Dylan J. Temples: Solution Set Seven

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Classical Mechanics, Third Ed.- Goldstein
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1 Goldstein 5.18.

A dumbbell is formed by connecting two small spherical masses of mass m with a massless rod of length $2b$. The rod is attached to an axle in such a way that it makes a constant angle ϕ with the axle. The dumbbell rotates about the axle at a rate ω, as shown in Figure 1. The angular momentum about this axis was determined in the last homework as

$$\mathbf{L} = m\omega b^2 \begin{bmatrix} -\sin 2\phi \cos \omega t \\ -\sin 2\phi \sin \omega t \\ 2\sin^2 \phi \end{bmatrix} . \tag{1}$$

The right handed coordinate system is set up so that $\boldsymbol{\omega}$ points along $\hat{\mathbf{z}}$, with the origin at the point the barbell crosses $\boldsymbol{\omega}$.

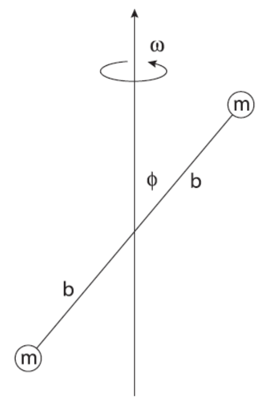

Figure 1: Depiction of the oblique dumbbell system examined in problem #1.

1.1 Components of Torque Along Principal Axes.

The principal axes of a body are the set of orthonormal eigenvectors found from diagonalizing the inertia tensor $\hat{I}$. The components of this tensor are listed in the previous homework; using Mathematica, the principal moments (eigenvalues) are

$$I_1 = 2mb^2 \qquad I_2 = 2mb^2 \qquad I_3 = 0 \ , \tag{2}$$

and the corresponding eigenvectors are

$$\mathbf{w}_1 = \begin{bmatrix} -\cot\phi \sec \omega t \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{w}_2 = \begin{bmatrix} \tan \omega t \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{w}_3 = \begin{bmatrix} \tan\phi \cos \omega t \\ \tan \phi \sin \omega t \\ 1 \end{bmatrix} , \tag{3}$$

which are not yet orthonormalized. Note that because $\mathbf{w}_3$ corresponds to a nondegenerate eigenvalue, it can be chosen as a principal axis, and the other two can be made orthogonal to it. The remaining two eigenvectors, since they are degenerate, are both orthogonal to $\mathbf{w}_3$, but not to each other. Therefore another vector can be chosen as a principal axis, for simplicity $\mathbf{w}_2$ is chosen, so that the only eigenvector that must be orthonormalized is $\mathbf{w}_1$. Before doing so, the other two vectors must be normalized. The norm of a vector $\mathbf{w}_i$ is $\sqrt{\mathbf{w}_i \cdot \mathbf{w}_i}$ (note that assuming $0 < \phi < \pi/2$ means that all trig functions are positive definite), so the first two principal axes are

$$\hat{\mathbf{e}}_2 = \frac{1}{\sec \omega t}\mathbf{w}_2 = \mathbf{w}_2 \cos \omega t = \begin{bmatrix} \sin \omega t \\ -\cos \omega t \\ 0 \end{bmatrix} \quad \hat{\mathbf{e}}_3 = \frac{1}{\sec\phi}\mathbf{w}_3 = \mathbf{w}_3 \cos\phi = \begin{bmatrix} \sin\phi \cos \omega t \\ \sin\phi \sin \omega t \\ \cos\phi \end{bmatrix} . \tag{4}$$

To find the orthonormal vector $\mathbf{w}_2$, the Gram-Schmidt procedure is used,

$$\hat{\mathbf{e}}_2 = \mathbf{w}_2 - \mathrm{proj}_{\hat{\mathbf{e}}_1}(\mathbf{w}_2) - \mathrm{proj}_{\hat{\mathbf{e}}_3}(\mathbf{w}_2) \ , \tag{5}$$

with the projection operator defined as

$$\mathrm{proj}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} \ . \tag{6}$$

