Dylan J. Temples: Solution Set Three

Northwestern University, Classical Mechanics
Classical Mechanics, Third Ed.- Goldstein
October 14, 2015

Contents

1 Problem #1: Speed Along an Orbit.

Given details about the shape of an elliptic orbit and an expression for energy in terms of eccentricity, the relationship between velocity and orbital radius can be found. The energy of the orbit as a function of eccentricity (ϵ) can be found by solving Goldstein Equation 3.57 for energy,

$$E = -\frac{\kappa^2 \mu}{2l^2}(1 - \epsilon^2) \ . \tag{1}$$

where μ is the reduced mass of the sun-planet system, κ is the attraction factor of the gravitational potential, and l is the angular momentum of the planet. The equation for the semi-latus rectum is

$$r_S = a(1 - \epsilon^2) = \frac{l^2}{\kappa \mu} \ . \tag{2}$$

where a is the semi-major axis of the orbit. Note the factor of $1 - \epsilon^2$ is just r_S/a, which using the far right expression for r_S and substituting in $1 - \epsilon^2 = r_S/a$, gives the energy to be

$$E = -\frac{\kappa^2 \mu}{2l^2} \frac{l^2}{a \kappa \mu} = -\frac{\kappa}{2a} \ , \tag{3}$$

which is just the sum of kinetic and potential energies,

$$-\frac{\kappa}{2a} = \frac{1}{2} m v^2 + \frac{(-\kappa)}{r} \ , \tag{4}$$

where m is the mass of the planet and r is the distance of the planet from the sun. This can be solved for the velocity squared, giving

$$v^2 = \left(\frac{2}{m}\right) \left[\frac{\kappa}{r} - \frac{\kappa}{2a}\right] = \frac{\kappa}{m} \left[\frac{2}{r} - \frac{1}{a}\right] \ . \tag{5}$$

2 Goldstein 3.10.

A planet of mass M is in an orbit around the Sun with eccentricity $\epsilon = 1-\alpha$ where $\alpha \ll 1$. Assume the motion of the Sun can be neglected, and that only gravitational forces act. When the planet is at its greatest distance from the Sun, it is struck by a comet of mass m, where $m \ll M$, travelling in a tangential direction. Assuming the collision is completely inelastic, find the minimum kinetic energy the comet must have to change the new orbit to a parabola. A planet in an elliptic orbit has energy $E_p < 0$, while a planet in a parabolic orbit has $E_p = 0$. In order for a comet with kinetic energy T_c to change a planet's orbit from elliptic to parabolic it must be that T_c is high enough to impart enough energy through an inelastic collision (where kinetic energy is not conserved) that will raise the planet's (now of mass $m+M$) total energy to zero. Note that while kinetic energy is not conserved in an inelastic collision, linear momentum is.

Before the collision the planet has energy $E_p = T+U$ and is located at aphelion. Conservation of linear momentum from just before the collision and just after gives

$$mv_c + Mv_p = (m+M)v_f \ , \tag{6}$$

where v_c and v_P are the velocities of the comet and planet, respectively, before the collision, and v_f is the final velocity of the new composite object. This makes the expression for the final velocity

$$v_f = \frac{m}{m+M}v_c + \frac{M}{m+M}v_p \ , \tag{7}$$

which if we approximate that $m+M \simeq M$ for $m \ll M$, this becomes

$$v_f = \frac{m}{M}v_c + v_p \ . \tag{8}$$

This expression can be used to get an expression for the total energy of the comet-planet composite object's orbit around the sun just after the collision at aphelion, r_a,

$$E_f = T+U = \frac{1}{2}(M+m)v_f^2 - \frac{\kappa}{r_a} = \frac{1}{2}(M+m)\left(\frac{m}{M}v_c + v_p\right)^2 - \frac{\kappa}{r_a} = \frac{1}{2}M\left(\frac{m}{M}v_c + v_p\right)^2 - \frac{\kappa}{r_a} \ , \tag{9}$$

where in the last step, the same approximation was used as in Equation 8. However, the velocity of the planet at aphelion is only due to its angular momentum and can be replaced by $v_p = l/(Mr_a)$. This equation expands to

$$E_f = \frac{1}{2}\frac{m^2v_c^2}{M} + mv_cv_p + \frac{1}{2}Mv_p^2 - \frac{\kappa}{r_a} = \frac{1}{2}\frac{m^2v_c^2}{M} + mv_cv_p + E_p \ , \tag{10}$$

