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Classical Mechanics, Third Ed.- Goldstein
November 30, 2015

Contents

1 Goldstein 8.3.

Consider a function $\mathcal{G}(p, \dot{p}; t)$ which is defined as a double Legendre transformation of the Lagrangian, $\mathcal{L}(q, \dot{q}; t)$, with p and $\dot{p}$ as independent variables.

1.1 Form of $\mathcal{G}$.

The Hamiltonian is constructed by performing a Legendre transformation on the variable $\dot{q}_i$ to the variable p_i. From a generic Lagrangian, $\mathcal{L}(q, \dot{q}; t)$, the Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ . \tag{1}$$

Now consider the total differential of the Lagrangian,

$$d\mathcal{L}(q, \dot{q}; t) = \frac{\partial \mathcal{L}}{\partial q_i} dq_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial \mathcal{L}}{\partial t} dt \ . \tag{2}$$

In order to properly transform $\dot{q} \to p$, and get the Hamiltonian, $\mathcal{H}(q, p; t)$, there must be no $d\dot{q}_i$ dependence in the total differential. Therefore the variable being transformed from is multiplied by the variable being transformed to, and the function that is being transformed is subtracted,

$$\mathcal{H} = \dot{q}_i p_i - \mathcal{L} \ . \tag{3}$$

Now consider the total differential of the Hamiltonian,

$$d\mathcal{H}(q, p; t) = \frac{\partial \mathcal{H}}{\partial q_i} dq_i + \frac{\partial \mathcal{H}}{\partial p_i} dp_i + \frac{\partial \mathcal{H}}{\partial t} dt = p_i d\dot{q}_i + \dot{q}_i dp_i - \frac{\partial \mathcal{L}}{\partial q_i} dq_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial t} dt \ , \tag{4}$$

after plugging in the Lagrangian. As stated before, the terms with $d\dot{q}_i$ dependence must cancel, so the following relationship for the transformation is obtained by equating the coefficients of these terms,

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ . \tag{5}$$

Hamilton's equations can be found by equating the coefficients of similar total differential terms:

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad \frac{\partial \mathcal{H}}{\partial q_i} = -\frac{\partial \mathcal{L}}{\partial q_i} \equiv -S \qquad \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \ . \tag{6}$$

Note that taking the time derivative of Equation 5 yields

$$\frac{d}{dt} p_i = \dot{p}_i = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} = S, \tag{7}$$

so $S = \dot{p}_i$ and the correct Hamilton equation is obtained (as expected). Now consider a second Legendre transformation, $q_i \to S = \dot{p}_i$. Following the procedure above, the terms with dq_i dependence must cancel in the total differential of the new double-transformed function,

$$\mathcal{G} = q_i \dot{p}_i \pm \mathcal{H} \quad \Rightarrow \quad d\mathcal{G} = q_i d\dot{p}_i + \dot{p}_i dq_i \pm \left[\dot{q}_i dp_i - \frac{\partial \mathcal{L}}{\partial q_i} dq_i - \frac{\partial \mathcal{L}}{\partial t} dt \right] \ , \tag{8}$$

so the plus form must be correct (to get the terms to cancel), so the new function is

$$\mathcal{G} = q_i \dot{p}_i + \mathcal{H} \quad \Rightarrow \quad d\mathcal{G} = q_i d\dot{p}_i + \dot{q}_i dp_i - \frac{\partial \mathcal{L}}{\partial t} dt = q_i d\dot{p}_i + \dot{q}_i dp_i + \frac{\partial \mathcal{H}}{\partial t} dt \ . \tag{9}$$

To get the equivalent of Hamilton's equations for this double-Legendre transformed function, consider the definition of the total differential of $\mathcal{G}$,

$$d\mathcal{G}(p_i, \dot{p}_i; t) = \frac{\partial \mathcal{G}}{\partial p_i} dp_i + \frac{\partial \mathcal{G}}{\partial \dot{p}_i} d\dot{p}_i + \frac{\partial \mathcal{G}}{\partial t} dt \ , \tag{10}$$

and equate the coefficients of like terms from the previous expression for the total differential of $\mathcal{G}$. The resulting equations are

$$\dot{q}_i = \frac{\partial \mathcal{G}}{\partial p_i} \qquad q_i = \frac{\partial \mathcal{G}}{\partial \dot{p}_i} \qquad \frac{\partial \mathcal{G}}{\partial t} = \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \ . \tag{11}$$

