Dylan J. Temples: Solution Set One

Quantum Field Theory I
October 3, 2016

Contents

1 Dimensional Transmutation.

Explain what is the phenomenon of "dimensional transmutation?"

The "dimension" of dimensional transmutation refers to the fact that in a natural unit system, every dimensionful parameter is measured in the same units, and it's dimension is the exponent of that unit. Dimensional transmutation is the emergence of a dimensional scale in a theory of only a dimensionless (scale-invariant) theory, *e.g.*, in quantum field theory, a mass scale is spontaneously generated through the renormalization procedure. A scale-invariant theory is a system (Hamiltonian/Lagrangian) with no dimensional dynamical parameter. In a scale-invariant theory, a dimensionally transmuted scale may appear spontaneously by means of a dimensionful parameter being introduced to characterize this scale.

2 Dimensional Analysis in Natural Units.

Here are a few exercises on dimensional analysis and the Natural Unit system $c = \hbar = 1$.

A) The proton mass in the SI unit is $m_p = 1.672 \times 10^{-27}$ kg. Convert m_p into the Natural Unit. The Large Hadron Collider (LHC) at CERN in Geneva is a proton-proton collider designed to have a center-of-mass energy of 14 TeV. What is the speed of the proton, expressed in terms of the speed of light c, when the LHC is operating at the designed CM energy?

In SI units, the rest energy of the proton is

$$E_0 = m_p c^2 = (1.6726219 \times 10^{-27}) \text{ kg } c^2 = 1.50536 \times 10^{-10} \text{ J} . \tag{1}$$

Using the definition of the electron volt: 1 J = 6.242×10^{18} eV, the rest energy is $E_0 = 9.39646 \times 10^8 \text{ eV} \simeq 940 MeV$. So the mass of the proton can be expressed as

$$m_p = \frac{940 \text{ MeV}}{c^2} , \tag{2}$$

but in the Natural Unit, $c = 1$, so

$$m_p = 940 \text{ MeV} . \tag{3}$$

The energy of a relativistic particle is simply the product of its rest energy and the Lorentz factor:

$$E = \gamma E_0 = \gamma m_p \quad \text{with} \quad \gamma = \left(1 - \left[\frac{v}{c}\right]^2\right)^{-1/2} . \tag{4}$$

These expressions can be massaged to give

$$\frac{v}{c} = \sqrt{1 - \left[\frac{m_p}{E}\right]^2} = \sqrt{1 - \left[\frac{940}{14} 10^{-6}\right]^2} \approx 1 . \tag{5}$$

The result is less than one, by construction: $1 - \delta < 1$ for $\delta > 0$, so that $\sqrt{1-\delta} < 1$, but no computational tool at my disposal has the precision required to resolve the solution from 1.0. The proton is effectively moving at the speed of light. The precision in Matlab rounds 0.99999 to 1.0, so $v > 0.9999c$.

B) In SI unit Maxwell's equations contain three dimensionful coupling constants: the electric charge of the electron $e = 1.602 \times 10^{-19}$ C, the permittivity of free space $\epsilon_0 = 8.853 \times 10^{-12}$ F/m, and the permeability of free space $\mu_0 = 1/(\epsilon_0 c^2)$ which can be traded in for the speed of light c. Can you generate a quantity out of e, ϵ_0, c, and $\hbar$ that is *dimensionless?* Is there more than one dimensionless quantity that can be generated?

Let us note the SI units of the four constants e, ϵ_0, c, and $\hbar$:

$$[[e]] = \text{C} \qquad [[\epsilon_0]] = \frac{\text{F}}{\text{m}} = \frac{\text{s}^2 \cdot \text{C}^2}{\text{m}^3 \cdot \text{kg}} \quad \Rightarrow \quad \left[\left[\frac{e^2}{\epsilon_0}\right]\right] = \frac{\text{m}^3 \cdot \text{kg}}{\text{s}^2} \tag{6}$$

$$[[\hbar]] = \frac{\text{m}^2 \cdot \text{kg}}{\text{s}^2} \qquad [[c]] = \frac{\text{m}}{\text{s}} \quad \Rightarrow \quad [[\hbar c]] = \frac{\text{m}^3 \cdot \text{kg}}{\text{s}^2} . \tag{7}$$

The combinations $\hbar c$ and e^2/ϵ_0 both have the same dimensions, and therefore the quantities:

$$\frac{e^2}{\epsilon_0 \hbar c} \quad \text{and} \quad \frac{\epsilon_0 \hbar c}{e^2} \ , \tag{8}$$

are both dimensionless. Furthermore, since these expressions are dimensionless, taking either to an arbitrary power is still dimensionless.

