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Contents

1 Goldstein 3.24.

1.1 Radial Speed of Elliptical Orbit

The total energy of a particle of mass m in a gravitational field with an elliptical orbit of eccentricity ϵ and semi-major axis a is given by

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{\kappa}{r} \ , \tag{1}$$

where μ is the reduced mass of m and the mass creating the gravitational field M. But the energy of an elliptical orbit, from solving Goldstein Equation 3.57 for energy, makes Equation 1

$$-\frac{\kappa^2\mu}{2l^2}(1-\epsilon^2) = \frac{1}{2}\mu(\dot{r}^2 + r^2\frac{l^2}{\mu^2 r^4}) - \frac{\kappa}{r} \ , \tag{2}$$

where l is the particle's angular momentum about the origin, and the definition of angular momentum was used, $\dot{\phi} = l/(\mu r^2)$. This can be solved for $\dot{r}$,

$$\dot{r}^2 = -\frac{\kappa^2}{l^2}(1-\epsilon^2) + \frac{2\kappa}{\mu r} - \frac{l^2}{\mu^2 r^2} \ , \tag{3}$$

the two last terms can be combined to be $(2\kappa\mu r - l^2)/(\mu^2 r^2)$. At this point note that $\kappa = GmM$, which in terms of the reduced mass is

$$\kappa = G(m+M)\mu \qquad \Rightarrow \qquad \frac{\kappa}{\mu} = G(m+M) \ . \tag{4}$$

It is also necessary to note that the Harmonic Law can be written as,

$$\frac{\tau^2}{a^3} = \frac{4\pi^2}{G(M+m)} \qquad \Rightarrow \qquad G(M+m) = \frac{4\pi^2}{\tau^2}a^3 = \omega^2 a^3 \ , \tag{5}$$

by noting that the frequency is 2π divided by the orbital period τ. With these facts, the ratio of the coupling constant to the reduced mass can be written as

$$\frac{\kappa}{\mu} = \omega^2 a^3 \ , \tag{6}$$

which makes Equation 3 into

$$\dot{r}^2 = -\frac{\kappa^2}{l^2}(1-\epsilon^2) + \frac{2\mu^2\omega^2 a^3 r - l^2}{\mu^2 r^2} \ . \tag{7}$$

By introducng the semilatus rectum $r_s = l^2/(\kappa\mu)$ the above equation becomes

$$\begin{aligned}
\dot{r}^2 &= -\frac{\kappa}{r_s\mu}(1-\epsilon^2) + \frac{2\mu^2\omega^2 a^3 r - r_s\kappa\mu}{\mu^2 r^2} && (8)\\
&= -\frac{\omega^2 a^3\mu}{r_s\mu}(1-\epsilon^2) + \frac{2\mu^2\omega^2 a^3 r - r_s\omega^2 a^3\mu^2}{\mu^2 r^2} && (9)\\
&= -\frac{\omega^2 a^3}{r_s}(1-\epsilon^2) - \frac{r_s\omega^2 a^3 - 2\omega^2 a^3 r}{r^2} && (10)\\
&= \omega^2 a^3\left[-\frac{1}{r_s}(1-\epsilon^2) - \frac{r_s - 2r}{r^2}\right] \ . && (11)
\end{aligned}$$

The semilatus rectum can be removed from this equation by using its definition $r_s = a(1-\epsilon^2)$,

$$\dot{r}^2 = \omega^2 a^3 \left[-\frac{1}{a} - \frac{a(1-\epsilon^2) - 2r}{r^2} \right] \tag{12}$$

$$= \omega^2 a^3 \left[\frac{-r^2 - a^2 + a^2\epsilon^2 + 2ar}{ar^2} \right] \tag{13}$$

$$= \frac{\omega^2 a^2}{r} [a^2\epsilon^2 - (r-a)^2] \ , \tag{14}$$

which gives the equation for radial speed,

$$\dot{r} = \frac{\omega a}{r} \sqrt{a^2\epsilon^2 - (r-a)^2} \ . \tag{15}$$

By introducing the eccentric anomaly ψ in place of r, the differential equation immediately yields the Kepler equation.