where E_p is the total energy of the planet before the collision, still in its elliptical orbit. Just before the collision, as well as just after, the planet is located at its aphelion distance, r_a. The energy of an elliptical orbit was found in Section 1 to be $E_p = \frac{-\kappa}{2a}$, where a is the semi-major axis. Making this substitution makes the planet-comet final energy

$$E_f = \frac{1}{2}\frac{m^2v_c^2}{M} + mv_cv_p - \frac{\kappa}{2a} = mv_cv_p - \frac{\kappa}{2a} \ , \tag{11}$$

where the m^2/M term was neglected because $m/M << 1$ so it is effectively zero, regardless of the constant it is multiplied by. For a parabolic orbit, the final energy must be zero, which is what is desired for this question. Setting E_f to zero gives and solving the above for v_c give

$$v_c = -\frac{\kappa}{2amv_p} \ . \tag{12}$$

Using this expression and the expression for the planetary velocity squared, Equation 5, allows the minimum necessary kinetic energy of the comet to be calculated,

$$T_c = \frac{1}{2} m v_c^2 = \frac{1}{8} \frac{m\kappa^2}{a^2 m^2 v_p^2} = \frac{1}{8} \frac{\kappa^2}{a^2 m} \left(\frac{M}{\kappa} \frac{a r_a}{2a - r_a} \right) = \frac{1}{8} \frac{\kappa M}{am} \left(\frac{r_a}{2a - r_a} \right) \ . \tag{13}$$

It can be shown geometrically that the aphelion distance $r_a = a(1+\epsilon)$. This makes the comet's kinetic energy

$$T_c = \frac{1}{8} \frac{\kappa M}{am} \left(\frac{a(1+\epsilon)}{2a - a(1+\epsilon)} \right) = \frac{1}{8} \frac{\kappa M}{am} \left(\frac{1 + (1-\alpha)}{2 - (1 + (1-\alpha))} \right) = \frac{1}{8} \frac{\kappa M}{am} \left(\frac{2-\alpha}{\alpha} \right) \ , \tag{14}$$

giving the final expression for the minimum kinetic energy required for the comet to bump the planet from an elliptical orbit to a parabolic one, to be

$$T_c = \frac{1}{8} \frac{\kappa M}{am} \left(\frac{2}{\alpha} - 1 \right) \ , \tag{15}$$

which is correct dimensionally. This kinetic energy varies inversely with α, which would imply that the closer the orbit is to hyperbolic (smaller values of α) means the comet would need infinite energy to bump the elliptic orbit up to a hyperbolic one, which does not seem right. Since the semi-major axis is also in the denominator this problem is fixed. As the orbit becomes more and more elliptical on its way to being a hyperbola, the semi-major axis increases. So as $\alpha \to 0$, $a \to \infty$, which fixes the problem.

3 Goldstein 3.13.

Consider a particle of mass m in the circular orbit shown, where the force is directed toward a point on the edge of the circle as shown in Figure 1. As the particle orbits, the force varies as the inverse power of distance from the origin. In order to show, this the first step is to determine coordinates for the orbit, obviously polar coordinates is the best choice, but the points on the orbit are not described by the standard polar coordinates. Start with the equation for a circle with an arbitrary center and a radius of a in Cartesian coordinates,

$$(x - x_0)^2 + (y - y_0)^2 = a^2 \ , \tag{16}$$

in this case the center is located at $(x, y) = (a, 0)$, making the circle equation

$$(x - a)^2 + y^2 = a^2 \ . \tag{17}$$

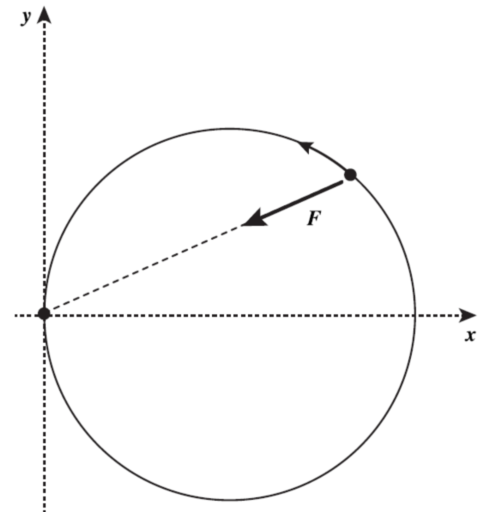

Figure 1: Depiction of the force and particle's orbit used in Problem #3.

