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Contents

1 Problem #1: Coupled Mass Oscillator System.

Consider a linear arrangement of three beads, connected by 4 springs a shown in Figure 1. They are constrained to move back and forth in 1 dimension. Let x_1, x_2, and x_3 be the displacement of the leftmost, middle, and rightmost masses, respectively, from their equilibrium positions. Let positive x_i be to the right.

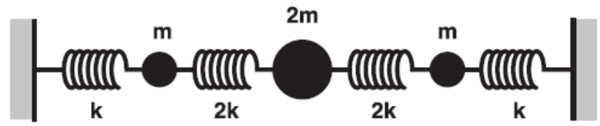


Figure 1: Depiction of the mass-spring system examined in problem #1.

1.1 Normal Modes

The kinetic energy of the system is determined by the three masses, and the potential is determined by the compression of the four springs. The potential energy of the springs joining the masses of mass m to the walls only depend on the position of the respective mass, while the energy of the other two springs depends on the relative displacement between neighboring masses. Therefore the energies can be written as

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 \tag{1}$$

$$U = \frac{1}{2}kx_1^2 + \frac{1}{2}(2k)(x_1 - x_2)^2 + \frac{1}{2}(2k)(x_3 - x_2)^2 + \frac{1}{2}kx_3^2 \ . \tag{2}$$

From this, the kinetic energy matrix and the potential energy matrix, denoted by $\hat{T}$ and $\hat{V}$, respectively, can be determined. The components of each are defined as

$$T_{ij} = \frac{\partial T}{\partial x_i \partial x_j} \qquad \text{and} \qquad V_{ij} = \frac{\partial U}{\partial x_i \partial x_j} \ , \tag{3}$$

where $, j$ span 1 to 3, because there are three generalized coordinates. This also implies there will be three normal modes of oscillation, each with its own frequency. From the form of T it is easy to tell $\hat{T}$ it will be diagonal, similarly from the form of U, $\hat{V}$ will be symmetric. Therefore whether i, j is columns, rows or rows, columns will not matter. The components of each matrix can be found easily, which yields

$$\hat{T} = \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \qquad \text{and} \qquad \hat{V} = \begin{bmatrix} 3k & -2k & 0 \\ -2k & 4k & -2k \\ 0 & -2k & 3k \end{bmatrix} \ . \tag{4}$$

To find the frequencies of oscillation for this system it is simply a matter of solving the following equation for ω, the frequency,

$$\det[\hat{v} - \omega^2 \hat{T}] = 0 \ , \tag{5}$$

which will result in a cubic equation for ω^2 whose three roots will be the normal mode frequencies, or eigenfrequencies. Plugging in yields

$$\begin{vmatrix} 3k - \omega^2(m) & -2k & 0 \\ -2k & 4k - \omega^2(2m) & -2k \\ 0 & -2k & 3k - \omega^2(m) \end{vmatrix} = 0 = 2\left(3k - m\omega^2\right)\left(2k^2 - 5km\omega^2 + m^2\omega^4\right) \ , \tag{6}$$

and solving for ω^2 gives three values. To find the frequencies, not the square of the frequencies, the positive root is chosen for physical reasons (frequencies must be positive), they are

$$\omega_{(1)} = \sqrt{\frac{3k}{m}} \ , \quad \omega_{(2)} = \sqrt{\frac{k}{2m}(5-\sqrt{17})} \ , \quad \omega_{(3)} = \sqrt{\frac{k}{2m}(5+\sqrt{17})} \ . \tag{7}$$

Next, the vectors describing normal mode motion must be found. These vectors describe relative motions of the masses, and are similar in concept to eigenvectors. To find these normal mode vectors $a_{(l)}$, which corresponds to the normal mode frequency $\omega_{(l)}$, the equations of motion must be solved, in summation notation this is

$$(V_{ij} - \omega_{(l)}^2 T_{ij})a_{j(l)} = 0 \ , \tag{8}$$

which for each eigenfrequency, $\omega_{(l)}$ is a system of three equations, that will result in the components of the un-normalized "eigenvectors" for the motion, which will be the normal mode vectors up to a factor of a constant. To normalize these vectors, a condition is imposed that they satisfy the congruency transformation, which in summation notation is

$$a_{i(l)} T_{ij} a_{j(h)} = \delta_{(l)(h)} \ , \tag{9}$$

where $\delta_{(l)(h)}$ is the Kronecker delta function. So that for one specific three component a vector, this is

$$1 = T_{11}(a_{1(l)})^2 + T_{22}(a_{2(l)})^2 + T_{33}(a_{3(l)})^2 \ . \tag{10}$$

Most of this calculation can be carried out generally (for any eigenfrequency). First, Equation 8 is written out in matrix form,

$$\begin{bmatrix} 3k - \omega_{(l)}^2(m) & -2k & 0 \\ -2k & 4k - \omega_{(l)}^2(2m) & -2k \\ 0 & -2k & 3k - \omega_{(l)}^2(m) \end{bmatrix} \begin{bmatrix} a_{1(l)} \\ a_{2(l)} \\ a_{3(l)} \end{bmatrix} = 0 \ , \tag{11}$$

which after doing out the matrix multiplication yields the system of equations

$$[3k - \omega_{(l)}^2(m)]a_{1(l)} + [-2k]a_{2(l)} = 0 \tag{12}$$

$$[-2k]a_{1(l)} + [4k - \omega_{(l)}^2(2m)]a_{2(l)} + [-2k]a_{3(l)} = 0 \tag{13}$$

$$[-2k]a_{2(l)} + [3k - \omega_{(l)}^2(m)]a_{3(l)} = 0 \ , \tag{14}$$

these equations will be solved for each $\omega_{(l)}$ independently.