1.2 Eccentric Anomaly and the Kepler Equation

The eccentric anomaly's relationship to the radial coordinate r is given by

$$r = a(1 - \epsilon \cos\psi) \ , \tag{16}$$

where a is the semi major axis of the elliptical orbit and ϵ is the eccentricity of the ellipse. The time derivative of the radial coordinate is $\dot{r} = a\epsilon\dot{\psi}\sin\psi$. Performing this substitution on Equation 15 and squaring both sides yields

$$\begin{aligned} a^2\epsilon^2\dot{\psi}^2 \sin^2\psi &= \frac{\omega^2 a^2}{a^2(1-\epsilon\cos\psi)^2} [a^2\epsilon^2 - (a(1-\epsilon\cos\psi) - a)^2] \\ &= \frac{\omega^2}{1 - 2\epsilon\cos\psi + \epsilon^2\cos^2\psi} [a^2\epsilon^2 - (a\epsilon\cos\psi)^2] \\ &= \frac{\omega^2 a^2 \epsilon^2}{1 - 2\epsilon\cos\psi + \epsilon^2\cos^2\psi} [1 - \cos^2\psi] \ , \end{aligned}$$

which by noting $1 - \cos^2\psi = \sin^2\psi$ becomes

$$\dot{\psi}^2 = \frac{\omega^2}{(1-\epsilon\cos\psi)^2} \ , \tag{17}$$

which by taking the square root and making the substitution $\dot{\psi} = \frac{d\psi}{dt}$, then rearragning gives the equation

$$d\psi(1 - \epsilon\cos\psi) = \omega dt \ . \tag{18}$$

This equation can be integrated directly,

$$\int d\psi(1 - \epsilon\cos\psi) = \int \omega dt \tag{19}$$

$$\psi - \epsilon\sin\psi = \omega t \ , \tag{20}$$

which is the Kepler equation, $\mathcal{M} = \psi - \epsilon\sin\psi$.

2 Goldstein 3.21.

Consider the potential field

$$V(r) = -\frac{\kappa}{r} + \frac{h}{r^2} \ , \tag{21}$$

where h is a constant. In this potential the motion of a particle of mass m is the same as that of the motion under the Kepler potential alone, when expressed in terms of a coordinate system rotating or precessing around the center of force.

2.1 Kepler Potential in Precessing Coordinate System

The Lagrangian for a particle in this potential is

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + \frac{\kappa}{r} - \frac{h}{r^2} = \frac{1}{2}m\dot{r}^2 + \frac{l^2 - 2mh}{2mr^2} + \frac{\kappa}{r} \ , \tag{22}$$

using the definition of anglar momentum. Now consider a particle moving only in the Kepler potential but in a rotating frame. In Cartesian coordinates, this transformation is just a rotation,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \ , \tag{23}$$

where the primed coordinates are the rotated coordinates. The rotated coordinates give a polar radial coordinate

$$r'^2 = x'^2 + y'^2 = (x\cos\phi - y\sin\phi)^2 + (x\sin\phi + y\cos\phi)^2 = x^2 + y^2 = r \ , \tag{24}$$

so the radial coordinate is unchanged, but the angular momentum is. The angular momentum in this frame is the angular momentum of the particle plus the angular momentum imparted by the rotating frame, alternatively, the angular momentum of the orbit's precession. This gives the new coordinates (using angular momentum as an analog for the angular coordinate) to be

$$r' = r \ , \qquad l' = l - l_{frame} \ , \tag{25}$$

the minus sign is due to the frame moving in the same direction as the particle, so in this coordinate frame the particle would have less angular momentum. This gives the Lagrangian of a particle in the Kepler potential in a rotating frame to be

$$\mathcal{L}' = \frac{1}{2}m\dot{r}'^2 + \frac{l'^2}{2mr'^2} + \frac{\kappa}{r'} = \frac{1}{2}m\dot{r}^2 + \frac{l^2 + l_f^2 - 2ll_f}{2mr^2} + \frac{\kappa}{r} \ . \tag{26}$$

This is the same form as Equation 22, and by comparison it is easy to see

$$h = -\frac{1}{2m}(l_f^2 - 2ll_f) \qquad \Rightarrow \qquad 0 = l_f^2 - 2ll_f + 2hm \ . \tag{27}$$