Using the transformation for Cartesian coordinates from polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$, the equation for the orbit of the particle becomes

$$a^2 = (r\cos\theta - a)^2 + (r\sin\theta)^2 = r^2\cos^2\theta + a^2 - 2ar\cos\theta + r^2\sin^2\theta \ , \tag{18}$$

which gives the particles distance from the origin as a function of θ,

$$r^2 = 2ar\cos\theta \Rightarrow r = 2a\cos\theta \ , \tag{19}$$

from this it can be seen that for each full cycle of $\theta = 0 \to 2\pi$, there are two full orbits of the particle. When θ points to the left of the y-axis, it gives a negative r value, which points to the correct spot on the orbit on the right of the $y - axis$.

3.1 Finding the Force.

Following the derivation of the shape equation in class, let $u \equiv 1/r$ so,

$$u = \frac{1}{2a}\cos^{-1}\theta \tag{20}$$

$$u' = \frac{1}{2a}(-\cos^{-2}\theta)(-\sin\theta) = \frac{1}{2a}\frac{\sin\theta}{\cos^2\theta} \tag{21}$$

$$u'' = \frac{1}{2a}(\cos\theta)(\cos^{-2}\theta) + (\sin\theta)(-2\cos^{-3}\theta)(-\sin\theta) = \frac{1}{2a}\left[\frac{1}{\cos\theta} + \frac{2\sin^2\theta}{\cos^3\theta}\right] \ . \tag{22}$$

Goldstein Equation 3.12 gives the equation of motion for a particle in orbital motion under an arbitrary force in the radial direction, $f(r)$, which is

$$m\ddot{r} = \frac{l^2}{mr^3} + f(r) \ , \tag{23}$$

where l is the angular momentum. To solve this for f requires knowing the second time derivative of the r coordinate. Use the chain rule to redefine the time derivative operator,

$$\frac{d}{dt} = \frac{d}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d}{d\theta} = \frac{l}{mr^2}\frac{d}{d\theta} \ , \tag{24}$$

using the definition of angular momentum, Goldstein Equation 3.8. To find $\dot{r}$ use the redefined operator on r,

$$\dot{r} = \frac{d}{dt}[r] = \frac{l}{mr^2}\frac{d}{d\theta}[r] = \frac{lu^2}{m}\frac{d}{d\theta}\left[\frac{1}{u}\right] = \frac{lu^2}{m}\left[\frac{-1}{u^2}\frac{du}{d\theta}\right] = -\frac{l}{m}u' \ , \tag{25}$$

to find the second time derivative repeat the above operation on $\dot{r}$,

$$\ddot{r} = \frac{d}{dt}[\dot{r}] = \frac{l}{mr^2}\frac{d}{d\theta}[\dot{r}] = \frac{lu^2}{m}\frac{d}{d\theta}\left[-u'\frac{l}{m}\right] = -\frac{l^2u^2}{m^2}u'' \ . \tag{26}$$

Using these derivatives, Equation 23 becomes

$$\ddot{r} = \frac{l^2}{m^2r^3} + \frac{1}{m}f(r) \Rightarrow -\frac{l^2u^2}{m^2}u'' = \frac{l^2u^3}{m^2} + \frac{1}{m}f(r) \ , \tag{27}$$

which simplifies to

$$u'' = -u + \frac{m}{l^2u^2}f(r) \ . \tag{28}$$

To find $f(r)$, the quantity $u'' + u$ must be calculated from Equations 20 and 22,

$$u'' + u = \frac{1}{2a}\frac{1}{\cos\theta} + \frac{1}{2a}\left[\frac{1}{\cos\theta} + \frac{2\sin^2\theta}{\cos^3\theta}\right] = \frac{1}{2a\cos\theta}\left[1 + 1 + \frac{2\sin^2\theta}{\cos^2\theta}\right] = \frac{1}{a\cos\theta}\left[1 + \tan\theta\right] \ . \tag{29}$$