Note that $\mathbf{w}_1$ is already orthogonal to $\mathbf{w}_3$, so $\text{proj}_{\hat{\mathbf{e}}_3}(\mathbf{w}_1) = 0$. Therefore the last orthogonal vector is

$$\mathbf{e}_1 = \begin{bmatrix} -\cot\phi\cos\omega t \\ -\cot\phi\sin\omega t \\ 1 \end{bmatrix} \quad \Rightarrow \quad \hat{\mathbf{e}}_1 = \frac{1}{\sqrt{\csc^2\phi}} \begin{bmatrix} -\cot\phi\cos\omega t \\ -\cot\phi\sin\omega t \\ 1 \end{bmatrix} = \begin{bmatrix} -\cos\phi\cos\omega t \\ -\cos\phi\sin\omega t \\ \sin\phi \end{bmatrix} . \tag{7}$$

Now that the three principal axes have been determined, the torque projections along each of these axes can be found by knowing the angular velocity projected onto the appropriate axis. The components of $\boldsymbol{\omega}$ in the coordinate basis are $(0, 0, \omega)$, so the transformed angular velocity in the principal axis basis is

$$\boldsymbol{\omega}' = \text{proj}_{\hat{\mathbf{e}}_1}(\boldsymbol{\omega}) + \text{proj}_{\hat{\mathbf{e}}_2}(\boldsymbol{\omega}) + \text{proj}_{\hat{\mathbf{e}}_3}(\boldsymbol{\omega}) = \begin{bmatrix} (\boldsymbol{\omega}\cdot\hat{\mathbf{e}}_1) \\ (\boldsymbol{\omega}\cdot\hat{\mathbf{e}}_2) \\ (\boldsymbol{\omega}\cdot\hat{\mathbf{e}}_3) \end{bmatrix} = \omega \begin{bmatrix} \sin\phi \\ 0 \\ \cos\phi \end{bmatrix} . \tag{8}$$

With these components, the torque can be found using Euler's equations, given by Goldstein Equation 5.39. Since none of the components of $\boldsymbol{\omega}'$ are explicity time dependent, the first term in each torque vanishes. Additionally, two of the components depend on the value of ω_2', which in this case is zero, so the only projection that is nonzero is along $\hat{\mathbf{e}}_2$. Therefore the torque along the principal axes is

$$\boldsymbol{\tau}' = \begin{bmatrix} 0 \\ -\omega_3\omega_1(I_3 - I_1) \\ 0 \end{bmatrix} = \omega^2\sin\phi\cos\phi(2mb^2) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = m\omega^2 b^2 \sin(2\phi)\hat{\mathbf{e}}_2 . \tag{9}$$

The torque along the coordinate axis, from this, is just this multiplied through by $\hat{\mathbf{e}}_2$, which has components in the coordinate basis:

$$\boldsymbol{\tau} = \tau_1'\hat{\mathbf{e}}_1 + \tau_2'\hat{\mathbf{e}}_2 + \tau_3'\hat{\mathbf{e}}_3 = m\omega^2 b^2 \sin 2\phi \begin{bmatrix} \sin\omega t \\ -\cos\omega t \\ 0 \end{bmatrix} . \tag{10}$$

1.2 Components of Torque Along Cartesian Axes.

The components of the torque in the coordinate basis can be computed directly, without knowing the principal moments. If the angular momentum is known (it was calculated in the last homework), the net torque can be found by Goldstein Equation 5.37,

$$\boldsymbol{\tau} = \boldsymbol{\omega} \times \mathbf{L} = \begin{bmatrix} b^2 m\omega^2 \sin 2\phi \sin\omega t \\ -b^2 m\omega^2 \sin 2\phi\cos\omega t \\ 0 \end{bmatrix} = m\omega^2 b^2 \sin 2\phi \begin{bmatrix} \sin\omega t \\ -\cos\omega t \\ 0 \end{bmatrix} , \tag{11}$$

which agrees with the result from the previous section.