1.2 Test Case: Simple Harmonic Oscillator.

Consider the simple case of a one dimensional horizontal harmonic oscillator consisting of a mass m attached to a spring of constant k. This can be solved with the function $\mathcal{G}$ as defined above. The Lagrangian for the simple one dimensional oscillator is trivial to write down, but using Equations 7 and 5, it can be shown that

$$p = m\dot{q} \qquad \dot{p} = -m\omega^2 q \ , \tag{12}$$

where $\omega = \sqrt{k/m}$ is the frequency of oscillation. Therefore,

$$\mathcal{G} = -\frac{\dot{p}^2}{m\omega^2} + \frac{p^2}{2m} + \frac{m\omega^2}{2}\left(-\frac{\dot{p}}{m\omega^2}\right)^2 = \frac{p^2}{2m} - \frac{\dot{p}^2}{2m\omega^2} \ . \tag{13}$$

So the Hamilton-esque equations are

$$\dot{q} = p/m \qquad q = -\frac{\dot{p}}{m\omega^2} \ , \tag{14}$$

so by differentiating the left most equation, solving the right most equation for $\dot{p}$, and plugging that into the expression for $\ddot{q}$, the equation

$$\ddot{q} = -\omega^2 q \ , \tag{15}$$

is obtained, which is easily recognizable as the equation for a simple harmonic oscillator in one dimension. Note that the potential term in $\mathcal{G}$ does not explicitly depend on velocity, so the Hamiltonian is the total energy. Additionally because the partial derivative with respect to time of $\mathcal{L}, \mathcal{H}$, and $\mathcal{G}$ are all zero, the Hamiltonian is conserved, and therefore total energy is a constant of the motion.

2 Problem #2: A 3D Central Potential.

Consider a full three dimensional central potential $U_E(r)$, using standard spherical polar coordinates $\{r, \theta, \phi\}$ as the generalized coordinates. In terms of the spherical coordinates, the Cartesian coordinates are

$$x = r\sin\theta\cos\phi \;\rightarrow\; \dot{x} = \dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi \tag{16}$$

$$y = r\sin\theta\sin\phi \;\rightarrow\; \dot{y} = \dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi \tag{17}$$

$$z = r\cos\theta \;\rightarrow\; \dot{z} = \dot{r}\cos\theta - \dot{\theta}r\sin\theta \; . \tag{18}$$

Note that v^2 is given by $\dot{x}^2 + \dot{y}^2 + \dot{z}^2$, so

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 \; . \tag{19}$$

2.1 The Hamiltonian.

Given v^2 and the general potential, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\sin^2\theta\dot{\phi}^2 - U_E(r) \; . \tag{20}$$

The conjugate momenta are given by $P_q = \partial\mathcal{L}/\partial\dot{q}$, so

$$P_r = \frac{\partial\mathcal{L}}{\partial\dot{r}} = m\dot{r} \quad \Rightarrow \quad \dot{r} = P_r/m \tag{21}$$

$$P_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mr^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = P_\theta/mr^2 \tag{22}$$

$$P_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mr^2\sin^2\theta\dot{\phi} \quad \Rightarrow \quad \dot{\phi} = P_\phi/mr^2\sin^2\theta \; . \tag{23}$$

Therefore the Hamiltonian is given by

$$\mathcal{H} = \dot{r}P_r + \dot{\theta}P_\theta + \dot{\phi}P_\phi - \mathcal{L} \tag{24}$$

$$= \frac{P_r^2}{m} + \frac{P_\theta^2}{mr^2} + \frac{P_\phi^2}{mr^2\sin^2\theta} - \frac{P_r^2}{2m} - \frac{P_\theta^2}{2mr^2} - \frac{P_\phi^2}{2mr^2\sin^2\theta} + U_E \tag{25}$$

$$= \frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} + \frac{P_\phi^2}{2mr^2\sin^2\theta} + U_E = E_{tot} \; , \tag{26}$$

and is equal to the total energy of the system.

