag{47}$$

where $p_i$ is the $i$th component of a particle's linear momentum. We want to maximize the function $G$ with the constraints listed above, which is given by

$$G = F - \alpha \sum N_{kl} - \beta \sum E_{kl} N_{kl} - \gamma \sum \ell_{kl} N_{kl} \ , \tag{48}$$

where $F$ is given by Garrod Equation 2.12. Let us note the dimensions of the Lagrange multipliers, whose values will be determined later. The expression for $G$ must be dimensionless, so $\alpha$ is dimensionless, $\beta$ has units of inverse energy, and $\gamma$ has units of inverse momentum. To maximize $G$ we take the partial derivative with respect to $N_{kl}$ and set it to zero,

$$0 = \frac{\partial G}{\partial N_{kl}} = -\log N_{kl} - \alpha - \beta E_{kl} - \gamma \ell_{kl} \ . \tag{49}$$

We can now solve this for $N_{kl}$, the occupancy number for the $kl$th bin in phase space,

$$N_{kl} = \exp\left[-\alpha - \beta E_{kl} - \gamma \ell_{kl}\right] = C \exp\left[-\beta\left(E_{kl} + \frac{\gamma}{\beta}\ell_{kl}\right)\right] \ , \tag{50}$$

where $C$ is the exponential of negative $\alpha$. Note that $\gamma/\beta$ has units of momentum per energy, which is just inverse time, so we will call it a frequency $\Omega$. We can write the distribution function in terms of the components of a particles momentum and position, using Garrod Equation 2.6. In our case, there is no potential, so we get the result

$$f(\mathbf{r}; \mathbf{v}) = C \exp[-\beta(E_{kl} + \Omega \ell_{kl})] \tag{51}$$

and using Equation 47, and the definition of the linear kinetic energy, we see that the equilibrium distribution function is

$$f(\mathbf{r}; \mathbf{v}) = C \exp[-\beta(mv^2/2 + m\Omega(xv_y - yv_x))] \ . \tag{52}$$

## 5.2 Average Particle Velocity.

Calculate the average velocity of particles at position $(x, y, z)$ and show that it is identical to what would be obtained for rigid body rotation about the z axis with angular velocity $\Omega$.

Let us expand the argument of the exponential,

$$-\beta(mv^2/2 - m\Omega(xv_y - yv_x)) = \frac{-\beta m}{2}\left(v_x^2 + v_y^2 + v_z^2 - 2\Omega x v_y + 2\Omega y v_x\right) \ , \tag{53}$$

and we can complete the squares for $v_x$ and $v_y$, such that

$$\frac{-\beta m}{2}\left(v_x^2 + 2\Omega y v_x + (\Omega y)^2 - (\Omega y)^2 + v_y^2 - 2\Omega x v_y + (\Omega x)^2 - (\Omega x)^2 + v_z^2\right) \tag{54}$$

$$= \frac{-\beta m}{2}\left[(v_x + \Omega y)^2 - (\Omega y)^2 + (v_y - \Omega x)^2 - (\Omega x)^2 + v_z^2\right] \ . \tag{55}$$

We can put this back into the exponential, and separate terms with $v_i$,

$$f(\mathbf{r};\mathbf{v}) = C\exp\left[\frac{-\beta m}{2}[(v_x+\Omega y)^2 - (\Omega y)^2]\right]\exp\left[\frac{-\beta m}{2}[(v_y - \Omega x)^2 - (\Omega x)^2]\right]\exp\left[\frac{-\beta m}{2}v_z^2\right] \ , \tag{56}$$

we can now move the position variables to their own exponential,

$$f = C\exp\left[\xi(v_x+\Omega y)^2\right]\exp\left[\xi[(v_y-\Omega x)^2]\right]\exp\left[\xi v_z^2\right]\exp\left[-\xi((\Omega x)^2 + (\Omega y)^2)\right] \ , \tag{57}$$

where $\xi = -\beta m/2$. We can note that each term with velocity dependence is in the form of a Gaussian. It is important to note that the average value in a Gaussian is the value at which the peak is centered. By inspection, we can see the average values for the velocity components are

$$\langle v_x(x,y,z)\rangle = -\Omega y \quad \langle v_y(x,y,z)\rangle = \Omega x \quad \langle v_z(x,y,z)\rangle = 0 \ . \tag{58}$$

Consider a rigid body rotating the $z$ axis in the $x - y$ plane with frequency $\Omega$. At a position $(x, y)$ the rotator has tangential velocity given by

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r} = \Omega\hat{\mathbf{z}}\times(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}) = (-\Omega y)\hat{\mathbf{x}} + (\Omega x)\hat{\mathbf{y}} \ , \tag{59}$$

which is exactly what we found as the average velocity of the particles in the cylinder.

## 5.3 Particle Density.

Show that the particle density $n(\mathbf{r})$ has the form $C\exp[-\phi(\mathbf{r}) = kT]$, where $\phi = -\frac{1}{2}m\Omega^2(x^2+y^2)$ is the centrifugal potential.