1.1.1 First Eigenfrequency, $l = 1$.

For the first eigenfrequency, Equations 12 through 14 become

$$\left[3k - \frac{3k}{m}(m)\right] a_{1(1)} + [-2k]a_{2(1)} = 0 \ \Rightarrow \ a_{2(1)} = 0 \tag{15}$$

$$[-2k]a_{1(1)} + \left[4k - \frac{3k}{m}(2m)\right] a_{2(1)} + [-2k]a_{3(1)} = 0 \ \Rightarrow \ -a_{1(1)} + a_{2(1)} - a_{3(1)} = 0 \tag{16}$$

$$[-2k]a_{2(1)} + \left[3k - \frac{3k}{m}(m)\right] a_{3(1)} = 0 \ \Rightarrow \ a_{2(1)} = 0 \ , \tag{17}$$

therefore $\vec{a}_{(1)} = A_1(1, 0, -1)$, where A_1 is a constant to be determined by normalization. Applying the congruency transformation, from Equation 10, gives

$$1 = m(A_1)^2 + m(-A_1)^2 \quad \Rightarrow \quad A_1 = \frac{1}{\sqrt{2m}} \ , \tag{18}$$

The first normal mode of oscillation is $\vec{a}_{(1)} = (1/\sqrt{2m}, 0, -1/\sqrt{2m})$, with frequency $\omega_{(1)} = \sqrt{3k/m}$. This motion is the two outer masses moving completely out of phase while the inner mass remains completely still.

1.1.2 Second Eigenfrequency, $l = 2$.

For the second eigenfrequency, letting $\alpha_- = 5 - \sqrt{17}$, Equations 12 through 14 become

$$\left[3k - \frac{\alpha_- k}{2m}(m)\right] a_{1(2)} + [-2k]a_{2(2)} = 0 \ \Rightarrow \ a_{2(2)} = \frac{6-\alpha_-}{4} a_{1(2)} \tag{19}$$

$$[-2k]a_{1(2)} + \left[4k - \frac{\alpha_- k}{2m}(2m)\right] a_{2(2)} + [-2k]a_{3(2)} = 0 \ \Rightarrow \ a_{3(2)} = \left[2 - \frac{\alpha_-}{2}\right] a_{2(2)} - a_{1(2)} \tag{20}$$

$$[-2k]a_{2(2)} + \left[3k - \frac{\alpha_- k}{2m}(m)\right] a_{3(2)} = 0 \ \Rightarrow \ a_{2(2)} = \frac{6-\alpha_-}{4} a_{3(2)} \ , \tag{21}$$

Solving these in MATHEMATICA yields $\vec{a}_{(2)} = A_2(1, \frac{1}{4}(1+\sqrt{17}), 1)$, where A_2 is a constant to be determined by normalization. Applying the congruency transformation, from Equation 10, gives

$$1 = m(A_2)^2 + 2m\left(A_2\left[-\frac{1}{4}\left(-\sqrt{17}-1\right)\right]\right)^2 + m(A_2)^2 \quad \Rightarrow \quad A_2 = \frac{2}{\sqrt{17m + \sqrt{17}m}} \ , \tag{22}$$

The second normal mode of oscillation is

$$\vec{a}_{(2)} = \left(\frac{2}{\sqrt{17m+\sqrt{17}m}}, \frac{1+\sqrt{17}}{2\sqrt{17m+\sqrt{17}m}}, \frac{2}{\sqrt{17m+\sqrt{17}m}}\right) \ , \tag{23}$$

with frequency $\omega_{(2)} = \sqrt{\frac{k}{2m}(5-\sqrt{17})}$, the slowest oscillation. This motion is all masses moving in the same direction, with the middle mass moving the fastest.

1.1.3 Third Eigenfrequency, $l = 3$.

For the third eigenfrequency, letting $\alpha_+ = 5 + \sqrt{17}$, Equations 12 through 14 become

$$\left[3k - \frac{\alpha_+ k}{2m}(m)\right] a_{1(3)} + [-2k]a_{2(3)} = 0 \ \Rightarrow \ a_{2(3)} = \frac{6-\alpha_+}{4} a_{1(3)} \tag{24}$$

$$[-2k]a_{1(3)} + \left[4k - \frac{\alpha_+ k}{2m}(2m)\right] a_{2(3)} + [-2k]a_{3(3)} = 0 \ \Rightarrow \ a_{3(3)} = \left[2 - \frac{\alpha_+}{2}\right] a_{2(3)} - a_{1(3)} \tag{25}$$

$$[-2k]a_{2(3)} + \left[3k - \frac{\alpha_+ k}{2m}(m)\right] a_{3(3)} = 0 \ \Rightarrow \ a_{2(3)} = \frac{6-\alpha_+}{4} a_{3(3)} \ , \tag{26}$$

Solving these in MATHEMATICA yields $\vec{a}_{(3)} = A_3(1, \frac{1}{4}(1-\sqrt{17}), 1)$, where A_3 is a constant to be determined by normalization. Applying the congruency transformation, from Equation 10, gives

$$1 = m(A_3)^2 + 2m\left(A_3\left[\frac{1}{4}\left(1-\sqrt{17}\right)\right]\right)^2 + m(A_3)^2 \quad \Rightarrow \quad A_3 = \frac{2}{\sqrt{17m - \sqrt{17}m}} \ , \tag{27}$$

The second normal mode of oscillation is

$$\vec{a}_{(3)} = \left(\frac{2}{\sqrt{17m - \sqrt{17}m}}, \frac{1-\sqrt{17}}{2\sqrt{17m - \sqrt{17}m}}, \frac{2}{\sqrt{17m - \sqrt{17}m}} \right) \ , \tag{28}$$

with frequency $\omega_{(3)} = \sqrt{\frac{k}{2m}(5+\sqrt{17})}$, the fastest oscillation. This motion is the outer masses moving in the same direction, with the middle mass moving the fastest in the opposite direction.