2.2 Angular Speed of Precession

If the angular speed of precession of the elliptical orbit is given by $\dot{\Omega}$, then the angular momentum due to precession is $l_f = mr^2\dot{\Omega}$. Solving Equation 27 and equating it to the previous expression (in terms of $\dot{\Omega}$) gives

$$mr^2\dot{\Omega} = l \pm \sqrt{l^2 - 2hm} = l\left(1 \pm \sqrt{1 - (2hm/l^2)}\right) \ . \tag{28}$$

In the limit that the additional potential term is small compared to the Kepler potential term, $h \to 0$, which allows the above equation to be expanded to first order in h,

$$mr^2\dot{\Omega} = l\left[1 - \left(1 - \frac{hm}{l^2}\right)\right] \qquad \Rightarrow \qquad \dot{\Omega} = \frac{1}{mr^2}\frac{hm}{l} = \frac{h}{r^2 l} \ . \tag{29}$$

By introducing another factor of l, the r^2 factor can be removed from this expression. The definition of angular momentum is $l = mr^2\dot{phi}$, so by multiplying the numerator and denominator both by l, the angular speed of precession becomes

$$\dot{\Omega} = \frac{hl}{r^2 l^2} = \frac{hmr^2\dot{\phi}}{r^2 l^2} = \frac{hm\dot{\phi}}{l^2} = \frac{2\pi hm}{\tau l^2} \ , \tag{30}$$

by noting the orbital period t is 2π divided by the orbital speed.

2.3 Precession of Mercury

The perihelion of Mercury is observed to precess (after correction for known planetary perturbations) at a rate of about 40" per century. Show that this precession could be accounted for classically if the dimensionless quantity $\eta = h/(\kappa a) \simeq 7 \times 10^{-8}$. The eccentricity of Mercury's orbit is 0.206, and its period is 0.24 years. Using the two definitions of the semilatus rectum, $a(1-\epsilon^2) = l^2/(\kappa\mu)$, the speed of precession can be rewritten as

$$\dot{\Omega} = \frac{2\pi}{\tau}hm\frac{1}{a\kappa\mu(1-\epsilon^2)} \ , \tag{31}$$

but because the mass of Mercury is negligible compared to the sun, $\mu \simeq m$, making the above equation

$$\dot{\Omega} = \frac{2\pi}{\tau}hm\frac{1}{a\kappa m(1-\epsilon^2)} = \frac{2\pi}{\tau}\frac{\eta}{(1-\epsilon^2)} \ , \tag{32}$$

which after plugging in the appropriate values yields $\dot{\Omega} = 1.9138110^{-6}$ radians per year which is 39.4572" per century.

3 Problem #3: Yukawa Potentials.

A modified form of the Coulomb potential was explored in the 1930s by Japanese physicist and Nobel Laureate, Hideki Yukawa. Consider a particle moving in a central force field where the potential is of the Yukawa form

$$V(r) = -k\frac{\exp[-r/a]}{r} \ , \tag{33}$$

where k and a are positive constants.

3.1 Equations of Motion

This potential gives a Lagrangian,

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + k\frac{\exp[-r/a]}{r} \ , \tag{34}$$

which is cyclic in ϕ, yielding the standard form the relationship between angular momentum and angular vevlocity, $\dot{\phi} = \frac{l}{mr^2}$. The reduced, one dimensional equation of motion is

$$m\ddot{r} = -\frac{l^2}{mr^3} - k\frac{\exp[-r/a]}{r}\left[\frac{1}{r} + \frac{1}{a}\right] \ . \tag{35}$$

It is easy to see from the Lagrangian (the total energy would have a negative potential term), the effective potential is

$$V_{eff} = \frac{l^2}{2mr^2} - k\frac{\exp[-r/a]}{r} \ , \tag{36}$$

therefore the turning points are at $V_{eff} = E$ and minima and maxima are determined by

$$\frac{dV_{eff}}{dr} = 0 = -\frac{l^2}{mr^3} + \frac{ke^{-\frac{r}{a}}}{r^2} + \frac{ke^{-\frac{r}{a}}}{ar} \ , \tag{37}$$

which can be rewritten as

$$\frac{l^2}{m\kappa} = r\exp[-r/a]\left(1 + \frac{r}{a}\right) \ , \tag{38}$$

which does not have an analytic solution. To visualize the solutions to this, the derivative with respect to r is taken again,