C) Divide your answer(s) in (b) by 4π and call it α. What is the numerical value of α? Approximate α by a fraction $\alpha \approx 1/N$ where N is an integer. What is N? Can you recognize that α is a well-known fundamental constant?

The dimensionless quantity is

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} = 0.00729735 \ , \tag{9}$$

which we can approximate as $\alpha \simeq 1/N$. Let $\tilde{N}$ be the exact value defined by

$$\alpha = 1/\tilde{N} \quad \Rightarrow \quad \tilde{N} = \frac{1}{\alpha} = 137.036 \ , \tag{10}$$

so $N = 137$.

D) The Newton's constant $G_N = 6.674 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}. In the Natural Unit it has the mass dimension of -2, which is used to define a Planck mass $M_p = 1.22 \times 10^{19}$ GeV. Convert M_p into the SI unit by expressing it in terms of G_N, $\hbar$, and c.

Using the dimensions of $\hbar c$ from earlier,

$$\left[\left[\frac{G_N}{\hbar c}\right]\right] = \text{kg}^{-2} \ , \tag{11}$$

so the combination $\sqrt{\hbar c/G_N}$ has dimensions of mass. In the SI unit system, this has a value of $M_p = 2.1777 \times 10^{-8}$ kg. If we convert this to the Natural Unit system, we see

$$M_p = 2.1777 \times 10^{-8}\text{kg } c^2 = 1.95991 \times 10^9 \text{ J} = 1.22338 \times 10^{28} \text{ eV} = 1.22 \times 10^{19} \text{ GeV} \ , \tag{12}$$

as expected.

E) With a CM energy of 14 TeV, typical energy scales at the LHC will be at around TeV. For a typical quantum mechanical amplitude at the LHC, how large is the correction to the amplitude due to effects of gravity?

The correction due to gravity becomes appreciable when energies reach that of the Planck mass. At the TeV scale, the correction has magnitude

$$\frac{10^{12}}{10^{19+6}} = 10^{-13} \ . \tag{13}$$

3 Spinless Relativistic Particle and Causality Violation.

In this problem we will show that the quantum mechanical amplitude for a spinless relativistic particle propagating outside of the lightcone is non-vanishing, thereby violating causality.

A) The Hamiltonian for a free spinless particle in relativity is

$$H = \sqrt{|\mathbf{p}|^2 + m^2} \ . \tag{14}$$

Let's start with a particle localized at the origin, represented by the state $|\mathbf{x} = 0\rangle$. In Quantum Mechanics the time evolution of a state vector is given by $e^{-iHt}|\mathbf{x} = 0\rangle$. The amplitude for finding the particle at position $\mathbf{x}$ at a later time t is then given by $\langle \mathbf{x}|e^{-iHt}|\mathbf{x} = 0\rangle$. Show

$$\langle \mathbf{x}|e^{-iHt}|\mathbf{x} = 0\rangle = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{-ik\cdot x} \ . \tag{15}$$

(Notice I am using the relativistic notation, where $k \cdot x = k_\mu x^\mu$.)

The identity can be expressed as an integral over all momentum states in 3-space:

$$\mathbb{1} = \int \mathrm{d}^3 k \, |\mathbf{k}\rangle \langle \mathbf{k}| \ . \tag{16}$$

We can insert the identity into the matrix element $\langle \mathbf{x}|e^{-iHt}|\mathbf{x} = 0\rangle$ before the time evolution operator:

$$\langle \mathbf{x}|e^{-iHt}|\mathbf{x} = 0\rangle = \int \mathrm{d}^3 k \, \langle \mathbf{x}|\mathbf{k}\rangle \langle \mathbf{k}|e^{-iHt}|\mathbf{x} = 0\rangle \ . \tag{17}$$