Simplifying the above equation, plugging in to Equation 28, and solving for the force yields

$$\frac{1}{a\cos^3\theta}\frac{l^2u^2}{m} = f(r) \ , \tag{30}$$

using the identity $1 + \tan\theta = \cos^{-2}\theta$. The first term in the above equation looks like the third power of u, times some constant coefficient:

$$\frac{1}{a\cos^3\theta} = Au^3 = \frac{A}{8a^3}\frac{1}{\cos^3\theta} \ , \tag{31}$$

so the coefficient is $8a^2$. Substituting this expression for the first term in Equation 30 yields the expression for $f(u)$,

$$f(u) = (8a^2u^3)\frac{l^2u^2}{m} \ . \tag{32}$$

Using the substitution that $u = 1/r$, the force as a function of distance from the origin is given by

$$f(r) = \frac{8l^2a^2}{m}r^{-5} \equiv \kappa r^{-5} \ , \tag{33}$$

which scales as the inverse fifth power of distance.

3.2 Energy of the Particle

The total energy of the particle is given by

$$E = T + U = T + \int f(r)dr = T - \kappa \frac{1}{4} r^{-4} \ . \tag{34}$$

The kinetic energy of a particle moving in both r and θ is given by,

$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) \ , \tag{35}$$

which will simplify after plugging in the definition of r from Equation 19. Note that

$$\dot{r} = 2a(-\sin\theta)\dot{\theta} = -\frac{2al}{mr^2}\sin\theta \Rightarrow \dot{r}^2 = \frac{4a^2l^2}{m^2r^4}\sin^2\theta \ , \tag{36}$$

which plugging in to the equation for total energy yields

$$E = \frac{1}{2}m\left(\frac{4a^2l^2}{m^2r^4}\sin^2\theta + 4a^2\cos^2\theta\frac{l^2}{m^2r^4}\right) - \frac{\kappa}{4}r^{-4} = \frac{2a^2l^2}{mr^4} - \frac{\kappa}{4r^4} = \frac{2a^2l^2}{mr^4} - \frac{8l^2a^2}{m}\frac{1}{4r^4} = 0 \ . \tag{37}$$

3.3 Period of the Orbit

To calculate the period of the orbit, we have to integrate the angular velocity over half a cycle of θ. Note that in the beginning of the problem, it was explained that for full cycle in θ the particle completes two orbits. This is due to the fact that when $0 \leq \theta \leq \frac{\pi}{2}$ the radial component traces out the half of the orbit with positive y values. During the region $\frac{\pi}{2} < \theta \leq \pi$ the radial component is negative so it traces out the bottom half of the circular orbit. For $\pi \leq \theta < \frac{\pi}{2}$ the radial component is still negative so it points to the positive x and positive y quadrant, tracing out the top half of the orbit again. Finally, on $\frac{3\pi}{2} \leq \theta < 2\pi$, the bottom half of the orbit is traced out again. Therefore a full orbit takes place from $0 \leq \theta \leq \pi$.

The angular velocity's relationship with angular momentum is given by

$$\frac{d\theta}{dt} = \dot{\theta} = \frac{l}{mr^2} = \frac{l}{4a^2m\cos^2\theta} = \frac{l}{2a^2m(1+\cos 2\theta)} \ , \tag{38}$$

which after rearranging and applying an integral becomes

$$\int_0^\tau dt = \frac{2a^2m}{l}\int_0^\pi d\theta(1+\cos 2\theta) \ . \tag{39}$$

Taking the integral yields an expression for the period,

$$\tau = \frac{2a^2m\pi}{l} \ . \tag{40}$$

3.4 Cartesian Velocities at Origin

The velocity of the particle and it's Cartesian projections are given by

$$\dot{x} = \frac{d}{dt}[2a\cos^2\theta] = -4a\cos(\theta)\sin(\theta)\dot{\theta} = -2a\dot{\theta}\sin(2\theta) \tag{41}$$

$$\dot{y} = \frac{d}{dt}[2a\cos\theta\sin\theta] = \frac{d}{dt}[a\sin(2\theta)] = 2a\dot{\theta}\cos(2\theta) \tag{42}$$

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} = 2a\dot{\theta} \ . \tag{43}$$