1.1.4 Normal Mode Coordinates.

Let ξ_i be the i^{th} normal mode coordinate. The generalized coordinates can be expressed as linear combinations of these coordinates. The coefficients of which are determined by the i^{th} component of each "eigenvector",

$$x_1 = a_{1(1)}\xi_1 + a_{1(2)}\xi_2 + a_{1(3)}\xi_3 \tag{29}$$

$$x_2 = a_{2(1)}\xi_1 + a_{2(2)}\xi_2 + a_{2(3)}\xi_3 \tag{30}$$

$$x_3 = a_{3(1)}\xi_1 + a_{3(2)}\xi_2 + a_{3(3)}\xi_3 \ , \tag{31}$$

where again, the subscript inside the parenthesis denotes which eigenvector/value to use and the other subscript is the specific component of the eigenvector. Solving this system of equations for ξ_i in MATHEMATICA yields

$$\xi_1 = \sqrt{\frac{m}{2}}[x_1 - x_3] \tag{32}$$

$$\xi_2 = -\frac{\sqrt{m(17+\sqrt{17})}}{136}\left[\left(\sqrt{17}-17\right)x_1 - 8\sqrt{17}x_2 + \left(\sqrt{17}-17\right)x_3\right] \tag{33}$$

$$\xi_3 = \frac{\sqrt{m(17-\sqrt{17})}}{136}\left[\left(\sqrt{17}+17\right)x_1 - 8\sqrt{17}x_2 + \left(\sqrt{17}+17\right)x_3\right] \ , \tag{34}$$

which may make it easier to visualize the motion. The velocities of these normal coordinates are the same as above, but all coordinates are replaced with their first time derivative. The normal modes occur when $\xi_i = 0$, which is at

$$\xi_1 = 0: \ x_1 = x_3 \tag{35}$$

$$\xi_2 = 0: \ x_2 = \frac{\sqrt{17}-17}{8\sqrt{17}}(x_1+x_3) = \frac{1-\sqrt{17}}{8}(x_1+x_3) \tag{36}$$

$$\xi_3 = 0: \ x_2 = \frac{\sqrt{17}+17}{8\sqrt{17}}(x_1+x_3) = \frac{1+\sqrt{17}}{8}(x_1+x_3) \tag{37}$$

1.2 Motion Due to Initial Conditions.

Suppose at $t = 0$ all three particles are in their equilibrium positions with the two particles on the ends at rest and the one in the middle moving with velocity v. (This can happen as a result of an impulse acting on the middle particle). These initial conditions are

$$\begin{cases} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{cases} \Rightarrow \begin{cases} \xi_1 &= 0 \\ \xi_2 &= 0 \\ \xi_3 &= 0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{x}_1 &= 0 \\ \dot{x}_2 &= v \\ \dot{x}_3 &= 0 \end{cases} \Rightarrow \begin{cases} \dot{\xi}_1 &= 0 \\ \dot{\xi}_2 &= +v\sqrt{m}\sqrt{1+\frac{1}{\sqrt{17}}} \\ \dot{\xi}_3 &= -v\sqrt{m}\sqrt{1-\frac{1}{\sqrt{17}}} \end{cases} . \tag{38}$$

The Euler-Lagrange equations for the normal coordinates are given by Goldstein Equation 6.46: $\ddot{\xi}_k + \omega_k^2\xi_k = 0$, so the solutions can be represented as a sum of sine and cosine, depending on initial conditions.

1.2.1 Normal Mode 1, ξ_1.

Imposing initial conditions on $\xi_1(t)$ gives

$$\xi_1(t) = A_1 \sin(\omega_1 t) + B_1 \cos(\omega_1 t) \tag{39}$$
$$\xi_1(0) = 0 = A_1 \sin(0) + B_1 \cos(0) \;\Rightarrow\; B_1 = 0 \tag{40}$$
$$\dot{\xi}_1(t) = A_1\omega_1 \cos(\omega_1 t) \tag{41}$$
$$\dot{\xi}_1(0) = 0 = A_1\omega_1 \;\Rightarrow\; A_1 = \frac{-v\sqrt{m}}{\omega_1\sqrt{2}} \ , \tag{42}$$

so the final equation of motion is for ξ_1 is

$$\xi_1(t) = 0 \qquad \omega_1 = \sqrt{\frac{3k}{m}} \ , \tag{43}$$

1.2.2 Normal Mode 2, ξ_2.

Imposing initial conditions on $\xi_2(t)$ gives

$$\xi_2(t) = A_2 \sin(\omega_2 t) + B_2 \cos(\omega_2 t) \tag{44}$$
$$\xi_2(0) = 0 = A_2 \sin(0) + B_2 \cos(0) \;\Rightarrow\; B_2 = 0 \tag{45}$$
$$\dot{\xi}_2(t) = A_2\omega_2 \cos(\omega_2 t) \tag{46}$$
$$\dot{\xi}_2(0) = +v\sqrt{m}\sqrt{1+\frac{1}{\sqrt{17}}} = A_2\omega_2 \;\Rightarrow\; A_2 = \frac{v\sqrt{m}}{\omega_2}\sqrt{1+\frac{1}{\sqrt{17}}} \ , \tag{47}$$

so the final equation of motion is for ξ_2 is

$$\xi_2(t) = \frac{v\sqrt{m}}{\omega_2}\sqrt{1+\frac{1}{\sqrt{17}}}\sin(\omega_2 t) \qquad \omega_2 = \sqrt{\frac{k}{2m}(5-\sqrt{17})} \ , \tag{48}$$

1.2.3 Normal Mode 3, ξ_3.

Imposing initial conditions on $\xi_3(t)$ gives

$$\xi_3(t) = A_3 \sin(\omega_3 t) + B_3 \cos(\omega_3 t) \tag{49}$$
$$\xi_3(0) = 0 = A_3 \sin(0) + B_3 \cos(0) \;\Rightarrow\; B_3 = 0 \tag{50}$$
$$\dot{\xi}_3(t) = A_3\omega_3 \cos(\omega_3 t) \tag{51}$$
$$\dot{\xi}_3(0) = -v\sqrt{m}\sqrt{1-\frac{1}{\sqrt{17}}} = A_3\omega_3 \;\Rightarrow\; A_3 = \frac{-v\sqrt{m}}{\omega_3}\sqrt{1-\frac{1}{\sqrt{17}}} \ , \tag{52}$$

so the final equation of motion is for ξ_3 is

$$\xi_3(t) = -\frac{v\sqrt{m}}{\omega_3}\sqrt{1-\frac{1}{\sqrt{17}}}\sin(\omega_3 t) \qquad \omega_3 = \sqrt{\frac{k}{2m}(5+\sqrt{17})} \ , \tag{53}$$

2 Problem #2: The Pendulum of Doom.

A mass m is attached to pendulum of length ℓ. The pivot point of the pendulum is attached to the rim of a wheel of radius a, and the wheel is rotating with constant angular velocity ω. Let the generalized coordinate θ be the angle the pendulum makes with the vertical, as shown in Figure 2. The system is subject to a downwards uniform gravitational acceleration g.