The projection of a state vector in the position basis into the momentum basis is a plane wave:

$$\langle \mathbf{x}|\mathbf{k}\rangle = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \ , \tag{18}$$

where the three-halves factors of 2π come from the normalization of the plane wave to a three-dimensional delta function. We can now act the time evolution operator to the left on the momentum state vector:

$$\langle \mathbf{k}| \, e^{-iHt} = \left[e^{iHt} \, |\mathbf{k}\rangle \right]^\dagger \ . \tag{19}$$

The operator in the above expression can be expanded:

$$e^{iHt} = 1 + (iHt) + \mathcal{O}(H^2) \ , \tag{20}$$

which acts on a state vector:

$$e^{iHt} \, |\mathbf{k}\rangle = |\mathbf{k}\rangle + (iHt) \, |\mathbf{k}\rangle + \mathcal{O}(H^2) \, |\mathbf{k}\rangle = |\mathbf{k}\rangle + (iE_k t) \, |\mathbf{k}\rangle + \mathcal{O}(E_k^2) \, |\mathbf{k}\rangle = e^{iE_k t} \, |\mathbf{k}\rangle \ , \tag{21}$$

where $E_k = \sqrt{|\mathbf{k}|^2 + m^2}$, so that E_k is a C-number, and the exponential can therefore be pulled out of the matrix element. Taking the conjugate transpose of this, as indicated by Equation 19, and inserting it back into the matrix element yields

$$\langle \mathbf{x}|e^{-iHt}|\mathbf{x} = 0\rangle = \int \mathrm{d}^3 k \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} e^{-iE_k t} \, \langle \mathbf{k}|\mathbf{x} = 0\rangle \ . \tag{22}$$

As shown in Equation 18, the inner product $\langle \mathbf{k}|\mathbf{x}=0\rangle = (2\pi)^{-3/2} e^{i\mathbf{k}\cdot 0} = (2\pi)^{-3/2}$, and we are left with

$$\langle \mathbf{x}|e^{-iHt}|\mathbf{x}=0\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3} e^{-i(E_k t - \mathbf{k}\cdot\mathbf{x})} \ . \tag{23}$$

The four-momentum is $k^\mu = (E_k, k_1, k_2, k_3)$ and the four-position is $x^\mu(t, x_1, x_2, x_3)$, so that

$$k_\mu x^\mu = g_{\mu\nu} k^\nu x^\mu = E_k t - k_1 x_1 - k_2 x_2 - k_3 x_3 = E_k t - \mathbf{k}\cdot\mathbf{x} \ . \tag{24}$$

Using this, we obtain the result

$$\langle \mathbf{x}|e^{-iHt}|\mathbf{x}=0\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3} e^{-ik_\mu x^\mu} = \int \frac{\mathrm{d}^3k}{(2\pi)^3} e^{-ik\cdot x} \ . \tag{25}$$

B) Perform the angular integration to arrive at

$$\frac{-i}{(2\pi)^2|\mathbf{x}|} \int_{-\infty}^{\infty} |\mathbf{k}|\mathrm{d}|\mathbf{k}| e^{-i\omega_k t} e^{i|\mathbf{k}||\mathbf{x}|} \ , \tag{26}$$

where $\omega_k = \sqrt{|\mathbf{k}|^2 + m^2}$.

Using Equation 24, we can express Equation 25 as

$$\langle \mathbf{x}|e^{-iHt}|\mathbf{x}=0\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3} e^{-i\omega_k t} e^{i\mathbf{k}\cdot\mathbf{x}} \ , \tag{27}$$

where $\omega_k = E_k = \sqrt{|\mathbf{k}|^2 + m^2}$. This can be expressed in spherical coordinates so that $\mathrm{d}^3k = |\mathbf{k}|^2 \sin\theta \mathrm{d}|\mathbf{k}|\mathrm{d}\theta\mathrm{d}\phi$. We can immediately perform the azimuthal (ϕ) integral, acquiring a factor of 2π, so that

$$\langle \mathbf{x}|e^{-iHt}|\mathbf{x}=0\rangle = \frac{1}{(2\pi)^2} \int_0^{\infty} |\mathbf{k}|^2 \mathrm{d}|\mathbf{k}| e^{-i\omega_k t} \int_0^{\pi} \sin\theta \mathrm{d}\theta e^{i|\mathbf{k}||\mathbf{x}|\cos\theta} \ . \tag{28}$$