These simplify to functions of r and θ by eliminating $\dot{\theta}$,

$$\dot{x} - \frac{2al}{mr^2}\sin(2\theta) \tag{44}$$

$$\dot{y} = \frac{2al}{mr^2}\cos(2\theta) \tag{45}$$

$$v = \frac{2al}{mr^2} \ . \tag{46}$$

The center of the force is located at $r = 0$ and $\theta = (n\pi/2)^+$ for odd n. Taking the limits of these as the coordinates approach their appropriate values gives

$$\begin{aligned}
\lim_{r\to 0}\lim_{\theta\to\pi/2}\dot{x} &= -\frac{2al}{m}\lim_{r\to 0}\lim_{\theta\to\pi/2}\frac{\sin(2\theta)}{r^2} \\
&= -\frac{2al}{m}\lim_{r\to 0}\lim_{\theta\to\pi/2}\frac{2\cos(2\theta)}{2r} = -\frac{2al}{m}\lim_{r\to 0}\frac{2\cos(\theta)}{r} = \infty \\
\lim_{r\to 0}\lim_{\theta\to\pi/2}\dot{y} &= \frac{2al}{m}\lim_{r\to 0}\lim_{\theta\to\pi/2}\frac{\cos(2\theta)}{r^2} = \frac{2al}{m}\lim_{r\to 0}\frac{-1}{r^2} = \infty \\
\lim_{r\to 0} v &= \frac{2al}{m}\lim_{r\to 0}\frac{1}{r^2} = \infty \ .
\end{aligned}$$

4 Problem #4: Force in Electromagnetism.

Electromagnetism is also an attractive central force with $f(r) = -\kappa/r^2$ when the charges are opposite, but when they are the same it is repulsive, $f(r) = +\kappa/r^2$. In this case, all orbits are hyperbolic, which can be seen by investigating the energy of the particle. According to the table on Goldstein page 94, the energy of a hyperbolic orbit is always positive, so for any positive energy the orbit is always hyperbolic. For an orbit in polar coordinates, the effective potential for a general force in the center of mass frame is given by Goldstein Equation 3.22',

$$V_{eff} = V + \frac{l^2}{2\mu r^2} \ , \tag{47}$$

where μ is the reduced mass of the two like-charged particles interacting. For repulsive electromagnetism, this becomes

$$V_{eff} = \frac{\kappa}{r} + \frac{l^2}{2\mu r^2} \ , \tag{48}$$

making the total energy of a repulsive electromagnetic orbit,

$$E = T + V_{eff} = \frac{1}{2}\mu v^2 + \frac{\kappa}{r} + \frac{l^2}{2\mu r^2} \ , \tag{49}$$

since κ, μ, and r are all defined to be positive, the energy can never be negative. This implies every orbit is hyperbolic. This can be seen from deriving the shape equation for the repulsive force. Beginning with the equation of motion (in $u(r)$) given in the class notes,

$$u'' = -u - \frac{\mu}{l^2 u^2} f(r) \tag{50}$$

and putting in the repulsive E&M force gives

$$u'' = -u - \frac{\kappa\mu}{l^2} \ , \tag{51}$$

which can be solved by making the substitution $x \equiv u + \frac{\kappa\mu}{l^2}$ such that

$$x'' = -x \ , \tag{52}$$

note that x is a function of u, which is a function of r, which is a function of ϕ due to the redefinition of the time derivative operator, just as in Equation 24. The above differential equation can be solved by

$$x(\phi) = A\cos(\phi + \delta) \ , \tag{53}$$

where δ is an arbitrary phase. Using the definition of x and dropping the phase gives

$$u(\phi) = -\frac{\kappa\mu}{l^2} + A\cos\phi = \frac{\kappa\mu}{l^2}[\epsilon\cos\phi - 1] \ , \tag{54}$$

where ϵ is the eccentricity. By defining a length scale $r_a = \frac{l^2}{\kappa\mu}$, the eccentricity becomes $\epsilon = Ar_a$. Substituting these facts in and inverting gives the shape equation,

$$r(\phi) = \frac{r_a}{\epsilon\cos\phi - 1} \ . \tag{55}$$

From the shape equation, the angle between asymptotes of the arms of the hyperbola can be found. The hyperbola approaches its asymptotic values as $r \to \infty$. Solving the shape equation for ϕ gives the angle from the central axis to one of the asymptotic arms,

$$\epsilon \cos\phi = \frac{r_a}{r} + 1 \Rightarrow \phi = \arccos\left[\frac{1}{\epsilon}\left(\frac{r_a}{r} + 1\right)\right] \ , \tag{56}$$

and taking the limit,

$$\lim_{r\to\infty} \phi = \lim_{r\to\infty} \arccos\left[\frac{1}{\epsilon}\left(\frac{r_a}{r} + 1\right)\right] = \arccos\left[\frac{1}{\epsilon}\right] \ . \tag{57}$$