2 Problem #2: Rotating Plate.

Consider a rectangular plate of sides a and b rotating about a diagonal axis (orner to corner) with angular velocity around that axis of ω. Align a Cartesian coordinate system so the rectangle lies in the $x-z$ plane, with the origin such that the rectangle extends to $z=\pm a/2$ and $x=\pm b/2$, as shown in Figure 2. The components inertia tensor for this plate are given by

$$I_{ij}=\rho(\mathbf{r})\int_V [(x^2+y^2+z^2)\delta_{ij}-r_ir_j]\ d^3\mathbf{r}=\sigma\int_0^a\int_0^b [(x^2+z^2)\delta_{ij}-r_ir_j]\ dx\ dz\ , \tag{12}$$

with $r_k\in\{x,y,z\}$, because a point in the body always has a zero y component of its position. This also means all cross terms with a y component vanish. The diagonal elements are

$$I_{xx}=\sigma\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}[(x^2+z^2)-x^2]\ dx\ dz\ =\frac{a^2M}{12} \tag{13}$$

$$I_{yy}=\sigma\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}[(x^2+z^2)]\ dx\ dz\ =\frac{1}{12}M\left(a^2+b^2\right) \tag{14}$$

$$I_{zz}=\sigma\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}[(x^2+z^2)-z^2]\ dx\ dz\ =\frac{b^2M}{12}\ , \tag{15}$$

and the only potentially non-zero off-diagonal terms are

$$I_{xz}=I_{zx}=\sigma\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}[-xz]\ dx\ dz\ =0\ . \tag{16}$$

Note that because the inertia tensor along these axes is diagonal, they must be a set of principal axes. The torque required to maintain the rotation is given by

$$\boldsymbol{\tau}=\frac{d\mathbf{L}}{dt}=\mathbf{L}\times\boldsymbol{\omega}\ . \tag{17}$$

The angular velocity vector points in a direction in the $x-z$ plane proportional to the ratio of side lengths of the rectangle. The corners are a distance $\sqrt{a^2+b^2}$ from each other and the components are b in the x direction and a in the z direction. This gives the unit vector pointing along $\boldsymbol{\omega}$,

$$\hat{\boldsymbol{\omega}}=\frac{1}{\sqrt{a^2+b^2}}(b\hat{\mathbf{i}}+a\hat{\mathbf{k}})\ , \tag{18}$$

given the vector has a magnitude ω, the complete vector is $\boldsymbol{\omega}=\omega\hat{\boldsymbol{\omega}}$.

The angular momentum of this rotation is given by

$$\mathbf{L}=\hat{I}\cdot\boldsymbol{\omega}=\frac{M}{12}\frac{\omega}{\sqrt{a^2+b^2}}\begin{bmatrix}a^2 & 0 & 0\\ 0 & a^2+b^2 & 0\\ 0 & 0 & b^2\end{bmatrix}\begin{bmatrix}b\\0\\a\end{bmatrix}=\frac{M}{12}\frac{\omega}{\sqrt{a^2+b^2}}\begin{bmatrix}a^2b\\0\\ab^2\end{bmatrix}\ , \tag{19}$$

where $\hat{I}$ is the inertia tensor. The torque can be now be found directly from Goldstein Equation 5.37

$$\boldsymbol{\tau}=\frac{d\mathbf{L}}{dt}+\boldsymbol{\omega}\times\mathbf{L}=\frac{M}{12}\frac{\omega}{\sqrt{a^2+b^2}}\frac{\omega}{\sqrt{a^2+b^2}}\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ b & 0 & a\\ a^2b & 0 & ab^2\end{vmatrix}\ , \tag{20}$$

working out the determinant and noting that the only surviving term is in the y direction, the torque is

$$\boldsymbol{\tau}=\frac{M}{12}\frac{\omega^2}{a^2+b^2}(a^3b-ab^3)\hat{\mathbf{j}}=\frac{M\omega^2ab}{12}\frac{b^2-a^2}{a^2+b^2}\hat{\mathbf{j}}\ . \tag{21}$$

3 Goldstein 5.16.

Three equal mass points (mass m) are located at $(a, 0, 0), (0, a, 2a)$, and $(0, 2a, a)$ for the coordinates (x, y, z). The components of the inertia tensor $\hat{I}$ about the origin are given by

$$I_{jk} = \sum_{i=1}^{3} m_{(i)} [r_{(i)}^2 \delta_{jk} - r_{j(i)} r_{k(i)}] \ , \tag{22}$$

where i is an index that corresponds to the i^{th} mass, while $j, k \in \{x, y, x\}$. The Cartesian distance for each mass is $\{a^2, 5a^2, 5a^2\}$. The first component is