If we define $w = \cos\theta$, the angular integral can be expressed as

$$\int_{-1}^{1} \mathrm{d}w e^{i|\mathbf{k}||\mathbf{x}|w} = \frac{-i}{|\mathbf{k}||\mathbf{x}|} \left(e^{i|\mathbf{k}||\mathbf{x}|} - e^{-i|\mathbf{k}||\mathbf{x}|} \right) \ , \tag{29}$$

and the matrix element can be written

$$\langle \mathbf{x}|e^{-iHt}|\mathbf{x}=0\rangle = \frac{-i}{(2\pi)^2|\mathbf{x}|} \int_0^{\infty} |\mathbf{k}|\mathrm{d}|\mathbf{k}| e^{-i\omega_k t} \left(e^{i|\mathbf{k}||\mathbf{x}|} - e^{-i|\mathbf{k}||\mathbf{x}|} \right) \ . \tag{30}$$

Consider only the integral:

$$I = \int_0^{\infty} |\mathbf{k}|\mathrm{d}|\mathbf{k}| e^{-i\omega_k t} \left(e^{i|\mathbf{k}||\mathbf{x}|} - e^{-i|\mathbf{k}||\mathbf{x}|} \right) = \int_0^{\infty} |\mathbf{k}|\mathrm{d}|\mathbf{k}| e^{-i\omega_k t} e^{i|\mathbf{k}||\mathbf{x}|} - \int_0^{\infty} |\mathbf{k}|\mathrm{d}|\mathbf{k}| e^{-i\omega_k t} e^{-i|\mathbf{k}||\mathbf{x}|} \ ,$$

in the second integral, we can make the change of variable $|\mathbf{k}| \to -|\mathbf{q}|$ (note this does not change ω_k because it depends on the square of $|\mathbf{k}|$:

$$\begin{aligned} I &= \int_0^{\infty} |\mathbf{k}|\mathrm{d}|\mathbf{k}| e^{-i\omega_k t} e^{i|\mathbf{k}||\mathbf{x}|} - \int_0^{-\infty} (-|\mathbf{q}|)(-\mathrm{d}|\mathbf{q}|) e^{-i\omega_k t} e^{i|\mathbf{q}||\mathbf{x}|} \\ &= \int_0^{\infty} |\mathbf{k}|\mathrm{d}|\mathbf{k}| e^{-i\omega_k t} e^{i|\mathbf{k}||\mathbf{x}|} + \int_{-\infty}^{0} |\mathbf{q}|\mathrm{d}|\mathbf{q}| e^{-i\omega_k t} e^{i|\mathbf{q}||\mathbf{x}|} \ . \end{aligned}$$

If we rename $|\mathbf{q}|$ as $\mathbf{k}$, then we are left with

$$I = \int_{-\infty}^{\infty} |\mathbf{k}| \mathrm{d}|\mathbf{k}| e^{-i\omega_k t} e^{i|\mathbf{k}||\mathbf{x}|} \ , \tag{31}$$

and we obtain the result

$$\langle \mathbf{x} | e^{-iHt} | \mathbf{x} = 0 \rangle = \frac{-i}{(2\pi)^2 |\mathbf{x}|} \int_{-\infty}^{\infty} |\mathbf{k}| \mathrm{d}|\mathbf{k}| e^{-i\omega_k t} e^{i|\mathbf{k}||\mathbf{x}|} \ . \tag{32}$$

C) The above integral can be performed using Cauchy's theorem. The only complication is the square-root in the exponent requires a branch cut. As a warm-up exercise, consider a complex function $f(z) = \sqrt{z}$ where z is a complex variable. If we choose the branch cut to be along the positive real axis, show that $f(z)$ is discontinuous across the branch cut:

$$f(x + i\epsilon) = -f(x - i\epsilon) \tag{33}$$

for a real number $x > 0$ and $\epsilon \to 0$. (*Hint: go to the the polar coordinate!*)

If we express the complex variable in the polar coordinate, we have $z = re^{i\varphi}$, as such

$$f(z) = \sqrt{z} = \sqrt{r} e^{i\varphi/2} \ , \tag{34}$$

which for a complex number $z = a + ib$,

$$r = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \arctan\left(\frac{b}{a}\right) \ . \tag{35}$$

