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Quantum Field Theory I
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Contents

1 Relativity exercises.

Do exercises a, e, and h in the handout by Professor Michael Dine at UC Santa Cruz on relativity at http://scipp.ucsc.edu/ dine/ph217/217relativity.pdf.

A) For a Lorentz transformation along the z axis, determine the components of Λ. Note that you can write this nicely by taking:

$$t' = \cosh(\omega)t + \sinh(\omega)z \qquad z' = \sinh(\omega)t - \cosh(\omega)z \ . \tag{1}$$

A Lorentz transformation along the z axis leaves x and y unchanged, so

$$x^{\mu\ \prime} = \Lambda^\mu_\nu x^\nu \quad \text{with} \quad x^{1\ \prime} = x^1,\ x^{2\ \prime} = x^2, \tag{2}$$

implying

$$\Lambda^i_1 = \Lambda^1_i = \delta_{i1} \quad \text{and} \quad \Lambda^i_2 = \Lambda^2_i = \delta_{i2}, \tag{3}$$

where δ_{i2} is the Kronecker delta. Performing the transformation results in two nontrivial equations:

$$x^{0\ \prime} = \Lambda^0_0 x^0 + \Lambda^0_3 x^3 \quad \Rightarrow \quad t' = \Lambda^0_0 t + \Lambda^0_3 z \tag{4}$$

$$x^{3\ \prime} = \Lambda^3_0 x^0 + \Lambda^3_3 x^3 \quad \Rightarrow \quad z' = \Lambda^3_0 t + \Lambda^3_3 z \ . \tag{5}$$

Furthermore, we know the transformation must leave the quantity $t^2 - z^2$ invariant:

$$t^2 - z^2 = (t')^2 - (z')^2 \tag{6}$$

$$= \left[(\Lambda^0_0)^2 - (\Lambda^3_0)^2\right] t^2 + \left[(\Lambda^0_3)^2 - (\Lambda^3_3)^2\right] z^2 + 2\left[(\Lambda^0_0)(\Lambda^0_3) - (\Lambda^3_3)(\Lambda^3_0)\right] tz \ , \tag{7}$$

yielding

$$1 = (\Lambda^0_0)^2 - (\Lambda^3_0)^2 \tag{8}$$

$$1 = (\Lambda^3_3)^2 - (\Lambda^0_3)^2 \tag{9}$$

$$(\Lambda^0_0)(\Lambda^0_3) = (\Lambda^3_3)(\Lambda^3_0) \ , \tag{10}$$

which are reminiscent of hyperbolic trig functions. We see the above relations are satisfied if we set $\Lambda^0_0 = \Lambda^3_3 = \cosh(\omega)$ and $\Lambda^0_3 = \Lambda^3_0 = -\sinh(\omega)$. Therefore a Lorentz transformation along the z axis can be represented as

$$\Lambda^\mu_\nu = \begin{pmatrix} \cosh\omega & 0 & 0 & -\sinh\omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\omega & 0 & 0 & \cosh\omega \end{pmatrix} , \tag{11}$$

where ω is a real constant.

B) Evaluate s in the lab frame for an electron-proton collision.

The total center of mass energy is given by

$$s = (p_1 + p_2)^2 = (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2) \cdot (\mathbf{p}_1 + \mathbf{p}_2) \ , \tag{12}$$

where

$$E_1 = \sqrt{|\mathbf{p}|_1^2 + m_e^2} \quad \text{and} \quad E_2 = \sqrt{|\mathbf{p}|_2^2 + m_p^2} \ . \tag{13}$$

In the lab frame, we will assume the proton is at rest so $\mathbf{p}_2 = 0$, so

$$s = E_1^2 + m_p^2 + 2m_pE_1 - |\mathbf{p}_1|^2 = m_p^2 + m_e^2 + 2m_pE_1 \ . \tag{14}$$

If we make the approximation $m_p \gg m_e$, this becomes

$$s = m_p^2 + 2m_pE_1 \ . \tag{15}$$

C) $F_{\mu\nu}$, being a tensor, transforms like the product $x_\mu x_\nu$ under Lorentz transformations. Work out the transformation of $\mathbf{E}$ and $\mathbf{B}$ under Lorentz transformations along the z axis using this fact.

A second rank tensor transforms under a Lorentz boost as

$$F'_{\mu\nu} = \Lambda_\mu^\alpha \Lambda_\nu^\beta F_{\alpha\beta} \ . \tag{16}$$

The field strength tensor has components

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \ . \tag{17}$$

Using the matrix representation of Λ_μ^ν for a boost along the z axis found previously, we can carry out the first transformation:

$$\Lambda_\nu^\beta F_{\alpha\beta} = \begin{pmatrix} \cosh\omega & 0 & 0 & -\sinh\omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\omega & 0 & 0 & \cosh\omega \end{pmatrix} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{18}$$

$$= \begin{pmatrix} E_z\mathcal{S}(\omega) & E_x\mathcal{C}(\omega) + B_y\mathcal{S}(\omega) & E_y\mathcal{C}(\omega) - B_x\mathcal{S}(\omega) & E_z\mathcal{C}(\omega) \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z\mathcal{C}(\omega) & -B_y\mathcal{C}(\omega) - E_x\mathcal{S}(\omega) & B_x\mathcal{C}(\omega) - E_y\mathcal{S}(\omega) & -E_z\mathcal{S}(\omega) \end{pmatrix} \ , \tag{19}$$

where $\mathcal{S}(\omega) = \sinh(\omega)$ and $\mathcal{C}(\omega) = \cosh(\omega)$. Performing the second transformation, we obtain

$$F'_{\mu\nu} = \begin{pmatrix} E_z\mathcal{S}(2\omega) & E_x\mathcal{C}(2\omega) + B_y\mathcal{S}(2\omega) & E_y\mathcal{C}(2\omega) - B_x\mathcal{S}(2\omega) & E_z\mathcal{C}(2\omega) \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z\mathcal{C}(2\omega) & -B_y\mathcal{C}(2\omega) - E_x\mathcal{S}(2\omega) & B_x\mathcal{C}(2\omega) - E_y\mathcal{S}(2\omega) & -2E_z\mathcal{C}(\omega)\mathcal{S}(\omega) \end{pmatrix} .$$

2 Two-dimensional quantum harmonic oscillator.

We would like to consider a two-dimensional quantum harmonic oscillator with the following Hamiltonian in Cartesian coordinate:

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{1}{2}\left(x_1^2 + x_2^2\right) \ , \tag{20}$$

where I have set $m = \omega = 1$ for simplicity. The operators satisfy $[x_i, p_j] = +i\delta_{ij}$.

A) Write down the Hamiltonian in terms of creation and annihilation operators $\{a_x^\dagger, a_y^\dagger, a_x, a_y\}$, which satisfy the commutation relation $[a_i, a_j^\dagger] = \delta_{ij}$. (All other commutators are zero.)