$$I_{11} = m \sum_{i=1}^{3} [r_{(i)}^2 \delta_{jk} - x_{(i)} x_{(i)}] \tag{23}$$

$$= m[(a^2 - a \cdot a) + (5a^2 - 0) + (5a^2 - 0)] \ , \tag{24}$$

and the other components follow. Therefore the inertia tensor is given by

$$I = ma^2 \begin{bmatrix} 10 & 0 & 0 \\ 0 & 6 & -4 \\ 0 & -4 & 6 \end{bmatrix} \ , \tag{25}$$

the principal moments are the eigenvalues of this matrix, and the principal axes are the normalized eigenvectors, assuming they are all orthogonal. Using MATHEMATICA the principal moments are

$$I_1 = 10ma^2 \qquad I_2 = 10ma^2 \qquad I_3 = 2ma^2 \ , \tag{26}$$

and the principal axes are

$$\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad \hat{\mathbf{e}}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{\mathbf{e}}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \ , \tag{27}$$

which are in fact orthogonal.

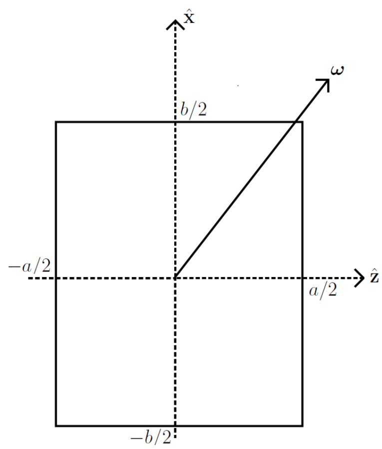

Figure 2: Depiction of the coordinate system and rectangle orientation for problem #2.

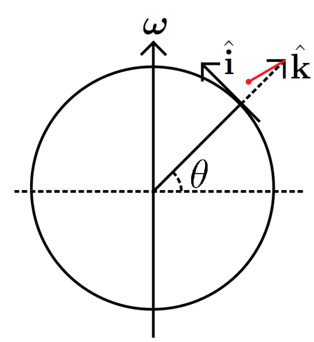

Figure 3: Depiction of the coordinate system for the Foucault pendulum in Problem #4.

4 Goldstein 4.23.

The Foucault pendulum experiment consists in setting a long pendulum in motion at a point on the surface of the rotating Earth with its momentum originally in the vertical plane containing the pendulum bob and the point of suspension. Consider a massless pendulum of length ℓ suspending a mass m in a uniform downwards gravitational acceleration. Pick a right handed reference frame on the surface of the earth such that the origin is set to the equilibrium position of the pendulum. From here the z axis points vertically (normal to the surface of the earth). In the small oscillation limit, the motion of the mass is confined to the horizontal $(x-y)$ plane. In this approximation, z and $\dot{z}$ can be set to zero. Using Newton's second law the vector equation of motion for the pendulum mass is

$$m\ddot{\mathbf{r}} = m\mathbf{g} + \mathbf{T} - 2m\boldsymbol{\omega} \times \dot{\mathbf{r}} \ , \tag{28}$$

as given by Thornton and Marion Equation 10.42. In this expression $\mathbf{r}$ is the distance vector, and $\mathbf{T}$ is the tension vector which always points towards the pivot. The final term is the force due to the Coriolis effect, with ω as the frequency of the Earth's rotation. The projections of $\mathbf{T}$ in the $x-y$ plane must always be negative because the mass will always be attracted to the equilibrium position at $(x, y) = (0, 0)$. The projections in this plane are small (in the small angle approximation) so the projection in the z axis is approximately $|\mathbf{T}|$. For this small angle approximation (to first order), the projections into x and y can be expressed as the ratio of the coordinate to the length of the pendulum, because $\sin\theta \simeq \theta$. Therefore the equation of motion for each projection becomes