For the points above and below the real axis, at a point x we have

$$f(x + i\epsilon) = (x^2 + \epsilon^2)^{1/4} e^{i \arctan(\epsilon/x)/2} \tag{36}$$

$$f(x - i\epsilon) = (x^2 + \epsilon^2)^{1/4} e^{i \arctan(-\epsilon/x)/2} = (x^2 + \epsilon^2)^{1/4} e^{-i \arctan(\epsilon/x)/2} \ , \tag{37}$$

and in the limit $\epsilon \to 0$, the arctangent approaches zero (or any integral multiple of 2π). Therefore, due to the branch cut, when $z = x - i\epsilon$, we have $\varphi = 2\pi$, and when $z = x + i\epsilon$, we have $\varphi = 0$, so:

$$f(x + i\epsilon) = (x^2 + \epsilon^2)^{1/4} (+1) \tag{38}$$

$$f(x - i\epsilon) = (x^2 + \epsilon^2)^{1/4} e^{-i(2\pi)/2} = (x^2 + \epsilon^2)^{1/4} (-1) \ , \tag{39}$$

yielding the result $f(x + i\epsilon) = -f(x - i\epsilon)$.

D) There are actually two branch cuts in Eq. 26. Choose them to be on the imaginary axis starting from $\pm im$. Show that the integrand approaches zero at infinity in the upper half-plane when $|\mathbf{x}| > t$, *i.e.*, when the final position x lies outside of the lightcone of the origin.

We are interested in the behavior of

$$e^{-i\sqrt{(|\mathbf{k}|^2 + m^2)} t} e^{i|\mathbf{k}||\mathbf{x}|} = e^{-i\left[\sqrt{(|\mathbf{k}|^2 + m^2)} t - |\mathbf{k}||\mathbf{x}|\right]} \ , \tag{40}$$

in the upper half-plane as our complex variable $|\mathbf{k}|$ approaches infinity. In the limit of large $|\mathbf{k}|$, the exponential becomes

$$e^{-i|\mathbf{k}|(t-|\mathbf{x}|)} , \tag{41}$$

and in the case $|\mathbf{x}| > t$, the quantity in the parenthesis in the exponential is negative, this factor scales like $e^{i|\mathbf{k}|\beta}$, where $\beta > 0$. Since $|\mathbf{k}|$ is approaching imaginary infinity in the upper half plane, this factor becomes $e^{-|\mathbf{k}|\beta}$, which is exponentially decaying for $|\mathbf{k}| \to \infty$. Therefore the intgrand vanishes at infinity in the upper half plane.

E) Now evaluate the integral by choosing a contour that runs along the real axis, the infinity in the upper half-plane, and around the branch cut. In particular, on the left-side (right-side) of the branch cut choose $\Im\sqrt{|\mathbf{k}|^2 + m^2} < 0$ (> 0). Then show the amplitude outside of the lightcone is

$$\langle \mathbf{x}|e^{iHt}|\mathbf{x} = 0\rangle = \frac{i}{2\pi^2}\frac{e^{-m|\mathbf{x}|}}{|\mathbf{x}|}\int_m^\infty \mathrm{d}\alpha e^{-(\alpha-m)|\mathbf{x}|}\sinh\left(\sqrt{\alpha^2 - m^2}\right) , \tag{42}$$

where we have a changed the variable $|\mathbf{k}| = i\alpha$.

The branch cut relative to this integration extends from $+im$ to psotive imaginary infinity. The contour C is along the real axis (the integral of interest $\mathscr{I}$), extends from real positiive infinity to imaginary positive infinity in a arc (A_1), then down the imaginary axis on the right of the branch cut (I_1), across the branch cut (this integral is zero because it is of infinitesimal extent), then up the imaginary axis to the left of the branch cut (I_2), and then a arc from positive imaginary infinity to negative real infiniy (A_2). By Cauchy's Theorem, since there are no poles contained in the contour C, the integral along the controur is

$$\oint_C f(z)\mathrm{d}z = \mathscr{I} + A_1 + I_1 + A_2 + I_2 = 0 . \tag{43}$$

In the previous section, it was shown that the integrand vanishes at infinity in the upper half plane, so $A_1 = A_2 = 0$, and we see that the integral of interest is equvalent to