$$0 = \frac{\exp[-r/a]}{a^2}(a^2 + ar + r^2) \ , \tag{39}$$

so for Equation 38 to have a solution, it must be that $a^2 + ar + r^2 = 0$, which occurs at $r_m/a = \frac{1\pm\sqrt{5}}{2}$. Since $\sqrt{5} > 1$ and $r > 0$, the solution must be the positive one. The right hand side of Equation 38 takes a maximum value when r/a is this solution,

$$a\frac{r_m}{a}\exp[-r_m/a]\left(1 + \frac{r_m}{a}\right) = 0.839962a \ , \tag{40}$$

thanks to Mathematica. Therefore if $l > \sqrt{0.839962am\kappa}$ there are no extrema and there are no bound states or circular orbits. At the critical value $l_c^2 = 0.839962am\kappa$ there exists one point where the derivative is zero, which would result in one unstable orbit. For $l < l_c$ there is both a local minimum and maximum of the effective potential which gives rise to a stable and unstable circular orbit, and bound orbits for energies between the local minimum and maximum of the effective potential. The limits on the energy are such that bound orbits exist for the condition previously mentioned, while for energies below the minimum and above the maximum, all motion is unbounded.

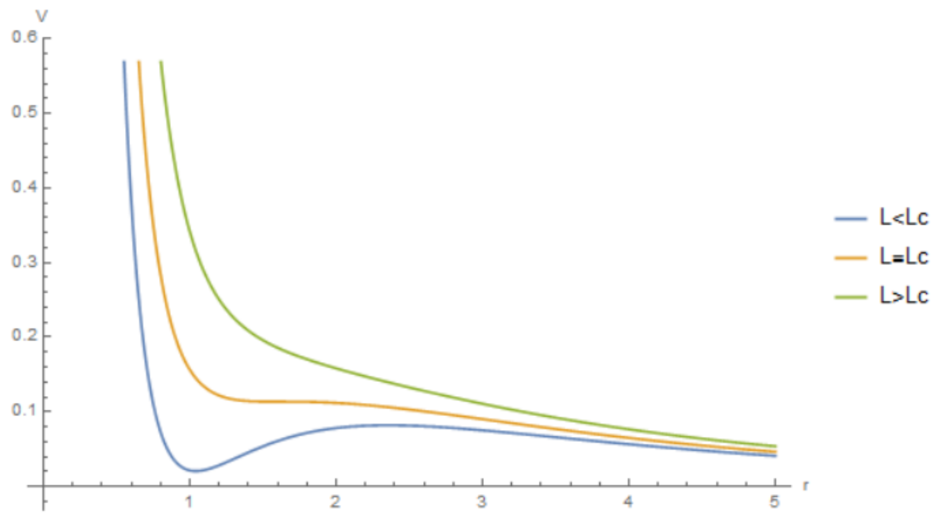

Figure 1: The three cases for l and its effect on V_{eff}.

3.2 Small Oscillations of Circular Orbits

As previously stated, circular orbits are possible only when the orbiting particle has energy exactly equal to the minimum of the potential (stable) or equal to the maximum of the potential (unstable). The angular momentum must be below the critical value for two circular orbits to exist. When it is at its critical value, only one unstable circular orbit exists. For values of $l < l_c$ there are two locations that allow circular orbits, and the stable orbit will occur with radius $\rho < a$. For this orbit Equation 38 will be satisfied, and can be rewritten as

$$\frac{l^2}{m\kappa} = \rho \exp[-\rho/a] \left(1 + \frac{\rho}{a}\right) \ , \tag{41}$$

these orbits occur at $E = V_{min}$. Orbits with energies slightly above this are still bound and their radial coordinate is given by $r = \rho + \delta r$, where δr is a small perturbation to the orbit. As in the derivation of the shape equation, the substitution $u = 1/r$ is made, so that $u = u_\rho + \delta u$, so that $u'' = \delta u''$. The orbit equation for the standard u coordinate is