2.2 Constants of Motion.

From this Hamiltonian, it is easy to determine some constants of motion. Noting that the Hamiltonian is not explicitly dependent on time means it is conserved, and therefore the total energy is as well. The conjugate momentum to ϕ is also conserved because from Hamilton's equations,

$$\dot{P}_\phi = -\frac{\partial\mathcal{H}}{\partial\phi} = 0 \quad \Rightarrow \quad P_\phi = cst. \tag{27}$$

2.3 Hamilton's Equations.

The remaining Hamilton equations are

$$\dot{P}_r = -\frac{\partial \mathcal{H}}{\partial r} = \frac{P_\theta^2}{mr^3} + \frac{P_\phi^2}{mr^3 \sin^2\theta} - \frac{\partial U_E}{\partial r} \tag{28}$$

$$\dot{P}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = \frac{P_\phi^2 \cos\theta}{mr^2 \sin^3\theta} \tag{29}$$

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial P_r} = P_r/m \quad \Rightarrow \quad m\ddot{r} = \dot{P}_r \tag{30}$$

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial P_\theta} = \frac{P_\theta}{mr^2} \quad \Rightarrow \quad \frac{d}{dt}[mr^2\dot{\theta}] = \dot{P}_\theta \tag{31}$$

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial P_\phi} = \frac{P_\phi}{mr^2 \sin^2\theta} \quad \Rightarrow \quad \frac{d}{dt}[mr^2 \sin^2\theta\dot{\phi}] = \dot{P}_\phi \ , \tag{32}$$

after differentiating the second set of equations with respect to time. Now the first three equations can be equated to the second three using the expressions on the right hand side of the equations above. Taking the indicated time derivatives and setting them equal to the corresponding momentum time derivative yields

$$m\ddot{r} = \frac{P_\theta^2}{mr^3} + \frac{P_\phi^2}{mr^3 \sin^2\theta} - \frac{\partial U_E}{\partial r} \tag{33}$$

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = \frac{P_\phi^2 \cos\theta}{mr^2 \sin^3\theta} \tag{34}$$

$$2mr\dot{r}\sin^2\theta\dot{\phi} + 2mr^2\dot{\theta}\sin\theta\cos\theta\dot{\phi} + mr^2\sin^2\theta\ddot{\phi} = 0 \ . \tag{35}$$

From here the definitions of the conjugate momenta (Equations 72 - 23) can be substituted in, and the expressions solved for the second time derivative terms,

$$m\ddot{r} = mr\dot{\theta}^2 + mr\sin^2\theta\dot{\phi}^2 - \frac{\partial U_E}{\partial r} \tag{36}$$

$$mr^2\ddot{\theta} = mr^2\sin\theta\cos\theta\dot{\phi}^2 - 2mr\dot{r}\dot{\theta} \tag{37}$$

$$mr^2\sin^2\theta\ddot{\phi} = -2mr\dot{r}\sin^2\theta\dot{\phi} - 2mr^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} \ , \tag{38}$$

which are the coupled equations of motion for r, θ, and ϕ.

2.4 Comparison to Euler-Lagrange Equations.

These equations of motion can now be compared to the equations of motion found by using the Lagrangian method. The Euler-Lagrange equations are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q} \ . \tag{39}$$

Applying this to the Lagrangian given in Equation 20 gives the three Lagrangian equations of motion,

$$\frac{d}{dt}[m\dot{r}] = mr\dot{\theta}^2 + mr\sin^2\theta\dot{\phi}^2 - \frac{\partial U_e}{\partial r} \quad \Rightarrow \quad m\ddot{r} = mr\dot{\theta}^2 + mr\sin^2\theta\dot{\phi}^2 - \frac{\partial U_e}{\partial r} \tag{40}$$

$$\frac{d}{dt}[mr^2\dot{\theta}] = mr^2\sin\theta\cos\theta\dot{\phi}^2 \quad \Rightarrow \quad mr^2\ddot{\theta} = mr^2\sin\theta\cos\theta\dot{\phi}^2 - 2mr\dot{r}\dot{\theta} \tag{41}$$

$$\frac{d}{dt}[mr^2\sin^2\theta\dot{\phi}] = 0 \quad \Rightarrow \quad mr^2\sin^2\theta\ddot{\phi} = -2mr\dot{r}\sin^2\theta\dot{\phi} - 2mr^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} \tag{42}$$

which are exactly the same equations of motion found through the Hamiltonian approach.