The creation and annihilation operators are defined as

$$a_i = \frac{1}{\sqrt{2}}\left(x_i + ip_i\right) \tag{21}$$

$$a_i^\dagger = \frac{1}{\sqrt{2}}\left(x_i - ip_i\right) \ , \tag{22}$$

such that they satisfy the commutation relation[1]. Their products (after moving the radical over) are

$$2a_i^\dagger a_i = (x_i - ip_i)(x_i + ip_i) = x_i^2 - (ip_i)^2 - ip_ix_i + ix_ip_i = x_i^2 + p_i^2 + i(x_ip_i - p_ix_i) \tag{26}$$

$$2a_ia_i^\dagger = (x_i + ip_i)(x_i - ip_i) = x_i^2 - (ip_i)^2 + ip_ix_i - ix_ip_i = x_i^2 + p_i^2 - i(x_ip_i - p_ix_i) \ . \tag{27}$$

We can identify the last term as the commutator $[x_i, p_i] = -i$, and so $2a_i^\dagger a_i = x_i^2 + p_i^2 + 1$, and $2a_ia_i^\dagger = x_i^2 + p_i^2 - 1$. If we sum, we find:

$$2a_i^\dagger a_i + 2a_ia_i^\dagger = 2x_i^2 + 2p_i^2 + 1 - 1 \quad \Rightarrow \quad a_i^\dagger a_i + a_ia_i^\dagger = x_i^2 + p_i^2 \ , \tag{28}$$

but since $1 = [a_i, a_i^\dagger] = a_ia_i^\dagger - a_i^\dagger a_i$, we have that $a_ia_i^\dagger = 1 + a_i^\dagger a_i$, so the above equation becomes

$$2a_i^\dagger a_i + 1 = x_i^2 + p_i^2 \ . \tag{29}$$

We can rearrange the Hamiltonian, and insert the above result:

$$H = \frac{1}{2}\left[(x_1^2 + p_1^2) + (x_2^2 + p_2^2)\right] = \frac{1}{2}\left[(2a_1^\dagger a_1 + 1) + (2a_2^\dagger a_2 + 1)\right] = a_1^\dagger a_1 + a_2^\dagger a_2 + 1 \ , \tag{30}$$

yielding the Hamiltonian entirely in terms of the creation and annihilation operator for each dimension.

[1]

$$[a_i, a_i^\dagger] = a_ia_i^\dagger - a_i^\dagger a_i = \frac{1}{2}(x_i + ip_i)(x_i - ip_i) - \frac{1}{2}(x_i - ip_i)(x_i + ip_i) \tag{23}$$

$$= \frac{1}{2}[(x_i^2 + p_i^2 - ix_ip_i + ip_ix_i) - (x_i^2 + p_i^2 + ix_ip_i - ip_ix_i)] = \frac{1}{2}i(-x_ip_i + p_ix_i - x_ip_i + p_ix_i) \tag{24}$$

$$= i(p_ix_i - x_ip_i) = i[p_i, x_i] = i(-i) = 1 \ . \tag{25}$$

B) The Hamiltonian obviously has rotational invariance in the two-dimensional space: $U(R)x_iU(R)^\dagger = D(R)_{ij}x_j$, where $D(R)$ is a 2×2 orthogonal matrix. A less obvious invariance is a complex rotation S in (a_1, a_2),

$$U(S)a_iU(S)^\dagger = D(S)_{ij}a_j, \tag{31}$$

where $D(S)$ is a complex 2×2 matrix. Work out the condition on $D(S)$ in order for the Hamiltonian to be invariant under S : $U(S)HU(S)^\dagger = H$. Show that different states related by an S transformation $|a\rangle = U(S)|b\rangle$ are degenerate in energy.

The Hamiltonian is invariant under S if

$$H = U(S)HU(S)^\dagger = U(S)a_1^\dagger a_1U(S)^\dagger + U(S)a_2^\dagger a_2U(S)^\dagger + U(S)1U(S)^\dagger \ . \tag{32}$$

The unitary operator $U(S)$ has the property $U(S)U(S)^\dagger = U(S)^\dagger U(S) = \mathbb{1}$, so the identity can be inserted between the creation and annihilation operators for each dimension, so the rotated Hamiltonian is

$$U(S)HU(S)^\dagger = \left(U(S)a_1^\dagger U(S)^\dagger\right)\left(U(S)a_1U(S)^\dagger\right) + \left(U(S)a_2^\dagger U(S)^\dagger\right)\left(U(S)a_2U(S)^\dagger\right) + 1 \ .$$

Assuming that the creation operator transforms similarly to the annihilation operator, this can be written

$$U(S)HU(S)^\dagger = \left(D(S)_{1j}a_j^\dagger\right)\left(D(S)_{1j}a_j\right) + \left(D(S)_{2j}a_j^\dagger\right)\left(D(S)_{2j}a_j\right) + 1 \ , \tag{33}$$

explicitly writing out the implied sum, we have

$$U(S)HU(S)^\dagger = \left(D(S)_{11}a_1^\dagger + D(S)_{12}a_2^\dagger\right)\left(D(S)_{11}a_1 + D(S)_{12}a_2\right) \tag{34}$$

$$+ \left(D(S)_{21}a_1^\dagger + D(S)_{22}a_2^\dagger\right)\left(D(S)_{21}a_1 + D(S)_{22}a_2\right) + 1 \ . \tag{35}$$

Performing the multiplication and combining terms yields

$$\begin{aligned} U(S)HU(S)^\dagger = 1 &+ \left\{D(S)_{11}^2 + D(S)_{21}^2\right\}a_1^\dagger a_1 + \left\{D(S)_{12}^2 + D(S)_{22}^2\right\}a_2^\dagger a_2 \\ &+ \left\{D(S)_{11}D(S)_{12} + D(S)_{21}D(S)_{22}\right\}a_1^\dagger a_2 + \left\{D(s)_{12}D(S)_{11} + D(S)_{22}D(S)_{21}\right\}a_2^\dagger a_1 \ . \end{aligned}$$

We must enforce that the Hamiltonian is invariant under S, so we are left with the conditions

$$D(S)_{11}^2 + D(S)_{21}^2 = 1 \qquad D(S)_{12}^2 + D(S)_{22}^2 = 1 \tag{36}$$

$$D(S)_{11}D(S)_{12} + D(S)_{21}D(S)_{22} = 0 \qquad D(s)_{12}D(S)_{11} + D(S)_{22}D(S)_{21} = 0 \ . \tag{37}$$

Consider the state $|b\rangle$ related to a state $|a\rangle$ by $|a\rangle = U(S)|b\rangle$, and thus

$$U(S)^\dagger|a\rangle = U(S)^\dagger U(S)|b\rangle = \mathbb{1}|b\rangle = |b\rangle \ , \tag{38}$$

additionally:

$$H = U(S)HU(S)^\dagger \tag{39}$$

$$U(S)^\dagger HU(S) = U(S)^\dagger U(S)HU(S)^\dagger U(S) \tag{40}$$

$$U(S)^\dagger HU(S) = H \ . \tag{41}$$

Now consider the action of the Hamiltonian on the state $|a\rangle$:

$$H|a\rangle = HU(S)|b\rangle \tag{42}$$
$$E_a|a\rangle = \left(U(S)U(S)^\dagger\right) HU(S)|b\rangle \tag{43}$$
$$E_a|a\rangle = U(S)H|b\rangle = E_b U(S)|b\rangle \tag{44}$$
$$E_a|a\rangle = E_b|a\rangle \ , \tag{45}$$

and as such $E_a = E_b$.