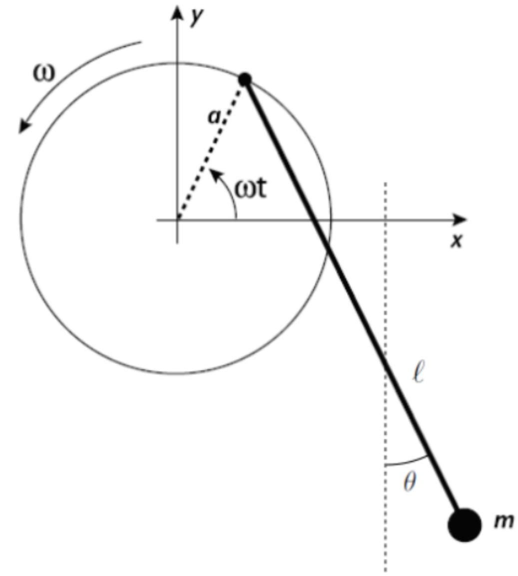


Figure 2: Depiction of the mass-spring system examined in problem #2.

2.1 Equation of Motion.

In Cartesian coordinates, the mass is located at

$$x(\theta,t) = a\cos[\omega t] + \ell\sin[\theta(t)] \tag{54}$$
$$y(\theta,t) = a\sin[\omega t] - \ell\cos[\theta(t)] \ , \tag{55}$$

with velocities given by the respective time derivatives

$$\dot{x} = -a\omega\sin[\omega t] + \ell\dot{\theta}\cos[\theta(t)] \tag{56}$$
$$\dot{y} = a\omega\cos[\omega t] + \ell\dot{\theta}\sin[\theta(t)] \ , \tag{57}$$

The kinetic and potential energies of this system, in Cartesian coordinates are given by

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \quad \text{and} \quad U = mgy \ , \tag{58}$$

note that the sum of squared velocities is

$$\begin{aligned}\dot{x}^2 + \dot{y}^2 &= [-a\omega\sin(\omega t)]^2 + [\ell\dot{\theta}\cos(\theta)]^2 + [a\omega\cos(\omega t)]^2 + [\ell\dot{\theta}\sin(\theta)]^2 \\ &\quad - 2a\omega\sin(\omega t)\ell\dot{\theta}\cos(\theta) + 2a\omega\cos(\omega t)\ell\dot{\theta}\sin(\theta) \\ &= (a\omega)^2 + (\ell\dot{\theta})^2 - 2a\omega\ell\dot{\theta}[\cos(\omega t)\sin(\theta) - \sin(\omega t)\cos(\theta)] \\ &= (a\omega)^2 + (\ell\dot{\theta})^2 + 2a\omega\ell\dot{\theta}\sin(\theta - \omega t) \ ,\end{aligned}$$

using trigonometric product and sum identities in the final step (read: Mathematica). This gives the Lagrangian to be

$$\mathcal{L} = \frac{ma^2\omega^2}{2} + \frac{m\ell^2\dot{\theta}^2}{2} + ma\omega\ell\dot{\theta}\sin(\theta - \omega t) - mga\sin(\omega t) + mg\ell\cos(\theta) \ , \tag{59}$$

note that the terms constant in $\theta, \dot{\theta}$ will drop out of the Euler Lagrange equations. Additionally, this Lagrangian is not cyclic in any coordinate, and it explicitly depends on time, so in this case, there are no conserved quantities. The Euler Lagrange equation for this system is

$$\frac{d}{dt}\left[m\ell^2\dot{\theta} + ma\omega\ell\sin(\theta - \omega t)\right] = ma\omega\ell\dot{\theta}\cos(\theta - \omega t) - mg\ell\sin(\theta) \tag{60}$$
$$m\ell^2\ddot{\theta} + ma\omega\ell(\dot{\theta} - \omega)\cos(\theta - \omega t) = ma\omega\ell\dot{\theta}\cos(\theta - \omega t) - mg\ell\sin(\theta) \ , \tag{61}$$

which after noting the two $\dot{\theta}$ terms cancel and isolating the $\ddot{\theta}$ term becomes

$$m\ell^2\ddot{\theta} = ma\omega^2\ell\cos(\theta - \omega t) - mg\ell\sin(\theta) \tag{62}$$

$$\ddot{\theta} = \frac{a\omega^2}{\ell}\cos(\theta - \omega t) - \frac{g}{\ell}\sin(\theta) \tag{63}$$

2.2 Simple Pendulum Limit.

In the $\omega \to 0$ limit, the equation of motion becomes

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta \ , \tag{64}$$

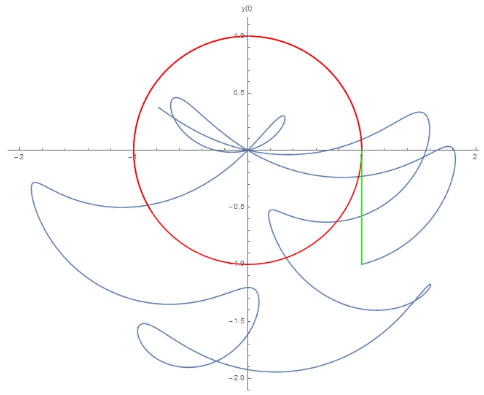
which is the equation for the simple pendulum equation. For small oscillations $\sin\theta \simeq \theta$, so the linearized equation gives a frequency of oscillation of $\omega_0 = \sqrt{g/\ell}$, which is the frequency of small oscillations for a simple pendulum.