3 Problem #3: Constrained Motion with Hamiltonians.

Consider a particle of mass m subject to a central force $\mathbf{F} = -k\mathbf{r}$, where $\mathbf{r}$ is a vector from the origin to the particle. The particle is constrained to move on a cylinder centered on the z-axis, defined by $x^2 + y^2 = R^2$. The transformation from cylindrical coordinates on this surface to Cartesian coordinates is given by

$$x = R\cos\theta \quad \Rightarrow \quad \dot{x} = -R\dot{\theta}\sin\theta \tag{43}$$

$$y = R\sin\theta \quad \Rightarrow \quad \dot{y} = R\dot{\theta}\cos\theta \tag{44}$$

$$z = z \quad \Rightarrow \quad \dot{z} = \dot{z}\ , \tag{45}$$

so the velocity squared is

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{\theta}^2 R^2 + \dot{z}^2\ , \tag{46}$$

from which the kinetic energy can be found directly, $T = \frac{1}{2}mv^2$. In order to find the Lagrangian, the potential energy must still be determined. A conservative force can be written as $\mathbf{F} = -\nabla U$, where U is the potential energy. The force described above is

$$\mathbf{F} = -kx\hat{\mathbf{i}} - ky\hat{\mathbf{j}} - kz\hat{\mathbf{k}} = -\frac{\partial U}{\partial x}\hat{\mathbf{i}} - \frac{\partial U}{\partial y}\hat{\mathbf{j}} - \frac{\partial U}{\partial z}\hat{\mathbf{k}}\ , \tag{47}$$

which implies the potential is $U = \frac{1}{2}k(x^2 + y^2 + z^2) = \frac{1}{2}k(R^2 + z^2)$. Hence, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m\dot{z}^2 - \frac{1}{2}kR^2 - \frac{1}{2}kz^2\ , \tag{48}$$

and the conjugate momenta are

$$P_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mR^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = P_\theta / mR^2 \tag{49}$$

$$P_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = m\dot{\theta} \quad \Rightarrow \quad \dot{z} = P_z/m. \tag{50}$$

The Hamilton, then, is

$$\mathcal{H} = \dot{z}P_z + \dot{\theta}P_\theta - \mathcal{L} = \frac{P_z^2}{m} + \frac{P_\theta^2}{mR^2} - \frac{1}{2}mR^2\left(\frac{P_\theta}{mR^2}\right)^2 - \frac{1}{2}m\left(\frac{P_z}{m}\right)^2 + \frac{1}{2}kR^2 + \frac{1}{2}kz^2 \tag{51}$$

$$= \frac{1}{2}\frac{P_\theta^2}{mR^2} + \frac{1}{2}\frac{P_z^2}{m} + \frac{1}{2}kR^2 + \frac{1}{2}kz^2\ . \tag{52}$$

Using Hamilton's equations,

$$\dot{P}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = 0 \qquad \dot{P}_z = -\frac{\partial \mathcal{H}}{\partial z} = -kz \tag{53}$$

$$\dot{\theta} = -\frac{\partial \mathcal{H}}{\partial P_\theta} = \frac{P_\theta}{mR^2} \qquad \dot{z} = -\frac{\partial \mathcal{H}}{\partial P_z} = \frac{P_z}{m}\ . \tag{54}$$

From this, it is easy to see P_θ is a constant of the motion. By differentiating the expression for $\dot{z}$ with respect to time and substituting in the expression for P_z, the Hamiltonian equations of motion are

$$\ddot{z} = -\frac{k}{m}z \tag{55}$$

$$\ell = mR^2\dot{\theta}\ , \tag{56}$$

where ℓ is the angular momentum. So the motion is a harmonic oscillator in the z direction (about the origin) while the particle moves around the z-axis in a circle, with conserved angular momentum.