C) Define the one-particle states $\{|i\rangle = a_i^\dagger |0\rangle\ ,\ i = 1, 2\}$. We discussed in class that any operator can be expressed in terms of creation and annihilation operators. Consider a set of operators $\{T^a,\ a = 1, 2, 3\}$ whose effects on the one-particle states are

$$T^a|0\rangle = 0 \qquad T^a|i\rangle = \frac{1}{2}|j\rangle [\sigma^a]_{ij} \tag{46}$$

where σ^a's are the Pauli matrices and $[\sigma^a]_{ij}$ are the matrix elements of Pauli matrices. Find the representation of T^a in terms of the creation and annihilation operators.

Considering the action of T^a on the vacuum, it can be expressed as

$$T^a|0\rangle = \Lambda^a a_i |0\rangle = \Lambda^a(0) = 0 \ , \tag{47}$$

where Λ^a is an undetermined operator. Now consider its action on the state $|i\rangle$:

$$T^a|i\rangle = \Lambda^a a_i |i\rangle = \Lambda^a |0\rangle = \frac{1}{2}|j\rangle [\sigma^a]_{ij} \ , \tag{48}$$

since the matrix elements $[\sigma^a]_{ij}$ are C-numbers, they can be moved around without consequence:

$$\Lambda^a|0\rangle = \frac{1}{2}[\sigma^a]_{ij}|j\rangle \ . \tag{49}$$

Let us now write $\Lambda^a = \Omega^a a_j^\dagger$, where Ω^a is another undetermined operator; its action on the vacuum is

$$\Lambda^a|0\rangle = \Omega^a a_j^\dagger |0\rangle = \Omega^a |j\rangle \ , \tag{50}$$

so now we can identify $\Omega^a = \frac{1}{2}[\sigma^a]_{ij}$, and see that

$$T^a = \frac{1}{2}[\sigma^a]_{ij}\, a_j^\dagger a_i \ . \tag{51}$$

D) Use your result in (c) to compute the commutators $[T^a, a_i^\dagger]$.

The commutator is

$$[T^a, a_i^\dagger] = \frac{1}{2}[\sigma^a]_{ij}\,[a_j^\dagger a_i, a_i^\dagger] = \frac{1}{2}[\sigma^a]_{ij}\left\{a_j^\dagger[a_i, a_i^\dagger] + [a_j^\dagger, a_i^\dagger]a_i\right\} \ , \tag{52}$$

using the commutator relations: $[a_i, a_i^\dagger] = 1$ and $[a_j^\dagger, a_i^\dagger] = 0$, we have

$$[T^a, a_i^\dagger] = \frac{1}{2}[\sigma^a]_{ij}\, a_j^\dagger \ . \tag{53}$$

3 Spinless boson.

Consider a free spinless boson $\phi(x)$ with the following plane-wave expansion:

$$\phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2\omega_k}} \left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x} \right) \ . \tag{54}$$

Be aware that I am using the non-relativistically normalized creation and annihilation operators!

A) Suppose we treat $\phi(x)$ as a quantum field and canonically quantize it using the commutation relations:

$$[\phi(\mathbf{x},t), \phi(\mathbf{y},t)] = [\partial_t\phi(\mathbf{x},t), \partial_t\phi(\mathbf{y},t)] = 0, \quad [\phi(\mathbf{x},t), \partial_t\phi(\mathbf{y},t)] = i\delta^{(3)}(x-y). \tag{55}$$

Express a_k and $a_k^\dagger$ in terms of $\phi(\mathbf{x},0)$ and $\partial_t\phi(\mathbf{x},0)$ and show they satisfy the commutation relations for creation and annihilation operators: $[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0$ and $[a_k, a_{k'}^\dagger] = \delta^{(3)}(k-k')$.

First, we can find the time derivative of the free spinless boson field:

$$\partial_t\phi(\mathbf{x},t) = \frac{\partial}{\partial t}\int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2\omega_k}} \left(a_k e^{-i\omega_k t} e^{i\mathbf{k}\cdot\mathbf{x}} + a_k^\dagger e^{i\omega_k t} e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \tag{56}$$

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2\omega_k}} \left((-i\omega_k) a_k e^{-ik\cdot x} + a_k^\dagger (i\omega_k) e^{ik\cdot x} \right) \tag{57}$$

$$= -i \int \frac{\mathrm{d}^3k}{(2\pi)^3} \sqrt{\frac{\omega_k}{2}} \left(a_k e^{-ik\cdot x} - a_k^\dagger e^{ik\cdot x} \right) \ , \tag{58}$$

where $\omega_k = k^0 = \sqrt{|\mathbf{k}|^2 + m^2}$. If we evaluate the field and its time derivative at $t=0$, we are left with

$$\phi(\mathbf{x},0) = \int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2\omega_k}} \left(a_k e^{i\mathbf{k}\cdot\mathbf{x}} + a_k^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \tag{59}$$

$$\partial_t\phi(\mathbf{x},0) = -i \int \frac{\mathrm{d}^3k}{(2\pi)^3} \sqrt{\frac{\omega_k}{2}} \left(a_k e^{i\mathbf{k}\cdot\mathbf{x}} - a_k^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \ , \tag{60}$$

but if we make the transformation $\mathbf{k} \to -\mathbf{k}$ in the second terms, these can be expressed as

$$\phi(\mathbf{x},0) = \int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2\omega_k}} \left(a_k + a_{-k}^\dagger \right) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{61}$$

$$\partial_t\phi(\mathbf{x},0) = -i \int \frac{\mathrm{d}^3k}{(2\pi)^3} \sqrt{\frac{\omega_k}{2}} \left(a_k - a_{-k}^\dagger \right) e^{i\mathbf{k}\cdot\mathbf{x}} \ . \tag{62}$$

Now consider taking the inverse Fourier transform of the field:

$$\int \mathrm{d}^3x \phi(\mathbf{x},0) e^{-i\mathbf{p}\cdot\mathbf{x}} = \iint \frac{\mathrm{d}^3k\mathrm{d}^3x}{(2\pi)^3\sqrt{2\omega_k}} \left(a_k + a_{-k}^\dagger \right) e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{x}} \tag{63}$$