2.3 Visualization.

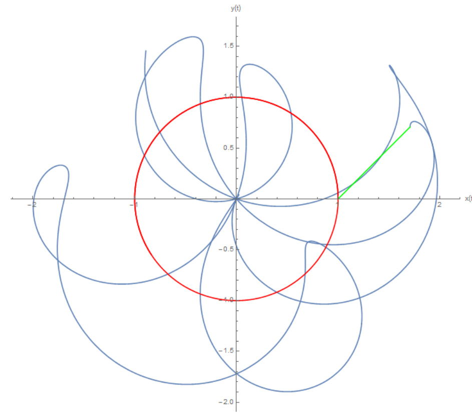
Equation 63 can be integrated numerically using MATHEMATICA, by selecting initial angular displacement `A`, and initial angular velocity `B`. After setting the values of constants and initial conditions, the following commands plot the motion in the $x - y$ plane:

```
sol=NDSolve[{\[Phi]''[t]==(a w/l)Cos[\[Phi][t]- w t]-(g/l)Sin[\[Phi][t]],
     \[Phi][0]==A,\[Phi]'[0]==B},\[Phi],{t,0,10}];
x[t_] := a Cos[w t] + l Sin[\[Phi][t]]
y[t_] := a Sin[w t] - l Cos[\[Phi][t]]
x0 = a Cos[w 0] + l Sin[A];
y0 = a Sin[w 0] - l Cos[A];
p1 = ParametricPlot[
     Evaluate[{a Cos[w t] + l Sin[\[Phi][t]], a Sin[w t] - l Cos[\[Phi][t]]} /. sol],
     {t, 0, 10}, AxesLabel -> {"x(t)", "y(t)"}];
p2 = ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 10}, PlotStyle -> Red];
p3 = ParametricPlot[{(x0 - t Sin[A]), y0 + t Cos[A]}, {t, 0, l}, PlotStyle -> Green];
p4 = Show[p1, p2, p3, PlotRange -> All]
```

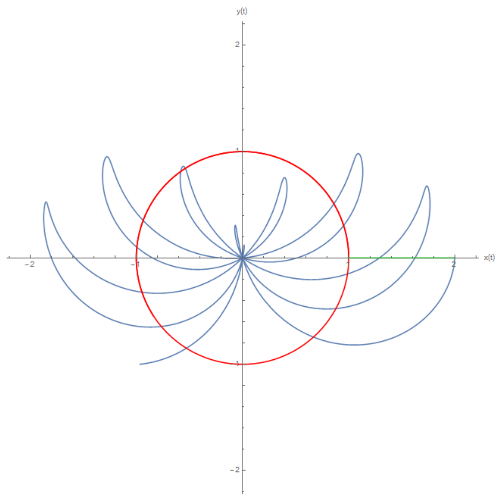
The plots in Figure 3 show the motion of this system with various values of ω, $\phi(0)$, and $\dot{\phi}(0)$, noted in the figure caption. For these plots the radius of the ring a, was set to 1 meter, the length of the pendulum ℓ, set to one meter, and the gravitational acceleration g set to 10 m/s^2. The ring the pendulum pivot moves around is noted in red, and the initial position of the pendulum is noted in green. All integrations took place over ten seconds.



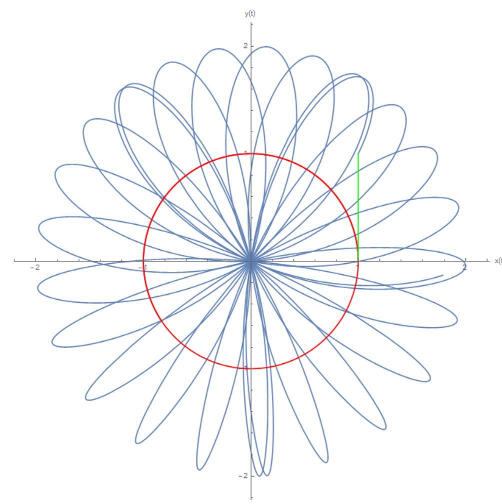
(a) Motion for $\omega = \pi/4$, $\phi(0) = 0$, and $\dot{\phi}(0) = \pi$.



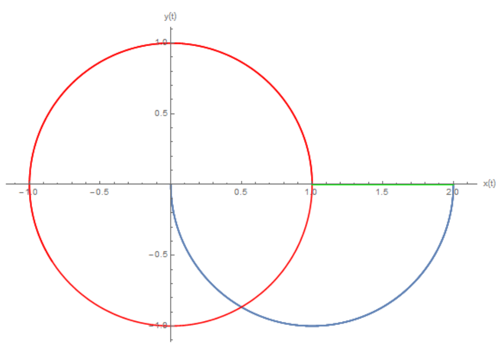
(b) Motion for $\omega = \pi/4$, $\phi(0) = 3\pi/4$, and $\dot{\phi}(0) = 0$.



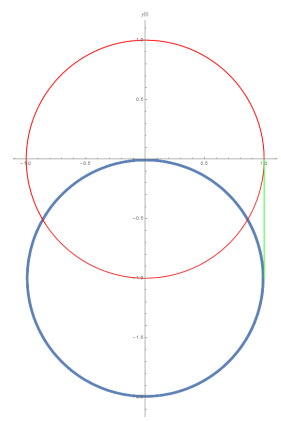
(c) Motion for $\omega = \pi/10$, $\phi(0) = \pi/2$, and $\dot{\phi}(0) = 0$.



(d) Motion for $\omega = 10$, $\phi(0) = \pi$, and $\dot{\phi}(0) = -5$.



(e) Motion for $\omega = 0$, $\phi(0) = \pi/2$, and $\dot{\phi}(0) = 0$.