4 Goldstein 8.26.

A particle of mass m can move in one dimension under the influence of two springs connected to fixed points a distance a apart, as shown in Figure 1. The springs obey Hooke's Law and have zero unstretched lengths, and force constants k_1 and k_2. Let the coordinate q be the distance the mass is from the left wall, with the positive direction to the right. This means the equilibrium position of the mass is at $q = a/2$.

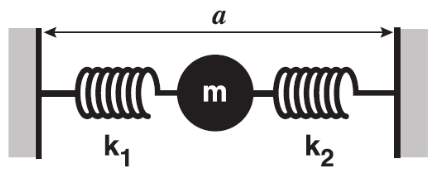

Figure 1: Depiction of the mass and spring system examined in problem #4.

4.1 Lagrangian and Hamiltonian.

When the mass is a distance x from the left wall, the left spring is compressed/extended to a length x while the right spring is compressed/expanded to a length $a - x$, therefore the potential energy of the system, using the generalized coordinate q, is

$$U = \frac{1}{2}k_1 q^2 + \frac{1}{2}k_2(a-q)^2 = \frac{1}{2}(k_1+k_2)q^2 - k_2 aq + \frac{1}{2}k_2 a^2 \ . \tag{57}$$

Note the last term is a constant, so the zero point of potential enery can be set so this term is zero (which will be done). The kinetic energy is the usual $T = \frac{1}{2}m\dot{q}^2$. Therefore the Lagrangian and, using the conjugate momentum to q: $p = m\dot{q}$, the Hamiltonian are given by

$$\mathcal{L} = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}(k_1+k_2)q^2 + k_2 aq \tag{58}$$

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}(k_1+k_2)q^2 - k_2 aq \ . \tag{59}$$

Note that there is no explicit time dependence in the Hamiltonian or Lagrangian, so the Hamiltonian is conserved. Additionally, the Hamilontian is easily recognizable as the total energy of the system, which is also conserved.

4.2 Change of Coordinates.

Consider the coordinate

$$Q = q - b\sin\omega t \quad \text{with} \quad b = \frac{k_2 a}{k_1+k_2} \ . \tag{60}$$

For the Lagrangian to be written in this coordinate, it can be manipulated starting with Equation 58,

$$\mathcal{L} = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}(k_1+k_2)\left(q^2 + \frac{k_2 a}{\frac{1}{2}(k_1+k_2)}q\right) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}(k_1+k_2)\left(q^2+2bq\right) \ , \tag{61}$$

noting that the last term in parenthesis can be written as $(q-b)^2 - b^2$ by completing the square. Again, the zero point of potential energy will be set so the constant term does not contribute. Therefore the Lagrangian, using the definition of Q can be written

$$\mathcal{L} = \frac{1}{2}m(\dot{Q} + \omega b\cos\omega t)^2 - \frac{1}{2}(k_1+k_2)(Q + b\sin\omega t - b)^2 \tag{62}$$

$$= \frac{1}{2}m\left(b^2\omega^2\cos^2\omega t + 2b\dot{Q}\omega\cos\omega t + \dot{Q}^2\right) - \frac{1}{2}(k_1+k_2)(Q + b\sin\omega t - b)^2 \ . \tag{63}$$

The conjugate momentum to Q is

$$P = \frac{\partial \mathcal{L}}{\partial \dot{Q}} = m\dot{Q} + mb\omega \cos \omega t \quad \Rightarrow \quad \dot{Q} = \frac{P}{m} - \omega b \cos \omega t \ , \tag{64}$$

and therefore the Hamiltonian is

$$\mathcal{H} = \dot{Q}P - \mathcal{L} = \frac{P^2}{m} - P\omega b \cos \omega t - \frac{1}{2} m \left(\frac{P}{m} \right)^2 + \frac{1}{2}(k_1 + k_2)(Q + b \sin \omega t - b)^2 \tag{65}$$

$$= \frac{P^2}{2m} - Pb\omega \cos \omega t + \frac{k_1 + k_2}{2}(Q + b \sin \omega t - b)^2 \ . \tag{66}$$

Notice now the Hamiltonian has explicit time dependence so it is not conserved. In these coordinates, the total energy is given by

$$E = T + U = \frac{1}{2} m(\dot{Q} + \omega b \cos \omega t)^2 + \frac{1}{2}(k_1 + k_2)(Q + b \sin \omega t - b)^2 = \frac{P^2}{2m} + \frac{1}{2}(k_1 + k_2)(Q + b \sin \omega t - b)^2 \ , \tag{67}$$

which also has explicit time dependence, so the energy is not conserved either.