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} + \frac{T}{m} \begin{bmatrix} -\frac{x}{\ell} \\ -\frac{y}{\ell} \\ 1 \end{bmatrix} - 2 \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} \ , \tag{29}$$

so all that remains is the components of angular velocity. As shown in Figure 3, $\boldsymbol{\omega}$ does not have any component in the y direction (out of the page). Given the co-latitude θ, the angular velocity vector is $\boldsymbol{\omega} = (-\omega\cos\theta, 0, \omega\sin\theta)$. So the cross product in Equation 28 becomes

$$\boldsymbol{\omega} \times \dot{\mathbf{r}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\omega\cos\theta & 0 & \omega\sin\theta \\ \dot{x} & \dot{y} & 0 \end{vmatrix} = \begin{bmatrix} 0 - \omega\dot{y}\sin\theta \\ -(0 - \omega\dot{x}\sin\theta) \\ -\omega\dot{y}\cos\theta - 0 \end{bmatrix} \ . \tag{30}$$

The three equations of motion then become

$$\ddot{x} = -\frac{T}{m}\frac{x}{\ell} - 2(-\omega\dot{y}\sin\theta) \tag{31}$$

$$\ddot{y} = -\frac{T}{m}\frac{y}{\ell} - 2(\omega\dot{x}\sin\theta) \tag{32}$$

$$\ddot{z} = 0 = -g + \frac{T}{m} - 2(-\omega\dot{y}\cos\theta) \ , \tag{33}$$

which in the limit that the Earth's rotational frequency is negligible compared to the frequency of the pendulum, the term with ω can be dropped in the last equation. Therefore the last equation can be solved with $T \simeq mg$. This substitution yields a pair of coupled second order equations

$$\ddot{x} = -\frac{g}{\ell}x + 2\omega\dot{y}\sin\theta \tag{34}$$

$$\ddot{y} = -\frac{g}{\ell}y - 2\omega\dot{x}\sin\theta \ . \tag{35}$$

If a coordinate ξ is defined to be $x + iy$, then

$$\ddot{\xi} = -\frac{g}{\ell}\xi - 2i\omega\dot{\xi}\sin\theta \quad \Rightarrow \quad \ddot{x} + i\ddot{y} = -\frac{g}{\ell}(x+iy) - 2i\omega(\dot{x}+i\dot{y})\sin\theta \tag{36}$$

$$= -\frac{g}{\ell}x - 2i\omega\dot{x}\sin\theta - \frac{g}{\ell}iy - 2\omega(i^2\dot{y})\sin\theta = \left[-\frac{g}{\ell}x + 2\omega\dot{y}\sin\theta\right] + i\left[-\frac{g}{\ell}y - 2\omega\dot{x}\sin\theta\right] \tag{37}$$

and the coupled equations can be merged to one, such that $\mathrm{Re}[\xi(t)] = x(t)$ and $\mathrm{Im}[\xi(t)] = y(t)$. The solutions to the differential equation in ξ (given by MATHEMATICA) are

$$\xi(t) = e^{-i\omega\sin\theta t}\left[c_1\exp\left(t\sqrt{-\frac{g^2}{l^2} - \omega^2\sin^2\theta}\right) + c_2\exp\left(-t\sqrt{-\frac{g^2}{l^2} - \omega^2\sin^2\theta}\right)\right] \ , \tag{38}$$

note that for $\omega = 0$ (no Earth rotation) the equation of motion reduces to the simple pendulum result, with frequency $\omega_0 = \sqrt{g/\ell}$. It has been previously stated that the rotational frequency of the Earth is negligible compared to the frequency of the pendulum, the solution can be rewritten using Euler trig identities

$$\xi(t) = e^{-i\omega\sin\theta t}\left[A\cos(\omega_0 t + \delta_1) + iB\sin(\omega_0 t + \delta_2)\right] \ . \tag{39}$$

Assume the pendulum's momentum is originally in the vertical plane of the pendulum mass and pivot point, so $\dot{x}_0 = v$ and $\dot{y}_0 = 0$, and it is in the equilibrium position at $t = 0$. Consider the first derivative,

$$\dot{x}(0) + i\dot{y}(0) = \left[(-i\omega\sin\theta)\left\{A\cos(\delta_1) + iB\sin(\delta_2)\right\} + \omega_0\left\{-A\sin(\delta_1) + iB\cos(\delta_2)\right\}\right] \ , \tag{40}$$

which implies $B = 0$. This is because the left side is real $\dot{\xi}(0) = v$, which is entirely in x, and equating real and imaginary parts means that either $B = 0$ or the two imaginary parts exactly cancel. This can not be the case because the assumption $\omega \ll \omega_0$ has been made. The constraints given by this and the initial conditions give

$$0 = A\cos(\delta_1) \tag{41}$$

$$v = (-i\omega\sin\theta)\left\{A\cos(\delta_1)\right\} + \omega_0\left\{-A\sin(\delta_1)\right\} \ . \tag{42}$$