So the angle from the central axis is ϕ as $r \to \infty$, but the angle between the asymptotes, ψ is twice this amount,

$$\psi = 2\arccos\left[\frac{1}{\epsilon}\right] \ . \tag{58}$$

5 Problem #5: Particles on Cones.

Consider a particle of mass m moving along the surface of a cone, as in Figure 2, under constant downwards gravitational acceleration g. In cylindrical coordinates the kinetic energy of the particle is given by

$$T = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2] \ , \tag{59}$$

but the z coordinate can be constrained using α, half the opening angle. Consider the triangle formed by the line connecting the mass to the axis of the cone, the line connecting the mass to the origin (along the cone surface), and the line connecting the origin to the height of the mass along the z axis. From this triangle it can be seen that $\tan\alpha = r/z$, or $z = r\cot\alpha$, making the kinetic energy,

$$\begin{aligned} T &= \frac{1}{2}m(1+\cot^2\alpha)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 \\ &= \frac{1}{2}m(\csc^2\alpha)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 \ , \end{aligned}$$

and the potential energy of the mass is

$$U = mgz = mgr\cot\alpha \ . \tag{60}$$

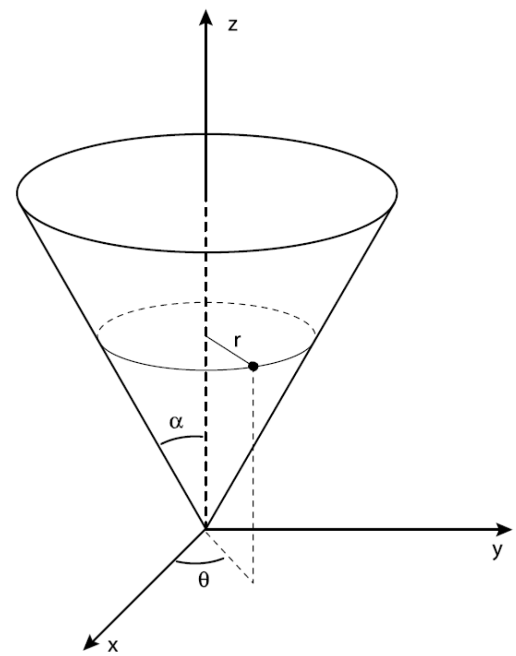

Figure 2: Depiction of coordinates for particle moving on the surface of a cone.

From this the following Euler-Lagrange equations can be found:

$$\frac{d}{dt}[mr^2\dot{\theta}] = 0 \tag{61}$$

$$\frac{d}{dt}[m(\csc^2\alpha)\dot{r} = mr\dot{\theta}^2 - mg\cot\alpha \ . \tag{62}$$

The first equation implies that $mr\dot{\theta}^2 = l$, where l is a constant angular momentum. This leads to an equation of motion for the particle on the surface of a cone,

$$\ddot{r} = \frac{l^2}{m^2r^3}\sin^2\alpha - g\sin\alpha\cos\alpha \ . \tag{63}$$

The total energy of this particle is $E = T + U$, which is

$$E = \frac{1}{2}m(\csc^2\alpha)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cot\alpha = \frac{1}{2}m(\csc^2\alpha)\dot{r}^2 + \frac{l^2}{2mr^2} + mgr\cot\alpha \ , \tag{64}$$

which leads directly to the effective potential, in this case the sum of the actual potential and the centrifugal terms,

$$V_{eff} = \frac{l^2}{2mr^2} + mgr\cot\alpha \ , \tag{65}$$

which is plotted in Figure 3, as well as a line depicting a constant energy E_0. Turning points are the points in space at which a particle with a specific energy will turn around because it cannot cross the potential barrier, *i.e.* when $E + V_{eff}$. For a constant energy E_0 this occurs if