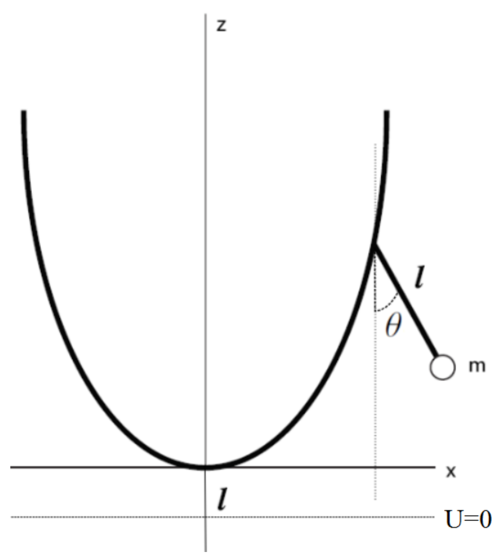

Figure 2: Depiction of the pendulum system and coordinates used in problem #5.

5 Goldstein 8.19.

The point of suspension of a simple pendulum of length ℓ and mass m is constrained to move on a parabola defined by $z = ax^2$ in the vertical plane, in a uniform downwards gravitational acceleration g. The position of the pivot point in the $x - z$ plane is given by (x, ax^2). Now, let a coordinate θ be the angle the pendulum makes with the vertical. The position of the mass is then $(x + \ell \sin\theta, ax^2 - \ell\cos\theta)$. So the generalized coordinates for this system are x and θ, and the kinetic energy is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) = \frac{m}{2}\left[\left(\dot{x} + \ell\dot{\theta}\cos\theta\right)^2 + \left(2ax\dot{x} + \ell\dot{\theta}\sin\theta\right)^2\right] \tag{68}$$

$$= \frac{m}{2}\left[(1 + 4a^2x^2)\dot{x}^2 + 2\ell(\cos\theta + 2ax\sin\theta)\dot{x}\dot{\theta} + \ell^2\dot{\theta}^2\right] \ . \tag{69}$$

Let the zero potential be defined such that when $x = \theta = 0$, $U = 0$. Therefore the zero potential surface is the plane defined by $z = -\ell$, so the potential energy is

$$U = mg(\ell + ax^2 - \ell\cos\theta) \ . \tag{70}$$

With this information, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + 2ma^2x^2\dot{x}^2 + m\ell\cos\theta\dot{x}\dot{\theta} + 2m\ell a\sin\theta x\dot{x}\dot{\theta} + \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell - mgax^2 + mg\ell\cos\theta \ , \tag{71}$$

so the conjugate momenta are

$$p_x = m\dot{x} + 4ma^2x^2\dot{x} + m\ell\cos\theta\dot{\theta} + 2m\ell a\sin\theta x\dot{\theta} \tag{72}$$

$$p_\theta = m\ell\cos\theta\dot{x} + 2m\ell a\sin\theta x\dot{x} + m\ell^2\dot{\theta} \ . \tag{73}$$

With these the Hamiltonian can be written as

$$\mathcal{H} = \dot{x}p_x + \dot{\theta}p_\theta - \mathcal{L} \ , \tag{74}$$

so Equations 72 and 73 must be solved simultaneously for $\dot{x}$ and $\dot{\theta}$. Using MATHEMATICA the solutions are

$$\dot{x} = -\frac{2ap_\theta x\sin\theta - \ell p_x\sin^2\theta - \ell p_x\cos^2\theta + p_\theta\cos\theta}{\ell m(\sin\theta - 2ax\cos\theta)^2} = \frac{\ell p_x - p_\theta(2ax\sin\theta + \cos\theta)}{\ell m(\sin\theta - 2ax\cos\theta)^2} \tag{75}$$