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2\omega_k}} \left(a_k + a_{-k}^\dagger \right) \int \mathrm{d}^3x e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{x}} \ , \tag{64}$$

now using the identity

$$\int \mathrm{d}^3x e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{x}} = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{p}) \ , \tag{65}$$

the previous equation may be written as

$$\int \mathrm{d}^3x\phi(\mathbf{x},0)e^{-i\mathbf{p}\cdot\mathbf{x}} = \int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}}\left(a_k + a^\dagger_{-k}\right)\delta^{(3)}(\mathbf{k}-\mathbf{p}) = \frac{1}{\sqrt{2\omega_p}}\left(a_p + a^\dagger_{-p}\right) \ . \tag{66}$$

We can do this same process for the time derivative of the field:

$$\int \mathrm{d}^3x\partial_t\phi(\mathbf{x},0)e^{-i\mathbf{p}\cdot\mathbf{x}} = -i\int \frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{\frac{\omega_k}{2}}\left(a_k - a^\dagger_{-k}\right)\int \mathrm{d}^3x e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{x}} \tag{67}$$

$$= -i\int \mathrm{d}^3k\sqrt{\frac{\omega_k}{2}}\left(a_k - a^\dagger_{-k}\right)\delta^{(3)}(\mathbf{k}-\mathbf{p}) \tag{68}$$

$$= -i\sqrt{\frac{\omega_p}{2}}\left(a_p - a^\dagger_{-p}\right) \ . \tag{69}$$

Solving Equations 66 and 69 for a_p and $a^\dagger_{-p}$ yields:

$$a_p = \frac{1}{2}\left(\sqrt{2\omega_p}\int \mathrm{d}^3x\phi(\mathbf{x},0)e^{-i\mathbf{p}\cdot\mathbf{x}} + i\sqrt{\frac{2}{\omega_p}}\int \mathrm{d}^3x\partial_t\phi(\mathbf{x},0)e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{70}$$

$$a^\dagger_{-p} = \frac{1}{2}\left(\sqrt{2\omega_p}\int \mathrm{d}^3x\phi(\mathbf{x},0)e^{-i\mathbf{p}\cdot\mathbf{x}} - i\sqrt{\frac{2}{\omega_p}}\int \mathrm{d}^3x\partial_t\phi(\mathbf{x},0)e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \ . \tag{71}$$

Switching from $\mathbf{p}\to\mathbf{k}$ fo a_p and $-\mathbf{p}\to\mathbf{k}$ in $a^\dagger_{-p}$ yields the result:

$$a_k = \frac{1}{2}\left(\sqrt{2\omega_k}\int \mathrm{d}^3x\phi(\mathbf{x},0)e^{-i\mathbf{k}\cdot\mathbf{x}} + i\sqrt{\frac{2}{\omega_k}}\int \mathrm{d}^3x\partial_t\phi(\mathbf{x},0)e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \tag{72}$$

$$a^\dagger_k = \frac{1}{2}\left(\sqrt{2\omega_k}\int \mathrm{d}^3x\phi(\mathbf{x},0)e^{i\mathbf{k}\cdot\mathbf{x}} - i\sqrt{\frac{2}{\omega_k}}\int \mathrm{d}^3x\partial_t\phi(\mathbf{x},0)e^{i\mathbf{k}\cdot\mathbf{x}}\right) \ . \tag{73}$$

Let us define the operators:

$$\bar{\phi}(\mathbf{p}) = \int \mathrm{d}^3x\phi(\mathbf{x},0)e^{-i\mathbf{p}\cdot\mathbf{x}} \tag{74}$$

$$\bar{\pi}(\mathbf{p}) = \int \mathrm{d}^3x\partial_t\phi(\mathbf{x},0)e^{-i\mathbf{p}\cdot\mathbf{x}} \ , \tag{75}$$

and thus

$$a_k = \frac{1}{2}\left(\sqrt{2\omega_k}\bar{\phi}(\mathbf{k}) + i\sqrt{\frac{2}{\omega_k}}\bar{\pi}(\mathbf{k})\right) \tag{76}$$

$$a^\dagger_k = \frac{1}{2}\left(\sqrt{2\omega_k}\bar{\phi}(-\mathbf{k}) - i\sqrt{\frac{2}{\omega_k}}\bar{\pi}(-\mathbf{k})\right) \ . \tag{77}$$

The canonical commutation relation

$$[\phi(\mathbf{x},0), \partial_t\phi(\mathbf{y},0)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{78}$$

multiplited by factors of $e^{-i(\mathbf{p}\cdot\mathbf{x})}$ and $e^{-i(\mathbf{p}'\cdot\mathbf{y})}$ and integrated over $\mathbf{x}$ and $\mathbf{y}$ is

$$\iint \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{y} e^{-i(\mathbf{p}\cdot\mathbf{x})}e^{-i(\mathbf{p}'\cdot\mathbf{y})}[\phi(\mathbf{x},0),\partial_t\phi(\mathbf{y},0)] = \iint \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{y} e^{-i(\mathbf{p}\cdot\mathbf{x})}e^{-i(\mathbf{p}'\cdot\mathbf{y})} i\delta^{(3)}(\mathbf{x}-\mathbf{y}). \quad (79)$$

The right-hand side of the above equation is

$$i\iint \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{y} e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{y})}\delta^{(3)}(\mathbf{x}-\mathbf{y}) , \quad (80)$$

and performing the integration over $\mathbf{y}$ yields

$$i\int \mathrm{d}^3\mathbf{x} e^{-i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} = i(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}') . \quad (81)$$

The left-hand side of Equation 79 is

$$\iint \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{y}\left\{(\phi(\mathbf{x},0))e^{-i(\mathbf{p}\cdot\mathbf{x})}(\partial_t\phi(\mathbf{y},0))e^{-i(\mathbf{p}'\cdot\mathbf{y})} - (\partial_t\phi(\mathbf{y},0))e^{-i(\mathbf{p}'\cdot\mathbf{y})}(\phi(\mathbf{x},0))e^{-i(\mathbf{p}\cdot\mathbf{x})}\right\} \quad (82)$$

$$= \bar{\phi}(\mathbf{p})\bar{\pi}(\mathbf{p}') - \bar{\pi}(\mathbf{p}')\bar{\phi}(\mathbf{p}) = [\bar{\phi}(\mathbf{p}),\bar{\pi}(\mathbf{p}')] \quad (83)$$

and so the canonical commutation relation for $\bar{\phi}$ and $\bar{\pi}$ is

$$[\bar{\phi}(\mathbf{p}),\bar{\pi}(\mathbf{p}')] = i(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{p}') . \quad (84)$$