(f) Motion for $\omega = 1000$, $\phi(0) = 0$, and $\dot{\phi}(0) = 0$, note integration only for 1 second.

Figure 3: Motion of the mass in the $x - y$ plane for various values of ω, $\phi(0)$, and $\dot{\phi}(0)$.

3 Problem #3: Everything is an SHO!

Consider the following potential energy function:

$$U(r) = U_0 \left(\frac{r}{R} + \lambda^2 \frac{R}{r} \right) , \tag{65}$$

where $\{U_0, R, \lambda\}$ are all constants, and the variable r can range $0 < r < \infty$. This potential is graphed in Figure 4.

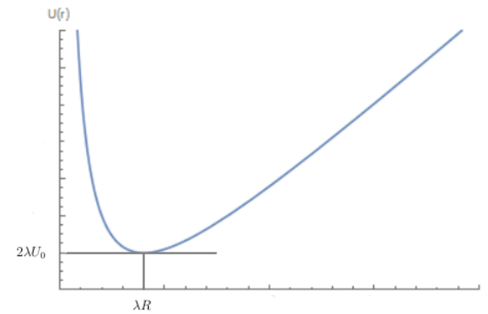


Figure 4: Sketch of the potential given by Equation 65.

3.1 Equilibrium Positions.

Equilibrium positions occur when the first derivative of the potential is zero. For this potential, these positions are at r_0, which can be found by solving the following equation for r_0,

$$\left.\frac{dU}{dr}\right|_{r_0} = 0 = U_0 \left(\frac{1}{R} + \lambda^2 R \frac{-1}{r_0^2} \right) \quad \Rightarrow \quad \frac{1}{R} = \lambda^2 \frac{R}{r_0^2} , \tag{66}$$

so $r_0 = \lambda R$. The potential at this location is

$$U(r_0) = U_0 \left(\frac{\lambda R}{R} + \lambda^2 \frac{R}{\lambda R} \right) = 2\lambda U_0 . \tag{67}$$

3.2 Harmonic Oscillator Approximation.

Consider the variable $x = r - r_0$. For small x values, the potential can be Taylor expanded around its equilibrium position r_0,

$$U(r) \simeq U(r_0) + (r - r_0) \left.\frac{dU}{dr}\right|_{r_0} + \frac{1}{2}(r - r_0)^2 \left.\frac{d^2U}{dr^2}\right|_{r_0} + O(r^3) , \tag{68}$$

but because x is small, it is close to the equilibrium position so the first derivative is zero. The Taylor series then becomes

$$U(x) \simeq 2\lambda U_0 + \frac{1}{2} x^2 \left(U_0 \lambda^2 R \frac{2}{r_0^3} \right) = 2\lambda U_0 + \frac{1}{2} \left(\frac{2U_0}{\lambda R^2} \right) x^2 , \tag{69}$$

which takes the form of the harmonic oscillator, $cst + \frac{1}{2}kx^2$.

3.3 Frequency of Small Oscillations.

A harmonic oscillator potential has the form $cst + \frac{1}{2}m\omega^2 x^2$, where m is the mass of a particle moving in the potential. The frequency of oscillation is ω, and its relation to k can be found,

$$k = m\omega^2 \quad \Rightarrow \quad \omega = \sqrt{\frac{k}{m}} , \tag{70}$$

therefore, for this potential, the frequency of small oscillations is

$$\omega = \sqrt{\frac{2U_0}{m\lambda R^2}} . \tag{71}$$

4 Problem #4: Physics as an Artform.

The two dimensional anisotropic oscillator makes interesting and pretty patterns from the trajectory the oscillator makes in space. Consider an oscillator whose trajectory is given by:

$$x(t) = \alpha \cos(\omega_x t) \qquad y(t) = \beta \cos(\omega_y t - \delta) \ . \tag{72}$$

4.1 Rational Frequency Ratio.

For rational frequencies, the motion repeats itself after a period and results in a closed path. Define the ratio of the frequencies to be the ratio of two integers, ensuring the ratio is a rational number,

$$\frac{\omega_x}{\omega_y} = \frac{m}{n} \quad \Rightarrow \quad \frac{m}{\omega_x} = \frac{n}{\omega_y} \tag{73}$$

and using the relationship of frequency and period, a single frequency can be defined,

$$\tau = \frac{2\pi m}{\omega_x} = \frac{2\pi n}{\omega_y} \ . \tag{74}$$

After this period, the x and y positions of the particle are given by

$$x(t+\tau) = \alpha \cos[\omega_x(t+\tau)] = \alpha \cos[\omega_x t + 2\pi m)] \tag{75}$$

$$y(t+\tau) = \beta \cos[\omega_y(t+\tau) - \delta] = \beta \cos[\omega_y t - \delta + 2\pi n] \ , \tag{76}$$

which, by noting that $\cos(a) = \cos(a + 2\pi k)$ for any integer k, implies the particle can be found at its starting location after one period, and the path is closed. The period of motion is given in Equation 74.

4.2 Irrational Frequency Ratio.

Consider the ratio $\omega_x/\omega_y = \xi$, where ξ is irrational, so it cannot be represented as the ratio of two integers, as in Section 4.1. Similarly to the previous section, if periodic motion were to occur, the period would be given by

$$\tau = \frac{2\pi}{\omega_x} = \frac{2\pi}{\xi\omega_y} \ . \tag{77}$$

By the definition of periodicity,

$$x(t+\tau) = \alpha \cos[\omega_x(t+\tau)] = \alpha \cos[\omega_x t + 2\pi)] = x(t) \tag{78}$$

$$y(t+\tau) = \beta \cos[\omega_y(t+\tau) - \delta] = \beta \cos\left[\omega_y t - \delta + \frac{2\pi}{\xi}\right] \neq y(t) \ , \tag{79}$$

for irrational ξ. Therefore, the motion may be periodic in one coordinate, but overall, the particle will never return to it's initial position and velocity in both coordinates. No periodic motion exists for irrational ξ, and the trajectory is not closed. It will cross every point in the region $-\alpha < x < \alpha$ and $-\beta < y < \beta$.