$$\dot{\theta} = -\frac{-4a^2p_\theta x^2 + 2a\ell p_x x\sin\theta + \ell p_x\cos\theta - p_\theta}{\ell^2 m(\sin\theta - 2ax\cos\theta)^2} = \frac{p_\theta(1 + 4a^2x^2) - \ell p_x(2ax\sin\theta - \cos\theta)}{\ell^2 m(\sin\theta - 2ax\cos\theta)^2} \ , \tag{76}$$

and the Hamiltonian is given by

$$\mathcal{H} = p_\theta\left[\frac{p_\theta(1 + 4a^2x^2) - \ell p_x(2ax\sin\theta - \cos\theta)}{\ell^2 m(\sin\theta - 2ax\cos\theta)^2}\right] + p_x\left[\frac{\ell p_x - p_\theta(2ax\sin\theta + \cos\theta)}{\ell m(\sin\theta - 2ax\cos\theta)^2}\right] - \mathcal{L} \tag{77}$$

$$= \frac{p_\theta^2\left(4a^2x^2 + 1\right) + 2gl^2m^2\left(-ax^2 + l\cos\theta + l\right)(\sin\theta - 2ax\cos\theta)^2 - 2lp_\theta p_x(2ax\sin\theta + \cos\theta) + l^2p_x^2}{2l^2m(\sin\theta - 2ax\cos\theta)^2} \ . \tag{78}$$

From this the final two of Hamilton's equations (the $\dot{x}$ and $\dot{\theta}$ Hamilton's equations are shown above) can be found,

$$\dot{p}_\theta = -\frac{\partial\mathcal{H}}{\partial\theta} \qquad \dot{p}_x = \frac{\partial\mathcal{H}}{\partial x} \ . \tag{79}$$

Using MATHEMATICA to take the derivatives of the Hamiltonian, these are

$$\dot{p}_\theta = \frac{1}{\ell^2 m(\sin\theta - 2ax\cos\theta)^3}\Bigg\{p_\theta^2\left(4a^2x^2+1\right)(2ax\sin\theta+\cos\theta) + g\ell^3 m^2\sin\theta(\sin\theta - 2ax\cos\theta)^3 + \ell^2 p_x^2(2ax\sin\theta+\cos\theta) + \frac{1}{2}\ell p_\theta p_x(4ax(ax(\cos(2\theta)-3)-\sin(2\theta))-\cos(2\theta)-3)\Bigg\}\,, \quad (80)$$

after simplifying,

$$\dot{p}_\theta = \frac{[p_\theta^2\left(4a^2x^2+1\right)+\ell^2p_x^2](2ax\sin\theta+\cos\theta)}{\ell^2 m(\sin\theta-2ax\cos\theta)^3} + g\ell m + \frac{\ell p_\theta p_x(4ax(ax[\cos(2\theta)-3]-\sin(2\theta))-\cos(2\theta)-3)}{\ell^2 m(\sin\theta-2ax\cos\theta)^3} \quad (81)$$

and

$$\dot{p}_x = \frac{-2a}{\ell^2 m(\sin\theta-2ax\cos\theta)^3}\Bigg\{g\ell^2m^2x(2ax\cos\theta-\sin\theta)^3 - \frac{1}{2}\ell p_\theta p_x(2ax\sin(2\theta)+\cos(2\theta)+3) + p_\theta^2(2ax\sin\theta+\cos\theta)+\ell^2p_x^2\cos\theta\Bigg\}\,, \quad (82)$$

after simplifying,

$$\dot{p}_x = 2agmx + \frac{2a}{\ell^2 m(\sin\theta-2ax\cos\theta)^3}\Bigg\{\frac{1}{2}\ell p_\theta p_x[2ax\sin(2\theta)+\cos(2\theta)+3] - p_\theta^2(2ax\sin\theta+\cos\theta)+\ell^2p_x^2\cos\theta\Bigg\}\,, \quad (83)$$

The equations of motion can then be found by differentiating Equations 75 and 76 and plugging in the above results, then substituting in the expressions for p_θ and p_x.