The same process can be done for the commutators $[\bar{\phi}(\mathbf{p}),\bar{\phi}(\mathbf{p}')]$ and $[\bar{\pi}(\mathbf{p}),\bar{\pi}(\mathbf{p}')]$, where the same integration is performed, but the right-hand side is zero:

$$[\bar{\phi}(\mathbf{p}),\bar{\phi}(\mathbf{p}')] = [\bar{\pi}(\mathbf{p}),\bar{\pi}(\mathbf{p}')] = 0 . \quad (85)$$

We are now ready to find the commutators of the creation and annihilation operators[2]:

$$[a_k, a^\dagger_{k'}] = \left[\frac{1}{2}\left(\sqrt{2\omega_k}\bar{\phi}(\mathbf{k}) + i\sqrt{\frac{2}{\omega_k}}\bar{\pi}(\mathbf{k})\right), \frac{1}{2}\left(\sqrt{2\omega_{k'}}\bar{\phi}(-\mathbf{k}') - i\sqrt{\frac{2}{\omega_{k'}}}\bar{\pi}(-\mathbf{k}')\right)\right] \quad (86)$$

$$= \frac{1}{4}\left\{\left[\sqrt{2\omega_k}\bar{\phi}(\mathbf{k}), -i\sqrt{\frac{2}{\omega_{k'}}}\bar{\pi}(-\mathbf{k}')\right] + \left[i\sqrt{\frac{2}{\omega_k}}\bar{\pi}(\mathbf{k}), \sqrt{2\omega_{k'}}\bar{\phi}(-\mathbf{k}')\right]\right\} \quad (87)$$

$$= \frac{1}{4}i\left\{2\left[\bar{\phi}(\mathbf{k}), -\bar{\pi}(-\mathbf{k}')\right] + 2\left[\bar{\pi}(\mathbf{k}),\bar{\phi}(-\mathbf{k}')\right]\right\} \quad (88)$$

$$= i\frac{1}{2}\left\{\left[\bar{\phi}(\mathbf{k}), -\bar{\pi}(-\mathbf{k}')\right] + \left[\bar{\pi}(\mathbf{k}),\bar{\phi}(-\mathbf{k}')\right]\right\} = \frac{1}{2}i\left\{2\left[\bar{\pi}(\mathbf{k}),\bar{\phi}(-\mathbf{k}')\right]\right\} \quad (89)$$

$$= i(i(2\pi)^3\delta^{(3)}(\mathbf{k}+(-\mathbf{k}'))) = -(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')) , \quad (90)$$

as expected. Similarly, we can find the commutator

$$[a_k, a_{k'}] = \left[\frac{1}{2}\left(\sqrt{2\omega_k}\bar{\phi}(\mathbf{k}) + i\sqrt{\frac{2}{\omega_k}}\bar{\pi}(\mathbf{k})\right), \frac{1}{2}\left(\sqrt{2\omega_{k'}}\bar{\phi}(-\mathbf{k}') + i\sqrt{\frac{2}{\omega_{k'}}}\bar{\pi}(-\mathbf{k}')\right)\right] \quad (91)$$

$$= \frac{1}{4}i\left\{2\left[\bar{\phi}(\mathbf{k}), \bar{\pi}(-\mathbf{k}')\right] + 2\left[\bar{\pi}(\mathbf{k}),\bar{\phi}(-\mathbf{k}')\right]\right\} \quad (92)$$

$$= \frac{1}{2}i\left\{\left[\bar{\phi}(\mathbf{k}), \bar{\pi}(-\mathbf{k}')\right] - \left[\bar{\phi}(-\mathbf{k}'),\bar{\pi}(\mathbf{k})\right]\right\} = 0 , \quad (93)$$

[2]Using the fact that $[A+B, C+D] = [A,C]+[A,D]+[B,C]+[B,D]$ with $A\sim\bar{\phi}(\mathbf{k})$, $B\sim\bar{\pi}(\mathbf{k})$, $C\sim\bar{\phi}(-\mathbf{k}')$, $D\sim\bar{\pi}(-\mathbf{k}')$. So that $[A,C]=[B,D]=0$.

and the commutator

$$[a_k^\dagger, a_{k'}^\dagger] = \left[\frac{1}{2}\left(\sqrt{2\omega_k}\bar{\phi}(\mathbf{k}) - i\sqrt{\frac{2}{\omega_k}}\bar{\pi}(\mathbf{k})\right), \frac{1}{2}\left(\sqrt{2\omega_{k'}}\bar{\phi}(-\mathbf{k}') - i\sqrt{\frac{2}{\omega_{k'}}}\bar{\pi}(-\mathbf{k}')\right)\right] \tag{94}$$

$$= \frac{1}{4}i\left\{-2\left[\bar{\phi}(\mathbf{k}), \bar{\pi}(-\mathbf{k}')\right] - 2\left[\bar{\pi}(\mathbf{k}), \bar{\phi}(-\mathbf{k}')\right]\right\} \tag{95}$$

$$= -\frac{1}{2}i\left\{\left[\bar{\phi}(\mathbf{k}), \bar{\pi}(-\mathbf{k}')\right] - \left[\bar{\phi}(-\mathbf{k}'), \bar{\pi}(\mathbf{k})\right]\right\} = 0\ . \tag{96}$$

B) Alternatively, treat $\phi(x)$ as the simplest quantum field constructed out of the creation and annihilation operators a_k and $a_k^\dagger$ and show that $\phi(x)$ and $\partial_t\phi(x)$ satisfy the correct commutation relations as required by the canonical quantization.

We are interested in showing

$$[\phi(\mathbf{x},0), \phi(\mathbf{y},0)] = [\partial_t\phi(\mathbf{x},0), \partial_t\phi(\mathbf{y},0)] = 0, \quad [\phi(\mathbf{x},0), \partial_t\phi(\mathbf{y},0)] = i\delta^{(3)}(x-y), \tag{97}$$

where

$$\pi(\mathbf{x}) \equiv \partial_t\phi(\mathbf{x},0) = -i\int\frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{\frac{\omega_k}{2}}\left(a_k e^{i\mathbf{k}\cdot\mathbf{x}} - a_k^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}\right)\ . \tag{98}$$

The first commutator is given by

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p 2\omega_q}}\left[\left(a_p e^{i\mathbf{p}\cdot\mathbf{x}} + a_p^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right), \left(a_q e^{i\mathbf{q}\cdot\mathbf{y}} + a_q^\dagger e^{-i\mathbf{q}\cdot\mathbf{y}}\right)\right] \tag{99}$$

and using the identity in footnote 2, and the fact that $[a_p, a_q] = [a_p^\dagger, a_q^\dagger] = 0$, this becomes

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p 2\omega_q}}\left(e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{q}\cdot\mathbf{y}}[a_p, a_q^\dagger] + e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{q}\cdot\mathbf{y}}[a_p^\dagger, a_q]\right)\ . \tag{100}$$