$$u'' + u = -\frac{m}{l^2 u^2}\frac{d}{dr}V(r) = -\frac{m}{l^2 u^2}\frac{1}{r^2}\frac{d}{du}V(r) = -\frac{m}{l^2}\frac{d}{du}V(1/u) \equiv f(u) \ , \tag{42}$$

and making the substitution for u becomes

$$\delta u'' + u_\rho \delta u = f(u_\rho + \delta u) = f(u_\rho) + \delta u f(u_\rho) \ , \tag{43}$$

where $f(u)$ was expanded around $u = u_\rho$. The above equation, to zeroth order in u says $u_\rho = f(u_\rho)$. Which makes the first order expansion

$$\delta u'' + \delta u = f'(u_\rho)\delta u \qquad \Rightarrow \qquad \delta u'' + (1 - f'(u_\rho))\delta u = 0 \ , \tag{44}$$

if $(1 - f'(u_\rho)) \equiv \omega_{osc}^2$, this takes the form of an oscillator with frequency ω_{osc}. This has the solution $\delta u = A\cos(\omega_{osc}\theta)$. The frequency of small oscillations of the orbit can be found by finding $f'(u_\rho)$, but first, the function f must be determined,

$$f(u) = -\frac{m}{l^2}\frac{d}{du}V(u) = -\frac{m}{l^2}\frac{d}{du}[-ku\exp(-1/au)] = \frac{mk}{l^2}\exp(-1/au)\left[1 + \frac{1}{au}\right] \ , \tag{45}$$

but from the first order approximation,

$$u_\rho = f(u_\rho) = \frac{mk}{l^2}\exp(-1/au_\rho)\left[1+\frac{1}{au_\rho}\right] \quad \Rightarrow \quad \exp(-1/au_\rho) = \frac{l^2}{mk}u_\rho\frac{1}{1+(1/au_\rho)} \ . \tag{46}$$

Next, the first derivative of f must be determined (using Mathematica),

$$f'(u) = \frac{mk}{l^2}\exp[-1/au]\frac{1}{a^2u^3} \ , \tag{47}$$

which evaluated at u_ρ says

$$f'(u_\rho) = \frac{mk}{l^2a^2u_\rho^3}\exp[-1/au] = \frac{mk}{l^2a^2u_\rho^3}\frac{l^2}{mk}u_\rho\frac{1}{1+(1/au_\rho)} = \left(\frac{1}{au_\rho}\right)^2\frac{1}{1+(1/au_\rho)} \ . \tag{48}$$

From all of the above steps, the frequency of small oscillations is determined to be

$$\omega_{osc} = \left[1-\left(\frac{1}{au_\rho}\right)^2\frac{1}{1+(1/au_\rho)}\right]^{1/2} = \left[1-\left(\frac{1}{au_\rho}\right)\frac{1}{au_\rho+1}\right]^{1/2} \ , \tag{49}$$

which is an exact solution. In order to get a more intelligible form, some approximations must be made, and will differ for the small, stable orbit, and the larger, unstable orbit.

3.2.1 Stable Orbit

For the small orbit, at or near the minimum of the effective potential, $au_\rho \gg 1$, so the last term in Equation 49 is approximately $1/(au_\rho)$. Using the binomial expansion, this becomes

$$\omega_{osc} = \left[1-\left(\frac{1}{a^2u_\rho^2}\right)\right]^{1/2} = 1-\left(\frac{1}{2a^2u_\rho^2}\right) \ . \tag{50}$$

For a complete revolution, $\Delta(\theta\omega_{osc}) = 2\pi$, the particle passes its original location by an angle ℓ, so that

$$\Delta\theta = 2\pi + \ell \ . \tag{51}$$

Note that because ω_{osc} is a constant, $\Delta(\theta\omega_{osc}) = \omega_{osc}\Delta\theta$, yielding

$$\frac{2\pi}{\omega_{osc}} \simeq 2\pi\left[1+\left(\frac{1}{2a^2u_\rho^2}\right)\right] = 2\pi + \frac{\pi\rho^2}{a^2} = 2\pi + \ell \ , \tag{52}$$

which says the major axis advances by an angle $\ell = \frac{\pi\rho^2}{a^2}$ each period of the particle's orbit.