5 Problem #5: Parametric Plots.

Mathematica can be used to parametrically plot various two and three dimensional trajectories.

5.1 Two Dimensional Trajectory.

A particular kind of two-dimensional trajectory can be parameterized by the equations

$$x(t) = a\cos t \qquad y(t) = b\sin t \ , \tag{80}$$

which is initialized using the Mathematica commands

```
x[a_] := a Cos[t];
y[b_] := b Sin[t];
z[\[Alpha]_] := \[Alpha] t;
```

This trajectory is plotted in Figure 5 for two cases for the time range $0 < t < 2\pi$. The red line shows the trajectory for the case where $a = 1$ and $b = 2$, while the blue line shows the trajectory for $a = b = 3$. From this figure it is easy to see that when the amplitudes are equal, the trajectory forms a circle with radius equal to the amplitude. But when they are not equal, the trajectory traces an ellipse. This ellipse width in each coordinate direction is twice the respective amplitude. The following Mathematica commands create this plot:

```
pa1 = ParametricPlot[{x[1], y[2]}, {t, 0, 2 \[Pi]}, PlotStyle -> Red];
pa2 = ParametricPlot[{x[3], y[3]}, {t, 0, 2 \[Pi]}];
Show[pa1, pa2, PlotRange -> All]
```

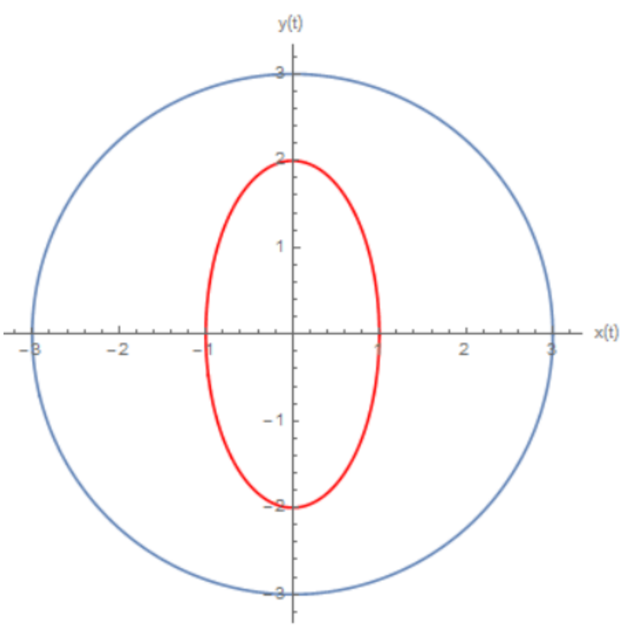


Figure 5: Motion of a two dimensional trajectory for $(a, b) = (1, 2)$, in red, and $(a, b) = (3, 3)$, in blue.

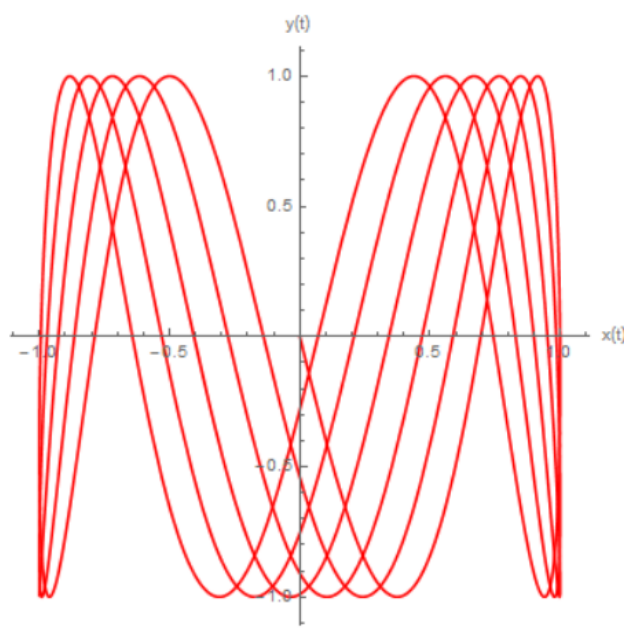


Figure 6: Anharmonic oscillator with parameters $(\beta, \omega, \delta) = (1, 45/11, \pi)$, over the time range $0 < t < 6\pi$.

5.2 Three Dimensional Trajectory

By adding a third parameterized equation to Equation 80 given by

$$z(t) = \alpha t \tag{81}$$

the trajectory becomes three dimensional. The motion of this trajectory is shown in Figure 7, for $(a, b, \alpha) = (1, 2, 2)$, for the time range $0 < t < 6\pi$. This trajectory is a corkscrew due to the z coordinate of the particle increasing linearly with time while the particle traces an ellipse in the $x - y$ plane. The following code creates the plot:

```
ParametricPlot3D[{x[1], y[2], z[2]},
     {t, 0, 6 \[Pi]},
     AxesLabel -> {"x(t)", "y(t)", "z(t)"}]
```

5.3 Anharmonic oscillator.

The anharmonic oscillator is parameterized by

$$x(t) = \sin(t) \qquad y(t) = \beta \sin(\omega t + \delta) \ , \tag{82}$$

The frequencies have been normalized to the frequency for the x motion: $\omega'_x = \omega_x/\omega_x = 1$ and $\omega'_y = \omega = \omega_y/\omega_x$. This is a matter of convenience, which allows the consequences of the frequency ratio having specific behaviors to be explored. For rational frequency ratios, the paths are periodic and closed, commonly called Lissajous figures. For irrational frequency ratios the paths are not closed.