We can now use the commutator $[a_k, a_{k'}^\dagger] = \delta^{(3)}(k-k') = -[a_k^\dagger, a_{k'}]$, making the commutator above

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p 2\omega_q}}\left(e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{q}\cdot\mathbf{y}} - e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{q}\cdot\mathbf{y}}\right)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\ .$$

If we carry out the integration over q, this becomes

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right) \tag{101}$$

$$= \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\ , \tag{102}$$

since $\omega_p = \sqrt{|\mathbf{p}|^2 + m^2}$ and is symmetric under $\mathbf{p} \to -\mathbf{p}$, and since we are integrating each component of momentum from $-\infty$ to ∞, the entire integrand is symmetric under $\mathbf{p} \to -\mathbf{p}$, so

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} = 0\ . \tag{103}$$

We can now do the same for the second commutator:

$$[\pi(\mathbf{x}),\pi(\mathbf{y})] = -\int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{\omega_p\omega_q}{2^2}}\left[\left(a_pe^{i\mathbf{p}\cdot\mathbf{x}} - a_p^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right),\left(a_qe^{i\mathbf{q}\cdot\mathbf{y}} - a_q^\dagger e^{-i\mathbf{q}\cdot\mathbf{y}}\right)\right]$$

using the commutation relations this is

$$\begin{aligned}[\pi(\mathbf{x}),\pi(\mathbf{y})] &= -\int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{\omega_p\omega_q}{2^2}}\left(e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{q}\cdot\mathbf{y}}[a_p,a_q^\dagger] + e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{q}\cdot\mathbf{y}}[a_p^\dagger,a_q]\right)\\ &= -\int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{\omega_p\omega_q}{2^2}}\left(e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{q}\cdot\mathbf{y}} + e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{q}\cdot\mathbf{y}}\right)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\\ &= -\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\omega_p}{2}\left(e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{q}\cdot\mathbf{y}} + e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{q}\cdot\mathbf{y}}\right)\ .\end{aligned}$$

Doing the same transformation on the second term as the previous commutator, we see that this commutator also vanishes:

$$[\pi(\mathbf{x}),\pi(\mathbf{y})] = 0\ . \tag{104}$$

The final commutator is

$$\begin{aligned}[\phi(\mathbf{x}),\pi(\mathbf{y})] &= -i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{\omega_q}{2^2\omega_p}}\left[\left(a_pe^{i\mathbf{p}\cdot\mathbf{x}} + a_p^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right),\left(a_qe^{i\mathbf{q}\cdot\mathbf{y}} - a_q^\dagger e^{-i\mathbf{q}\cdot\mathbf{y}}\right)\right]\\ &= -i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{\omega_q}{2^2\omega_p}}\left(e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{q}\cdot\mathbf{y}}[a_p^\dagger,a_q] - e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{q}\cdot\mathbf{y}}[a_p,a_q^\dagger]\right)\\ &= \frac{i}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{\omega_q}{\omega_p}}\left(e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{q}\cdot\mathbf{y}}[a_p,a_q^\dagger] - e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{q}\cdot\mathbf{y}}[a_p^\dagger,a_q]\right)\\ &= \frac{i}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{\omega_q}{\omega_p}}\left(e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{q}\cdot\mathbf{y}} + e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{q}\cdot\mathbf{y}}\right)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\\ &= \frac{i}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right)\\ &= \frac{i}{2}\left(2\delta^{(3)}(\mathbf{x}-\mathbf{y})\right)\ ,\end{aligned}$$

which yields the result

$$[\phi(\mathbf{x}),\pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})\ . \tag{105}$$

C) In canonical quantization the Hamiltonian of a free spinless boson can be written as

$$H = \int \mathrm{d}^3x\frac{1}{2}[(\partial_t\phi)^2 + (\nabla\phi)^2 + (m\phi)^2]\ . \tag{106}$$

Verify explicitly that

$$H = \int \mathrm{d}^3x\omega_k\left(a_ka_k^\dagger + \frac{1}{2}\delta^{(3)}(0)\right)\ . \tag{107}$$

The Hamiltonian can be written:

$$H = \int \mathrm{d}^3x\frac{1}{2}[(\pi)^2 + (|\mathbf{k}|\phi)^2 + (m\phi)^2] = \int \mathrm{d}^3x\frac{1}{2}[(\pi)^2 + (|\mathbf{k}|^2+m^2)\phi^2] \tag{108}$$

$$= \int \mathrm{d}^3x\frac{1}{2}[(\pi)^2 + \omega_k^2\phi^2]\ , \tag{109}$$

where π is the momentum of the field: $\partial_t \phi$. If we note that this is similar in form to the harmonic oscillator Hamiltonian, we see

$$\begin{aligned}
(\pi(\mathbf{x}) + i\omega_k\phi(\mathbf{x}))\,(\pi(\mathbf{x}) + i\omega_k\phi(\mathbf{x})) &= \pi(\mathbf{x})^2 + \omega_k^2\phi(\mathbf{x}^2) + i\omega_k\phi(\mathbf{x})\pi(\mathbf{x}) - i\omega\pi(\mathbf{x})\phi(\mathbf{x}) && (110)\\
&= \pi(\mathbf{x})^2 + \omega_k^2\phi(\mathbf{x}^2) + i\omega_k[\phi(\mathbf{x}), \pi(\mathbf{x})] && (111)\\
&= \pi(\mathbf{x})^2 + \omega_k^2\phi(\mathbf{x}^2) + i\omega_k(i\delta^{(3)}(\mathbf{x}-\mathbf{x})) && (112)\\
&= \pi(\mathbf{x})^2 + \omega_k^2\phi(\mathbf{x}^2) - \omega_k\delta^{(3)}(0) && (113)
\end{aligned}$$

If we use the definitions of a_k and $a_k^\dagger$:

$$a_k = \pi(\mathbf{x}) + i\omega\phi(\mathbf{x}) \quad \text{and} \quad a_k^\dagger = \pi(\mathbf{x}) - i\omega\phi(\mathbf{x}) \ , \tag{114}$$

then from above, we have

$$a_k a_k^\dagger = \pi(\mathbf{x})^2 + \omega_k^2\phi(\mathbf{x}^2) - \omega_k\delta^{(3)}(0) \ , \tag{115}$$

and so the Hamiltonian can be written

$$H = \int \mathrm{d}^3x\,\omega_k \left(a_k a_k^\dagger + \frac{1}{2}\delta^{(3)}(0) \right) \ . \tag{116}$$

4 Divergences in QFT.

The infinite constant in the Hamiltonian in Problem 3 (c),

$$H_{CC} = \int \mathrm{d}^3 k \omega_k \frac{1}{2} \delta^{(3)}(0), \tag{117}$$

actually contains two types of infinities:

A) The infinity in $\delta^{(3)}(0)$ comes about because the space in which our QFT lives is infinite in volume. To see this explicitly, recall that $\delta^{(3)}(0)$ arises from the commutator

$$[a_k, a^\dagger_{k'}] = \delta^{(3)}(k - k') = \int \mathrm{d}^3 x e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \ . \tag{118}$$

Use the above equation to show that if we had placed the QFT in a box with sides of length L, then $[a_k, a^\dagger_k] = L^3$ which is the volume of the box. Now take the length $L \to \infty$ and show that

$$H_{CC} = \int \mathrm{d}^3 k \frac{1}{2} \omega_k V \ , \tag{119}$$

where V is the volume of the infinite space. An infinity associated with an infinite volume is called the *infrared divergence*.