3.2.2 Unstable Orbit

For the large orbit, at or near the maximum of the effective potential, $au_\rho \ll 1$, so the last term in Equation 49 is approximately 1. Using the binomial expansion, this becomes

$$\omega_{osc} = \left[1-\left(\frac{1}{au_\rho}\right)\right]^{1/2} = 1-\left(\frac{1}{2au_\rho}\right) \ , \tag{53}$$

and using the same logic as for the stable orbit, the major axis advances by an angle $\ell = \frac{\pi\rho}{a}$ each period of the particle's orbit.

4 Goldstein 3.33.

A particle of mass m is constrained to move under gravity without friction on the inside of a vertical paraboloid of revolution.

4.1 Equation of Motion

The equation of a symmetric paraboloid in cylindrical coordinates is $z = ar^2$, where a is a constant to make the units match. It must have units of inverse length. The generalized coordinates for this system are r and *phi* because z is constrained. Therefore the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + (2ar\dot{r})^2) - mgar^2 \ , \tag{54}$$

which is cyclic in ϕ, so $\dot{\phi}$ is a constant, which allows the angular speed to be defined in terms of angular momentum, $\dot{\phi} = l/(mr^2)$. The Lagrangian is then reduced to one dimension,

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + 2a^2mr^2\dot{r}^2 - mgar^2 \ , \tag{55}$$

so the radial equation can be found using the Euler-Lagrange equations,

$$m\ddot{r} + \frac{d}{dt}\left[2a^2mr^2\dot{r}^2\right] = -\frac{l^2}{mr^3} + 4a^2mr\dot{r}^2 - 2amgr \tag{56}$$

$$m\ddot{r} + 4a^2mr\dot{r}^3 + 4a^2mr^2\dot{r}\ddot{r} = -\frac{l^2}{mr^3} + 4a^2mr\dot{r}^2 - 2amgr \tag{57}$$

$$m\ddot{r} = \frac{2am^2r^4\left(2a\left(\dot{r}-1\right)\dot{r}^2 + g\right) + l}{m^2r^3\left(4a^2r^2\dot{r}+1\right)} \ . \tag{58}$$

Note the effective potential is

$$V_{eff} = \frac{l^2}{2mr^2} + 2a^2mr^2\dot{r}^2 + amgr^2 \ . \tag{59}$$

4.2 Initial Velocity Conditions

In order to find circular orbits around the inside of the paraboloid, there must be some condition the initial velocity of the mass must satisfy. Obviously there can be no radial motion, which implies no motion in the z direction. Because this is a stable orbit, and the angular speed is constant, the time average of the kinetic energy is just the kinetic evergy, likewise for the potential, so the virial theorem holds. The potential energy scales as r^2, so $n = 1$, making the virial theorem,

$$\langle T\rangle = \frac{n+1}{2}\langle U\rangle \qquad \Rightarrow \qquad T = U \ . \tag{60}$$

Using the standard form for angular kinetic energy, this becomes

$$\frac{1}{2}m\dot{\phi}^2r^2 = amgr^2 \ , \tag{61}$$

which says the initial angular velocity must be $\dot{\phi} = \sqrt{2ag}$, which has the correct units. In order to get the initial tangetal velocity for the mass to maintain a circular orbit, it must have an equilibrium value of r, which can be found by minimizing the effective potential. Note that at the equilibrium radius, the radial velocity is zero because it is at a circular orbit so any $\dot{r}$ terms drop out of the effective potential,

$$0 = \left.\frac{dV_{eff}}{dr}\right|_{r_{eq}} = -\frac{l^2}{mr_{eq}^3} + 2amgr_{eq} \ , \tag{62}$$

which has a solution $r_{eq} = (l^2/2am^2g)^{1/4}$. So the initial tangental velocity is

$$v_i = r_{eq}\dot{\phi} = \left(\frac{2l^2ag}{m^2}\right)^{1/4} \ . \tag{63}$$

4.3 Small Oscillations of Circular Motion

For small oscillations in the circular orbit, δr, the generalized coordinates become $\rho + \delta r$, where ρ is the equilibrium point in the original radial coordinate. The effective potential can be expanded in a Taylor series about the stable circular orbit's radius ρ,

$$V(\delta r)|_\rho = V(\rho) + V'(\rho)[(\rho+\delta r) - \rho] + \frac{1}{2}V''(\rho)[(\rho+\delta r) - \rho]^2 + O(\delta r^3) \ , \tag{64}$$