The first constraint is that δ_1 is an odd integer multiple of $\pi/2$, so that $A = -v/\omega_0$. Therefore the solution is

$$\xi(t) = -e^{-i\omega\sin\theta t}\frac{v}{\omega_0}\cos\left(\omega_0 t + \frac{\pi}{2}\right) \tag{43}$$

$$= -\frac{v}{\omega_0}\cos\left(\omega_0 t + \frac{\pi}{2}\right)\left[\cos(\omega\sin\theta t) - i\sin(\omega\sin\theta t)\right] \ , \tag{44}$$

so the equations of motion in the original coordinates are

$$x(t) = -\frac{v}{\omega_0}\cos\left(\omega_0 t + \frac{\pi}{2}\right)\cos(\omega\sin\theta t) \tag{45}$$

$$y(t) = \frac{v}{\omega_0}\cos\left(\omega_0 t + \frac{\pi}{2}\right)\sin(\omega\sin\theta t) \ . \tag{46}$$

This gives the angle of precession of the pendulum plane to be $\Theta = \Omega t = \omega\sin\theta t$, thus the precessional frequency is $\Omega = \omega\sin\theta$, and the rotational frequency of Earth is 2π radians per day. Knowing the relationship between latitude (θ) and colatitude (θ_c) is $\theta + \theta_c = (\pi/2)$, and $\sin([\pi/2] - \alpha) = \cos(\alpha)$, the precessional frequency is $\Omega = 2\pi\cos\theta_c$ per day. Therefore in one day the plane of oscillation rotates $2\pi\cos\theta_c$ radians, the direction of rotation is based on the sign of the cosine. In the northern hemisphere $0 \le \theta_c < \pi/2$, so cosine is positive and the plane of oscillation rotates counterclockwise. Contrarily, in the southern hemisphere $\pi/2 < \theta_c \le \pi$, so the cosine is negative and the pendulum precesses clockwise.

5 Problem #5: Beads on Rods.

A bead of mass m is free to slide on a frictionless straight rod, which lies in a horizontal plane. The rod is spun with a constant angular velocity ω about a vertical axis through the midpoint of the rod.

5.1 The Hamiltonian.

Consider the polar coordinates r and $\theta = \omega t$, with the origin set in the horizontal plane of the rod, at the axis of rotation. The location of the bead is then $(r, \omega t)$, with speed $(\dot{r}, \omega)$. The kinetic energy of this system is given by

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2) = \mathcal{L} \ , \tag{47}$$

which is also the Lagrangian and total energy of the system because there is no potential. So the conjugate momentum for r is

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial}{\partial \dot{r}}\frac{1}{2}m(\dot{r}^2 + r^2\omega^2) = m\dot{r} \ . \tag{48}$$

The Hamiltonian is then

$$\mathcal{H} = \dot{q}p - \mathcal{L} = \dot{r}p - \frac{1}{2}m\dot{r}^2 - \frac{1}{2}mr^2\omega^2 = \frac{p}{m}p - \frac{1}{2}m\left(\frac{p}{m}\right)^2 - \frac{1}{2}mr^2\omega^2 \tag{49}$$

$$= \frac{1}{2m}p^2 - \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}m\dot{r}^2 - \frac{1}{2}m\omega^2 r^2 \neq T \ , \tag{50}$$

so the Hamiltonian is not equal to the total energy of the system.

5.2 Hamiltonian Equations.

For this system, Hamilton's equations are

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} \ \Rightarrow \ \dot{r} = \frac{p}{m} \qquad \text{and} \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} \ \Rightarrow \ \dot{p} = -\frac{\partial \mathcal{H}}{\partial r} = -\left(-m\omega^2 r\right) \ , \tag{51}$$

differentiating the first equation with respect to time and equating to the second yields

$$m\ddot{r} = m\omega^2 r \ , \tag{52}$$

which is the equation of motion for r.