Given that we have the distribution function

$$f(\mathbf{r},\mathbf{v}) = Ce^{-\frac{1}{2}\beta m(v_x+\Omega y)^2}e^{-\frac{1}{2}\beta m(v_y+\Omega x)^2}e^{-\frac{1}{2}\beta m v_z^2}e^{+\frac{1}{2}\beta m\Omega^2(x^2+y^2)} \ , \tag{60}$$

we can write the particle density as a function of position by integrating over the three dimensional momentum,

$$n(\mathbf{r}) = \int f(\mathbf{r},\mathbf{v})d^3v = g(\mathbf{r})\int f_{v_x}(\mathbf{r})dv_x \int f_{v_y}(\mathbf{r})dv_y \int f_{v_z}(\mathbf{r})dv_z \tag{61}$$

where $g$ is an arbitrary function of position only. If we define the centrifugal potential $\phi(\mathbf{r}) = -\frac{1}{2}m\Omega^2(x^2+y^2)$, we can write the particle density as

$$n(\mathbf{r}) = Ce^{-\beta\phi(\mathbf{r})} \int e^{-\frac{1}{2}\beta m(v_x+\Omega y)^2} dv_x \int e^{-\frac{1}{2}\beta m(v_y+\Omega x)^2} dv_y \int e^{-\frac{1}{2}\beta m v_z^2} dv_z \ , \tag{62}$$

which is now just a matter of solving elementary integrals. Using your favorite method to integrate these, we find that

$$n(\mathbf{r}) = Ce^{-\beta\phi(\mathbf{r})} \left(\frac{2\pi}{\beta m}\right)^{3/2} \ , \tag{63}$$

where $C$ is still some arbitrary constant.

# 6 Garrod 3.25: Ensemble of Distinguishable One-Dimensional Oscillators.

Using the uniform ensemble, evaluate the entropy $S(N, E)$ for a system of $N$ distinguishable one-dimensional harmonic oscillators. When the particles are distinguishable, the factor of $1/N!$ is left out of the definition of $S$ given in Garrod Eq. (3.84).

Consider a system of $N$ distinguishable one-dimensional harmonic oscillators, in the uniform ensemble. The entropy of a system is given by Garrod Equation 3.84, with the factor of $1/N!$ removed because the particles are distinguishable,

$$S = \log\left[\frac{1}{(2\pi\hbar)^N}\int \Theta(E - \mathcal{H}(x,p)) d^N x \; d^N p\right] \;, \tag{64}$$

where $\mathcal{H}$ is the Hamiltonian of the system, and we have set the Boltzmann constant $k_B = 1$. In this system, the Hamiltonian is

$$\mathcal{H} = \sum_{i=1}^{N}\frac{p_i^2}{2m} - \sum_{i=1}^{N}\frac{m\omega^2 x_i^2}{2} \;, \tag{65}$$

where $m$ is the mass of each oscillator, and $\omega$ is their frequency. Let us now make a change of variables such that

$$\begin{cases} \rho &= \frac{p}{\sqrt{2m}} \\ \chi &= \sqrt{\frac{m\omega^2}{2}}x \end{cases} \quad \Rightarrow \quad \begin{cases} d^N r &= \sqrt{2m}^N d^N \rho \\ d^N x &= \sqrt{\frac{2}{m\omega^2}}^N d^N \chi \end{cases} \;. \tag{66}$$

We can now write the integral in Equation 64

$$\left(\frac{2}{\omega}\right)^N \int \Theta\left(E - \sum_{i=1}^{N}\rho_i^2 - \sum_{i=1}^{N}\chi_i^2\right) d^N\chi \; d^N\rho \;, \tag{67}$$

but we can note that this is still of the form as the integral of a sphere in $2N$ dimensions, given by Garrod Equation A.30. Each term in the infinite series in the given integral is integrated independently from the other terms, which is exactly what is happening in the above integral. Therefore we can use the result given by Garrod Equation A.35, but as if we had evaluated the single infinite sum over $2N$ integrations. Therefore, the entropy of the system is

$$S = \log\left[\frac{1}{(2\pi\hbar)^N}\left(\frac{2}{\omega}\right)^N \frac{(\pi E)^N}{N!}\right] = \log\left[\left(\frac{E}{\hbar\omega}\right)^N \frac{1}{N!}\right] \;, \tag{68}$$

which we can take the logarithm to obtain

$$S = N\log\left[\frac{E}{\hbar\omega}\right] - \log[N!] \;. \tag{69}$$

If we assume $N$ is large, we can use Stirling's approximation to find the entropy:

$$S = N\log\left[\frac{E}{\hbar\omega}\right] - N\log[N] - N = N\log\left[\frac{E}{N\hbar\omega}\right] - N \;. \tag{70}$$

### 6.1 Temperature Dependence of Energy.

The temperature of the system is defined to be the inverse of the partial derivative of the entropy with respect to energy,

$$\frac{\partial S}{\partial E} = \beta = \frac{N}{E} = \frac{1}{T} \ , \tag{71}$$

so the energy of the system at a specific temperature is

$$E(T) = k_B N T \ , \tag{72}$$

where we have reinserted $k_B$ to get the correct units.