but the first derivative term is zero because the circular oribit is stable. Note that since the orbit is close to circular any $\dot{r}$ terms drop out of the effective potential. Terms of order δr^3 and higher are neglected, which makes the above potential into

$$V(\delta r)|_\rho = \left[\frac{l^2}{2m\rho^2} + amg\rho^2\right] + \frac{1}{2}(\delta r)^2\left[\frac{3l^2}{m\rho^4} + 2amg\right] = \left[\frac{l^2}{2m\rho^2} + amg\rho^2\right] + \frac{1}{2}m(\delta r)^2\left[\frac{3l^2}{m^2\rho^4} + 2ag\right] \ . \tag{65}$$

This potential takes the form of a harmonic oscillator potential with an offset, $V(r) = A + \frac{1}{2}m\omega^2 r^2$. Therefore the frequency of oscillation for small perturbations is the square root of the coefficient $(\delta r)^2$. The period then is 2π over this value

$$\tau = 2\pi\left[\frac{3l^2}{m^2\rho^4} + 2ag\right]^{(-1/2)} \ . \tag{66}$$

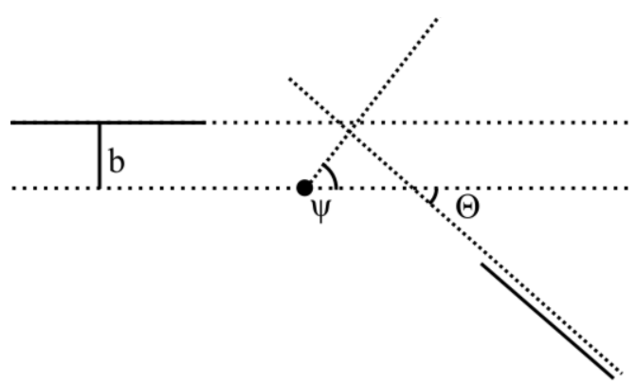

Figure 2: Diagram for Rutherford scattering from an attractive Coloumb potential for Problem # 5.

5 Goldstein 3.30.

Rutherford scattering from an attractive Coloumb potential should behave exactly the same quantitatively as a repulsive Coloumb potential because the physics behind both scenarios is the same expect for a sign change. The particle will still react to the source with the same magnitude, it will just be deflected towards the source instead of away, see Figure 2. To derive the scattering cross section for the attractive case, the method used by Goldstein for the repulsive case will be followed. Starting with Goldstein Equation 3.100, but using $k = +ZZ'e^2$ gives

$$\frac{1}{r} = -\frac{mk}{l^2}(\epsilon \cos\theta - 1) = \frac{mk}{l^2}(1 - \epsilon\cos\theta) \ . \tag{67}$$

The direction of the incoming asymptote, ψ, is determined by taking the limit as $r \to \infty$,

$$\cos\psi = \frac{1}{\epsilon} \ , \tag{68}$$

which only allows $\epsilon > 1$, so all orbits are hyperbolic. Using the fact that $\Theta = \pi - 2\psi$, this equation becomes

$$\frac{1}{\epsilon} = \cos\left[\frac{1}{2}(\pi - \Theta)\right] = \sin\frac{\Theta}{2} \ , \tag{69}$$

which can be inverted and squared,

$$\csc^2\frac{\Theta}{2} = \epsilon^2 \ , \tag{70}$$

now subtracting 1 from both sides and noting that $\csc^2\alpha - 1 = \cot^2\alpha$, this becomes

$$\cot^2\frac{\Theta}{2} = \epsilon^2 - 1 \ . \tag{71}$$

Using the relationship for eccentricity and energy E, Goldstein Equation 3.99, this becomes

$$\cot^2\frac{\Theta}{2} = \left[\frac{2Eb}{k}\right]^2 \qquad \Rightarrow \qquad \cot\frac{\Theta}{2} = \frac{2Eb}{k} \ , \tag{72}$$

which can be solved fro impact parameter b,

$$b = \frac{k\cot(\Theta/2)}{2E} \ , \tag{73}$$

which is needed, along with it's angular derivative, to find the differntial cross section,