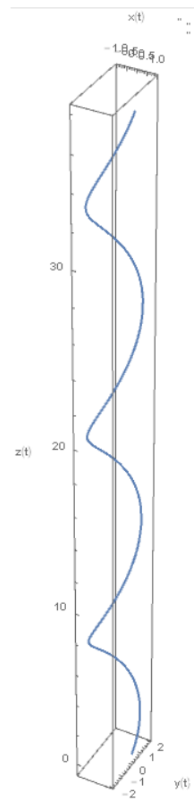
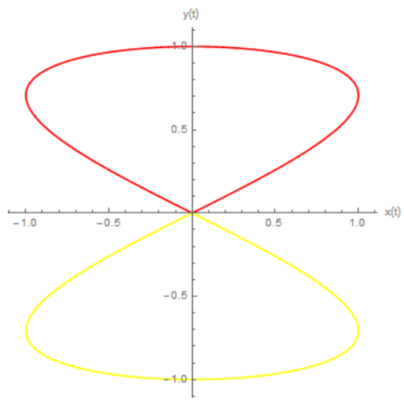


Figure 7: Motion of a three dimensional trajectory for $(a, b, \alpha) = (1, 2, 2)$.

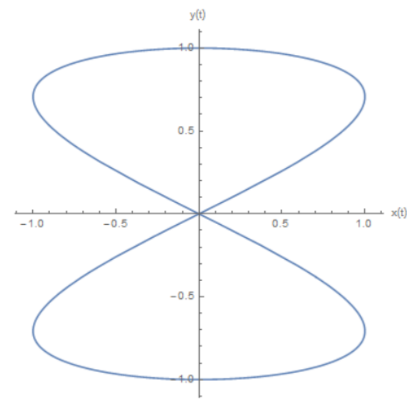
The code for all subsequent plots is:

```
xAO = Sin[t];
yAO[\[Beta]_, \[Omega]_, \[Delta]_] := \[Beta] Sin[\[Omega] t - \[Delta]];
pc1 = ParametricPlot[{xAO, yAO[1, 1/2, 0]}, {t, 0, 2 \[Pi]}, PlotStyle -> Red];
pc2 = ParametricPlot[{xAO, yAO[1, 1/2, 0]}, {t, 2 \[Pi], 4 \[Pi]}, PlotStyle -> Yellow];
pc3 = ParametricPlot[{xAO, yAO[1, 1/2, 0]}, {t, 0, 4 \[Pi]}];
Show[pc1, pc2, PlotRange -> All];
pc4 = ParametricPlot[{xAO, yAO[1, 3/5, 0]}, {t, 0, 20 \[Pi]}, PlotStyle -> Red]
pc5 = ParametricPlot[{xAO, yAO[1, Sqrt[2], 0]}, {t, 0, 6 \[Pi]}, PlotStyle -> Red];
pc6 = ParametricPlot[{xAO, yAO[1, Sqrt[2], 0]}, {t, 0, 60 \[Pi]}, PlotStyle -> Yellow];
pc7 = ParametricPlot[{xAO, yAO[1, Sqrt[2], 0]}, {t, 0, 600 \[Pi]}];
Show[pc7, pc6, pc5]
```

The anharmonic oscillator is shown in Figure 8 for the parameters $(\beta, \omega, \delta) = (1, 1/2, 0)$. In Figure 8a there are two time ranges plotted: red for $0 < t < 2\pi$ and yellow for $2\pi < t < 4\pi$. Figure 8b shows the motion for the same parameters over the time range $0 < t < 4\pi$. From this, it is easy to see the period of this motion is 4π.



(a) Anharmonic oscillator motion for time ranges $0 < t < 2\pi$ and $2\pi < t < 4\pi$.



(b) Anharmonic oscillator motion for the time range $0 < t < 4\pi$, which is one full period.

Figure 8: Anharmonic oscillator motion with parameters $(\beta, \omega, \delta) = (1, 1/2, 0)$.

For the parameters $(\beta, \omega, \delta) = (1, 3/5, 0)$, over the time range $0 < t < 20\pi$, the anharmonic oscillator motion is shown in Figure 9. This motion is also a closed, periodic path, with period 20π. If this motion is plotted only until $t = 10\pi$, the same path is traced out, but only half if it is plotted until $t = 5\pi$. So the particle completes this motion at $t = 10\pi$, but for it to get back to its initial position, it must trace this path again, so the period is 20π. Now the parameters were set to $(\beta, \omega, \delta) = (1, \sqrt{2}, 0)$ and three plots were overlaid in Figure 10. One over the time range $0 < t < 6\pi$ (red), one over $0 < t < 60\pi$ (yellow), and one over $0 < t < 600\pi$ (blue). As the time is increased, the particle traces out more and more of the 1×1 square where motion is possible. This is not a closed path, and the motion will never repeat. Figure 6 shows motion for the parameters $(\beta, \omega, \delta) = (1, 45/11, \pi)$ for the time range $0 < t < 6\pi$.

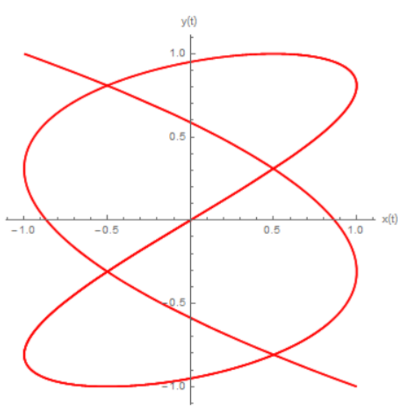


Figure 9: Anharmonic oscillator with parameters $(\beta, \omega, \delta) = (1, 3/5, 0)$, over the time range $0 < t < 20\pi$.

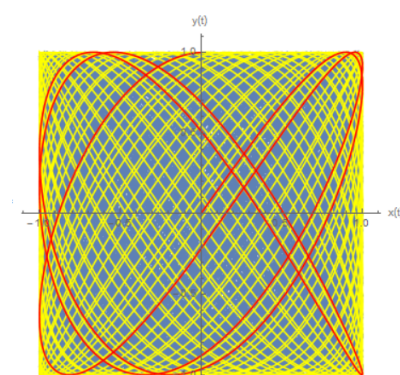


Figure 10: Anharmonic oscillator with parameters $(\beta, \omega, \delta) = (1, \sqrt{2}, 0)$, over three time ranges varying by 1 order of magnitude each.