$$\mathscr{I} = -(I_1 + I_2) . \tag{44}$$

Let the integrand be written

$$f(|\mathbf{k}|) = |\mathbf{k}|e^{-i\omega_k t}e^{i|\mathbf{k}||\mathbf{x}|} = |\mathbf{k}|e^{-i\sqrt{(|\mathbf{k}|^2+m^2)}t}e^{i|\mathbf{k}||\mathbf{x}|} , \tag{45}$$

which if we define $|\mathbf{k}| = i\alpha$, the integrals I_1 and I_2 are:

$$I_1 = \int_\infty^m (i\alpha)e^{-i\sqrt{(-\alpha^2+m^2)}t}e^{-\alpha|\mathbf{x}|}\mathrm{d}(i\alpha) = -\int_\infty^m \alpha e^{\sqrt{(\alpha^2-m^2)}t}e^{-\alpha|\mathbf{x}|}\mathrm{d}\alpha \tag{46}$$

$$I_2 = \int_m^\infty (i\alpha)e^{i\sqrt{(-\alpha^2+m^2)}t}e^{-\alpha|\mathbf{x}|}\mathrm{d}(i\alpha) = -\int_m^\infty \alpha e^{-\sqrt{(\alpha^2-m^2)}t}e^{-\alpha|\mathbf{x}|}\mathrm{d}\alpha , \tag{47}$$

making note of the choice that on the left-side (right-side) of the branch cut: $\Im\sqrt{|\mathbf{k}|^2 + m^2} < 0$ (> 0). If we use the negative sign to flip the lmits of integration on the first integral (right-side of the branch cut), we see

$$I_1 + I_2 = \int_m^\infty \alpha e^{-\alpha|\mathbf{x}|}\left[e^{\sqrt{(\alpha^2-m^2)}t} - e^{-\sqrt{(\alpha^2-m^2)}t}\right]\mathrm{d}\alpha . \tag{48}$$

The exponential with $|\mathbf{x}|$ vanishes at the infinite endpoint and is $e^{-m|\mathbf{x}|}$ at the other endpoint, so we may pull the endpoints out of the integral. Additionally, we can replace the difference of the exponentials with a hyperbolic sine and a factor of two:

$$I_1 + I_2 = 2e^{-m|\mathbf{x}|} \int_m^\infty \alpha e^{-(\alpha-m)|\mathbf{x}|} \sinh\left(\sqrt{(\alpha^2-m^2)}t\right) \mathrm{d}\alpha \ . \tag{49}$$

The negative of this result is equivalent to the integral of interest $\mathscr{I}$, so we can insert it into Equation 32 to find

$$\langle \mathbf{x}|e^{-iHt}|\mathbf{x}=0\rangle = \frac{i}{2\pi^2|\mathbf{x}|} e^{-m|\mathbf{x}|} \int_m^\infty \alpha e^{-(\alpha-m)|\mathbf{x}|} \sinh\left(\sqrt{(\alpha^2-m^2)}t\right) \mathrm{d}\alpha \ . \tag{50}$$

F) Without actually performing the above integral, argue it is non-vanishing. Therefore, the quantum mechanical amplitude is non-zero for a particle traveling outside of the lightcone.

Consider the fist exponential in the integrand: $\exp[-(\alpha-m)|\mathbf{x}|]$, as $\alpha \to +\infty$, for fixed m, the exponential is finite. However:

$$\lim_{\alpha\to+\infty} \sinh\left(\sqrt{\alpha^2-m^2}\right) = \lim_{\alpha\to+\infty} \cosh\left(\sqrt{\alpha^2-m^2}\right) = \infty \ . \tag{51}$$

The product of this with the decaying exponential in α changes the concavity of the curve, so that as $\alpha \to +\infty$, the integrand diverges very slowly to infinity. Additionally, it is clear that if $\alpha = m$, the integral will not have the same value as $\alpha \to \infty$. For these reasons, the integral is nonvanishing. More simply, since the argument of the hyperbolic sine is positive definite, so is the hyperbolic sine itself. Additionally, the exponential is positive definite, and due to the limits of integration, so is α. Therefore the entire integrand is positive definite, and will not have the same value at the endpoint, and therefore the integral is non-vanishing.