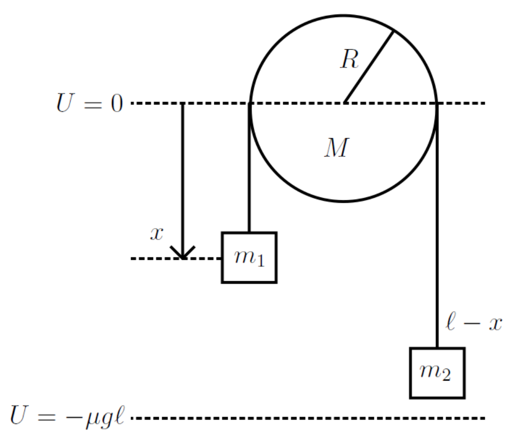

Figure 4: Depiction of the coordinate system for the Atwood in machine in Problem #6.

6 Problem #6: Massful Pulley.

Consider an Atwood machine with masses m_1 and m_2 suspended over a pulley of mass M and radius R in a uniform downwards gravitational acceleration g. This is a one dimensional problem because the motion of both masses and the pulley can be entirely represented by one coordinate x. Define the origin of the coordinate system to be the level of the center of mass of the pulley, with positive x the distance downwards, as shown in Figure 4. Therefore if one mass moves in a distance $+x$, the other moves a distance $-x$. Let the length of the string be $\ell + \pi R^2$, so that when the masses are at their farthest possible distance one is at $x = 0$ and one is at $x = \ell$. The potential energy of a mass μ at this position is $-\mu g \ell$, with $g > 0$. It is useful to note the moment of inertia for a uniform disk about an axis through its center, normal to the circular face, is $I = \frac{1}{2}MR^2$. This information allows the Lagrangian to be written down immediately,

$$\mathcal{L} = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_1(-\dot{x})^2 + \frac{1}{2}I\omega^2 - [-m_1gx + m_2g(\ell - x)] \ , \tag{53}$$

where ω is the rotational frequency of the pulley, in this case just $\dot{x}/R$. Simplification and substitution yields

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}\left[\frac{1}{2}MR^2\right]\left(\frac{\dot{x}}{R}\right)^2 + m_2g\ell + g(m_1 - m_2)x \tag{54}$$

$$\mathcal{L} = \frac{1}{2}\left(m_1 + m_2 + \frac{1}{2}M\right)\dot{x}^2 + m_2g\ell + g(m_1 - m_2)x \ . \tag{55}$$

Therefore the conjugate momentum for x is

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \left(m_1 + m_2 + \frac{1}{2}M\right)\dot{x} \ . \tag{56}$$

Additionally, the Hamiltonian is given by

$$\mathcal{H} = \dot{x}p - \mathcal{L} = \frac{p}{\left(m_1 + m_2 + \frac{1}{2}M\right)}p - \frac{1}{2}\frac{\left(m_1 + m_2 - \frac{1}{2}M\right)}{\left(m_1 + m_2 + \frac{1}{2}M\right)^2}p^2 - m_2g\ell - g(m_1 - m_2)x \tag{57}$$

$$= \frac{\frac{1}{2}p^2}{\left(m_1 + m_2 + \frac{1}{2}M\right)} - m_2g\ell - g(m_1 - m_2)x \ , \tag{58}$$

after substituting in Equation 56, note this is the total energy. Hamilton's equations of motion are

$$\dot{q} = \dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m_1 + m_2 + \frac{1}{2}M} \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = -[-g(m_1 - m_2)] \ , \tag{59}$$

which can be reduced for a second order differential equation in x. By differentiating $\dot{q}$ with respect to time and substituting in $\dot{p}$, this becomes

$$\ddot{x} = \frac{g(m_1 - m_2)}{m_1 + m_2 + \frac{1}{2}M} \ . \tag{60}$$

This answer makes sense because it is either monotonically decreasing or increasing, or zero depending on m_1/m_2, with constant acceleration. This is due to the heavier mass moving the system due to gravity, but fought by the other mass moving upward as well as the inertia of the pulley. The sign of the answer makes sense, for if $m_2 > m_1$, m_1 will accelerate upwards, which is the negative direction.