If our field theory exists in a cubic box of side length L, then the commutator $[a_k, a^\dagger_k]$ is given by

$$[a_k, a^\dagger_k] = \int \mathrm{d}^3 x e^{-i(0)\cdot\mathbf{x}} = \left(\int_0^L \mathrm{d}x \right)^3 = L^3 = V \ , \tag{120}$$

so the infinite constant becomes

$$H_{CC} = \int \mathrm{d}^3 k \omega_k \frac{1}{2} \int \mathrm{d}^3 x e^{-i(0)\cdot\mathbf{x}} = \int \mathrm{d}^3 k \omega_k \frac{1}{2} V \ , \tag{121}$$

which diverges to infinity as $L \to \infty$.

B) The infrared divergence comes about because we are computing the *total* energy of the system. We could instead compute the energy density $\mathcal{H}_{CC} \equiv H_{CC}/V$ to get around the infrared divergence. Show that there is still a divergence in $\mathcal{H}_{CC}$ because we assume the QFT is valid up to arbitrarily high energy and therefore integrate over arbitrarily high momentum $|\mathbf{k}|$. Such a divergence is called the *ultraviolet divergence*.

We define the energy density to be

$$\mathcal{H}_{CC} \equiv \frac{H_{CC}}{V} = \frac{1}{2} \int \sqrt{|\mathbf{k}|^2 + m^2} \mathrm{d}^3 k \ . \tag{122}$$

We can perform the integral in spherical coordinates, immediately integrating out the angular components:

$$\mathcal{H}_{CC} = \frac{4\pi}{2} \int_0^\infty \sqrt{|\mathbf{k}|^2 + m^2} (|\mathbf{k}|^2 \mathrm{d}|\mathbf{k}|). \tag{123}$$

The integrand scales as $\sim |\mathbf{k}|^3$, which divergres as $|\mathbf{k}| \to \infty$.

C) Since no one knows how to write down a consistent QFT for gravity, it is reasonable to assume that our QFT is valid only up to the Planck energy $M_{\rm pl}$ when the effect of gravity becomes important. Therefore we should cut off the $|\mathbf{k}|$ integral at $M_{\rm pl}$. Calculate the zero-point energy density $\mathcal{H}_{CC}$ in terms of $M_{\rm pl}$.

If we set our cut-off energy to be the Planck energy, the final integral from the part (B) becomes

$$\mathcal{H}_{CC} = 2\pi \int_0^{M_{\rm Pl}} \sqrt{|\mathbf{k}|^2 + m^2}(|\mathbf{k}|^2 {\rm d}|\mathbf{k}|). \tag{124}$$

Evaluating this integral in MATHEMATICA yields

$$\mathcal{H}_{CC} = \frac{\pi}{4}\left(M_{\rm Pl}\left(m^2 + 2M_{\rm Pl}^2\right)\sqrt{m^2 + M_{\rm Pl}^2} + m^4 \log\left(\frac{m}{\sqrt{m^2 + M_{\rm Pl}^2} + M_{\rm Pl}}\right)\right) \ . \tag{125}$$

However, we can assume $m \ll M_{\rm PL}$ because it is the rest mass of our spinless boson, which must be negligible compared to the Planck mass ($M_{\rm Pl} \sim 10^{19}$ GeV. Doing this, the energy density becomes

$$\mathcal{H}_{CC} = \frac{\pi}{4}\left(2M_{\rm Pl}^4 + m^4 \log\left(\frac{m}{2M_{\rm Pl}}\right)\right) \ . \tag{126}$$

Furthermore, in this limit, we may also neglect the entire logarithmic term because if $m \ll M_{\rm PL}$, then

$$M_{\rm PL} \gg m \log(m/M_{\rm PL}) \ , \tag{127}$$

so we get the result

$$\mathcal{H}_{CC} = \frac{\pi}{2} M_{\rm Pl}^4 \sim \ 3.5 \times 10^{112} \text{ eV}. \tag{128}$$

D) One way to remove the zero-point energy is to add a so-called *cosmological constant* term to the Klein-Gordon Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) + \Lambda_{CC} \ . \tag{129}$$

Show that the total zero-point energy density now becomes

$$\mathcal{E}_{\rm total} = \mathcal{H}_{CC} - \Lambda_{CC} \ . \tag{130}$$

The Hamiltonian density for this Lagrangian density is given by

$$\mathcal{H} = \pi(\mathbf{x})\partial_t\phi(\mathbf{x}) - \mathcal{L} = \pi(\mathbf{x})\partial_t\phi(\mathbf{x}) - \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) - \Lambda_{CC} \tag{131}$$

$$= \pi(\mathbf{x})\partial_t\phi(\mathbf{x}) - \frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 - \Lambda_{CC} \tag{132}$$

$$= \frac{1}{2}(\pi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 - \Lambda_{CC} \ , \tag{133}$$

but from Problem 3C, this can be written

$$\mathcal{H} = \omega_k a_k a_k^\dagger + \mathcal{H}_{CC} - \Lambda_{CC} \ , \tag{134}$$

so the total zero-point energy is

$$\mathcal{E}_{\rm total} = \mathcal{H}_{CC} - \Lambda_{CC} \ . \tag{135}$$

E) Over the last decade our colleagues in cosmology worked very hard and measured $\mathcal{E}_{\text{total}} \approx (10^{-3}\text{ eV})^4$ in our universe. Assuming that QFT is indeed only valid up to M_{pl}, what is the amount of cancellation needed between H_{CC} and Λ_{CC} in order to result in the observed value? One measure of the fine-tuning necessary is to estimate the order of magnitude of

$$\frac{\mathcal{H}_{CC} - \Lambda_{CC}}{\mathcal{H}_{CC} + \Lambda_{CC}} \ . \tag{136}$$

This is the famous cosmological constant problem!

For the total zero-point energy to be $\sim 10^{-12}$ eV, the $\mathcal{H}_{CC}$ must cancel with Λ_{CC} to 124 significant figures. Since $\mathcal{H}_{CC} \sim 10^{112}$ eV, the cosmological constant must cancel that, and the next 12 decimal places hence 124.