$$\sigma(\Theta) = \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right| = \frac{b}{2\sin(\Theta/2)\cos(\Theta/2)}\left|\frac{db}{d\Theta}\right| \ , \tag{74}$$

using the double angle formula. The derivative of b is given by

$$\frac{db}{d\Theta} = \frac{k}{2E}\left[-\frac{1}{2}\csc^2\frac{\Theta}{2}\right] \ , \tag{75}$$

which makes the differential cross section,

$$\sigma(\Theta) = \frac{k^2}{16E^2}\frac{\cot(\Theta/2)}{\sin(\Theta/2)\cos(\Theta/2)}\frac{1}{\sin^2(\Theta/2)} = \frac{k^2}{16E^2}\frac{\cos(\Theta/2)}{\sin^2(\Theta/2)\cos(\Theta/2)}\frac{1}{\sin^2(\Theta/2)} \ , \tag{76}$$

which simplifies completely to

$$\sigma(\Theta) = \frac{1}{4}\left(\frac{k}{2E}\right)^2\csc^4\frac{\Theta}{2} \ . \tag{77}$$

6 Goldstein 3.35 - Incomplete.

One version of the truncated Coulomb potential has the form

$$V = \frac{k}{r} - \frac{k}{a} \ , \tag{78}$$

for $r < a$, and is zero for all $r > a$. Define the origin to be the center of the potential, so that an incoming particle of mass m feels no potential for $r > a$. The particle's energy far away is completely determined by its kinetic energy. For $r > a$ the particle must travel in straight lines because it feels no potential. If the particle enters the potential moving only in the $\hat{x}$ direction, it will leave the potential at an angle Θ from its original trajectory, this is the scattering angle. Finding this angle as a function of the impact parameter b allows the differential cross section to be calculated. The impact parameter in this case is just the value of the y coordinate the particle has while it moves parallel to the x-axis during its approach to the potential. If the scattering angle is known it can be used to find the differential cross section as follows

$$\sigma(\theta) = \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right| \ , \tag{79}$$

where θ is the polar coordinate. The zero value of this coordinate is set to be when the particle is at its closest approach to the center of the potential. In order for the asymptotic behavior to connect, the particle must move in a hyperbola while inside the potential. This hyperbola would be given by the shape equation:

$$r(\theta) = \frac{l^2}{mk}\frac{1}{\epsilon\cos\theta - 1} \ , \tag{80}$$

where l is the particle's angular momentum about the center of the potential. The definition of $\theta = 0$ says

$$r_{min} = \frac{l^2}{mk}\frac{1}{\epsilon - 1} \tag{81}$$

is the closest approach the particle will make to the center of the potential.

Assume the particle starts at $x = -\infty$ and $y = b$, and it moves with velocity v_0 in the positive x direction. This determines the particle's angular momentum about the origin to be $l = mv_0 b$. Because the energy of the particle far away is completely determined by it's kinetic energy, the angular momentum can be written as

$$l^2 = m^2 v_0^2 b^2 = m(2E)b^2 \ . \tag{82}$$

When it interacts with the potential (at $r = a$), the angle ϕ from the origin to the point the particle hit the potential is defined by

$$\sin\phi = b/a \ . \tag{83}$$

Define another angle Φ which is the angle from the $\theta = 0$ direction to the point the particle leaves the potential. Then, the particle enters the potential at $(r, \theta) = (a, -\Phi)$ and leaves it at $(r, \theta) = (a, \Phi)$. This gives the identity

$$\pi = 2\phi + 2\Phi + \Theta \ , \tag{84}$$

which says the symmetry angle $\psi = \phi + \Phi$. Additionally, the shape equation evaluated at $\theta = \pm\Phi$ is a,

$$a = \frac{l^2}{mk}\frac{1}{\epsilon\cos\Phi - 1} = \frac{2Eb^2}{k}\frac{1}{\epsilon\cos\Phi - 1} \ , \tag{85}$$

which can be rearranged to

$$\epsilon \cos \Phi = \frac{2Eb^2}{ak} + 1 \ . \tag{86}$$

With these constraints there is presumably some way of finding an expression for Θ, and thusly $b(\Theta)$, which would allow the differential cross section to be known. See the following figure for an explanation of the geometry discussed.