$$E_0 = \frac{l^2}{2mr^2} + mgr\cot\alpha \ , \tag{66}$$

which becomes a cubic polynomial by multiplying through by r^2,

$$0 = mgr^3\cot\alpha - E_0 r^2 + \frac{l^2}{2m} \ . \tag{67}$$

This polynomial has three roots, but only two have a physical importance. The plot in Figure 3 shows that the constant energy line intersects with the effective potential twice, maximum (except if the energy is the absolute minimum of the effective potential, and energy less than that is not allowed). This must mean that there are two real roots and one imaginary root. The real roots then correspond to the two turning points in the well of the potential. Note that for a cone, $0 < \alpha < \pi/2$, so $\cot\alpha$ will always be a positive constant. Therefore, by looking at the constraint on the z coordinate, a maximum z value occurs at a maximum r value, likewise for minimum values. Figure 3 shows that, for a constant energy, a particle will be in a stable orbit between some r_{min} and some r_{max}. Which by reducing the system to two dimensions using the symmetry about the z axis implies that the particle orbits between the point (r_{max}, z_{max}) and a point (r_{min}, z_{min}) along the line connecting them through the center of the triangle. Now, reintroducing the third dimension requires the particle to travel between the points $(r_{max}, z_{max}, \theta_1)$ and $(r_{min}, z_{min}, \theta_2)$ along the surface of the cone. Projecting this orbit onto the $x - y$ plane would result in an ellipse.

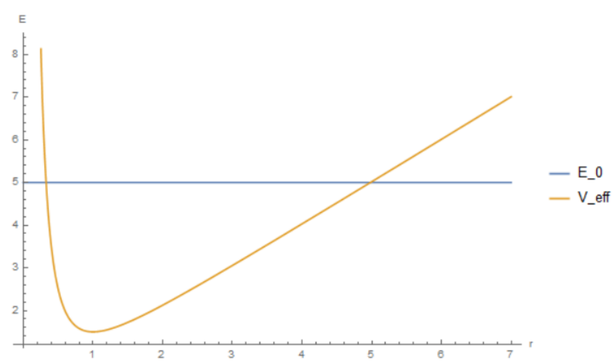

Figure 3: Plot of an effective potential (yellow) and a particle with constant energy (blue). Note the locations of the turning points r_{min} and r_{max} where the lines intersect.

6 Problem #6: Numerical Treatments of Orbits.

Consider a particle with mass m and angular momentum l in a central force field of form

$$F = -\frac{\kappa}{r^{5/2}} \ , \tag{68}$$

where κ is the coupling constant for the force, and r is the particle distance from the force center. The total energy for a particle moving in this central force, in polar coordinates is

$$E = T + U = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2] + \int(-F)dr = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2] - \int F dr \ . \tag{69}$$

because $-\nabla U = F$. Therefore the total energy is

$$E = T + U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \left[-\frac{-2}{3}\frac{\kappa}{r^{3/2}}\right] . \tag{70}$$

From this the effective potential can be read off as the sum of the true potential and centrifugal terms of the total energy,

$$V_{eff} = \frac{1}{2}mr^2\dot{\theta}^2 - \frac{2}{3}\frac{\kappa}{r^{3/2}} = \frac{l^2}{2mr^2} - \frac{2}{3}\frac{\kappa}{r^{3/2}} \ , \tag{71}$$

using the definition of angular momentum. From this point forward, the units will be changed such that, $m = l = \kappa = 1$, which makes the expression for effective potential

$$V_{eff} = \frac{1}{2r^2} - \frac{2}{3}\frac{1}{r^{3/2}} \ , \tag{72}$$

shown in Figure 4. At a distance r_0 this effective potential takes a minimum, which can be found by setting its derivative to zero,

$$\frac{dV_{eff}}{dr} = 0 = -\frac{1}{r_0^3} + \frac{1}{r_0^{5/2}} \Rightarrow 1 = r_0^{1/2} \ . \tag{73}$$