5 Conserved currents of infinitesimal Lorentz transforms.

A) In the class we showed that the conserved currents corresponding to spacetime translations $x^\alpha \to x^\alpha - a^\alpha$ are the energy-momentum tensor $T^{\mu\nu}$. Since we have been considering Lorentz-invariant quantum field theories, derive the conserved currents corresponding to infinitesimal Lorentz transformations $\Lambda^\alpha{}_\beta = \delta^\alpha{}_\beta + \omega^\alpha{}_\beta$.
(Hint: recall that in the case of translations, there are really four currents $T^{\mu\alpha} = (j^\mu)^\alpha$, one for each a^α. In this case there are really six conserved currents $M^{\mu\alpha}_\beta = (j^\mu)^\alpha{}_\beta$, one for each $\omega^\alpha{}_\beta$. You may wish to express $M^{\mu\alpha}_\beta$ in terms of $T^{\mu\alpha}$.)

A Noether current is defined by

$$J_\mu = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha} \ , \tag{137}$$

where α is some scalar, vector, or tensor parameter. When the equations of motion are satisfied, this current is conserved: $\partial_\mu J_\mu = 0$. In order to find this current, we need the variation of the field as the parameter α varies. We know the field is invariant under infinitesimal Lorentz transformations. Given our definition of the infinitesimal Lorentz transform

$$\Lambda^\alpha{}_\beta = \delta^\alpha{}_\beta + \omega^\alpha{}_\beta \ , \tag{138}$$

we may find the inverse transform:

$$\left(\Lambda^{-1}\right)^\mu{}_\nu = \Lambda^\nu{}_\mu = \delta^\nu{}_\mu + \omega^\nu{}_\mu = \delta^\mu{}_\nu - \omega^\mu{}_\nu \ , \tag{139}$$

where δ^α_β is the identity, and ω^α_β is a constant with respect to the coordinates, and is anti-symmetric. Consider now how the scalar field operator transforms

$$\phi(x) \to \phi(x') = \phi(\Lambda^{-1}x) = \phi(\delta^\mu{}_\nu x^\nu - \omega^\mu{}_\nu x^\nu) = \phi(x^\mu - \omega^\mu{}_\nu x^\nu) \ . \tag{140}$$

Since ω is an infinitesimal quantity, we may Taylor expand keeping only up to first order:

$$\phi(x) \to \phi(x') = \phi(x) - \omega^\beta{}_\alpha x^\alpha \partial_\beta \phi(x) \ , \tag{141}$$

after some dummy index manipulation. However, by analogy to the infinitesimal translation, we know there must be an additional term:

$$\phi(x) \to \phi(x') = \phi(x) - \omega^\beta{}_\alpha x^\alpha \partial_\beta \phi(x) + \omega^\beta{}_\alpha x_\beta \partial^\alpha \phi(x) \tag{142}$$

$$= \phi(x) + \omega^\beta{}_\alpha \left(x_\beta \partial^\alpha \phi(x) - x^\alpha \partial_\beta \phi(x)\right) \ . \tag{143}$$

Now we can identify, in the notation of Peskin (see equation 2.9), our infinitesimal parameter is $\alpha = \omega^\beta{}_\alpha$ and $\Delta\phi = x_\beta \partial^\alpha \phi(x) - x^\alpha \partial_\beta \phi(x)$. Knowing how a scalar transforms, we can say that the Lagrangian density transforms as

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \omega^\beta{}_\alpha \left(x_\beta \partial^\alpha \mathcal{L} - x^\alpha \partial_\beta \mathcal{L}\right) \ . \tag{144}$$

Inserting the identity yields

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \omega^\beta{}_\alpha \left(x_\beta \partial_\mu \delta^{\mu\alpha} \mathcal{L} - x^\alpha \partial_\mu \delta^\mu{}_\beta \mathcal{L}\right) \tag{145}$$

$$= \mathcal{L} + \omega^\beta{}_\alpha \partial_\mu \left(x_\beta \delta^{\mu\alpha} \mathcal{L} - x^\alpha \delta^\mu{}_\beta \mathcal{L}\right) \ , \tag{146}$$

and now we can identify

$$(J^\mu)^\alpha_{\ \beta} = \mathcal{L}\left(x_\beta \delta^{\mu\alpha} - x^\alpha \delta^\mu_{\ \beta}\right) \ . \tag{147}$$

Peskin equation 2.12 tells us

$$(j^\mu)^\alpha_{\ \beta} = M^{\mu\alpha}_\beta = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\left[x_\beta \partial^\alpha \phi(x) - x^\alpha \partial_\beta \phi(x)\right] - \mathcal{L}\left(x_\beta \delta^{\mu\alpha} - x^\alpha \delta^\mu_{\ \beta}\right) \ , \tag{148}$$

where $\partial_\mu (j^\mu)^\alpha_{\ \beta} = 0$ and is therefore conserved. Here we note the form of the stress-energy tensor:

$$T^\mu_{\ \nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi - \mathcal{L}\delta^\mu_{\ \nu} \ , \tag{149}$$

and then we can write the previous equation as

$$M^{\mu\alpha}_\beta = x_\beta \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial^\alpha \phi(x) - x^\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\beta \phi(x) - x_\beta \mathcal{L}\delta^{\mu\alpha} + x^\alpha \mathcal{L}\delta^\mu_{\ \beta} \tag{150}$$

$$= x_\beta T^{\mu\alpha} - x^\alpha T^\mu_{\ \beta} \ . \tag{151}$$

B) What is the physical interpretation for each of the conserved charges in (a)? Separate your discussions into those corresponding to rotations and those corresponding to Lorentz boosts.

A conserved charge is of the form

$$Q^\alpha_\beta = \int_{\text{all space}} j^{0\alpha}_{\ \ \beta} \mathrm{d}^3 x = \int \mathrm{d}^3 x \left(x_\beta T^{0\alpha} - x^\alpha T^0_{\ \beta}\right) \tag{152}$$

Now we consider just the components

$$Q^i_j = \int_{\text{all space}} j^{0i}_{\ \ j} \mathrm{d}^3 x = \int \mathrm{d}^3 x \left(x_j T^{0i} - x^i T^0_{\ j}\right) \ , \tag{153}$$

and we can identify the term $x_j T^{0i} - x^i T^0_{\ j}$ as angular momentum about the k axis. Therefore, Lorentz rotations conserve angular momentum. Now consider the component

$$Q^0_0 = \int_{\text{all space}} j^{00}_{\ \ 0} \mathrm{d}^3 x = \int \mathrm{d}^3 x \left(x_0 T^{00} - x^0 T^0_{\ 0}\right) \ , \tag{154}$$

the component T^{00} is the total energy density, and this says that energy is conserved. Similarly,

$$Q^0_i = \int_{\text{all space}} j^{00}_{\ \ i} \mathrm{d}^3 x = \int \mathrm{d}^3 x \left(x_0 T^{0i} - x^0 T^0_{\ i}\right) \ , \tag{155}$$

which are Lorentz boosts, give the center-of-mass theorem.