Therefore the distance r_0 that V_{eff} takes its minimum value is 1. It was shown in Section 5 that turning points are located at $E = V_{eff}(r_{min}) = V_{eff}(r_{max})$. For this potential, the turning points of a particle with $E = -0.1$ can be found by solving the equation,

$$-0.1 = \frac{1}{2r_m^2} - \frac{2}{3}\frac{1}{r_m^{3/2}} \ , \tag{74}$$

which can be done numerically using Mathematica. Issuing Mathematica the commands

```
U[r_] := 1/(2 r^2) - 2/(3 r^(3/2));
NSolve[U[rm] == -0.1, rm]
```

returns the two values of r_m that satisfy Equation 74, the smaller of which is $r_{min} = 0.667079$ (note that $r_{max} = 2.22214$). By working with angle ϕ in the orbital plane rather than time, the equation of motion can be transformed into the form

$$u'' = -u - \frac{\mu}{l^2u^2}F \ , \tag{75}$$

where $u(\phi) \equiv 1/r(\phi)$ and the primes denote derivatives with respect to ϕ. Which for the force given in Equation 68, and again setting constants to 1, becomes

$$u'' = -u + \frac{1}{1u^2} u^{5/2} = -u + u^{1/2} \; . \tag{76}$$

This can be solved numerically given the initial conditions. Assume that when $\phi = 0$, $r = r_{min}$, this gives $u' = \frac{d}{d\phi}(1/r) = -(1/r^2)r'$. There is no radial velocity (with respect to time or angle) when the particle is at $r = r_{min}$ because at this point the effective potential is equal to its energy. From this, the initial conditions are

$$u[0] = 1/r_{min} \qquad ; \qquad u'[0] = 0 \; . \tag{77}$$

This can be done using the MATHEMATICA command `NDSolve[]` as follows:

```
s = NDSolve[{u''[\[Phi]] == -u[\[Phi]] + (u[\[Phi]]^(5/2)/u[\[Phi]]^2),
            u[0] == 1/0.667079, u'[0] == 0}, u, {\[Phi], 0, 7 \[Pi]}];
r = 1/Evaluate[u[\[Phi]] /. s];
```

The solution to this is a numerical list of u values for a range of ϕ's. The orbit can be plotted in the $x - y$ plane by noting that

$$x = r\cos\phi = \frac{\cos\phi}{u(\phi)} \qquad ; \qquad y = r\sin\phi = \frac{\sin\phi}{u(\phi)} \; , \tag{78}$$

as shown in Figure 5. This was plotted using the MATHEMATICA command

```
ParametricPlot[Evaluate[ {Cos[\[Phi]] / u[\[Phi]], Sin[\[Phi]] / u [\[Phi]]} /.s] ,
                 {\[Phi], 0, 7 \[Pi]}]
```

and shown in Figure 5.

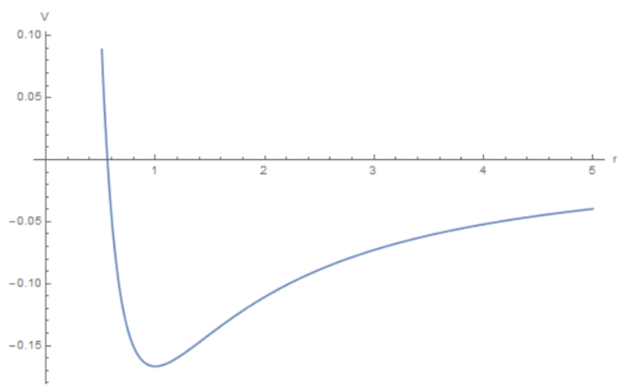

Figure 4: Effective potential for a particle moving under a force $F = -\kappa \; r^{5/2}$.

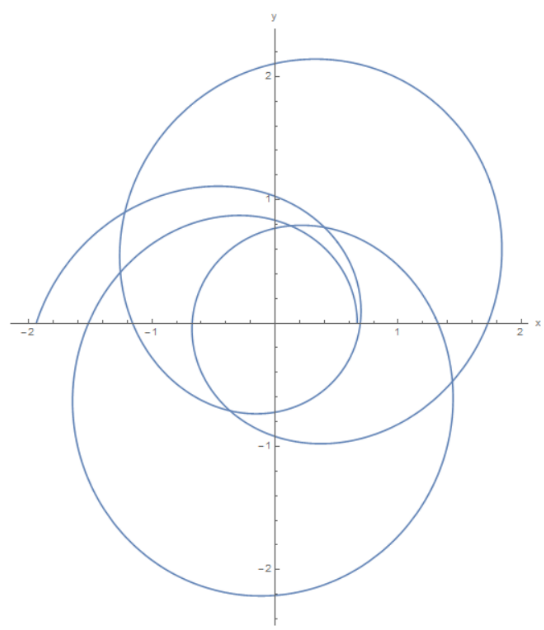

Figure 5: Projection of the orbit into the $x - y$ plane.