Dylan J. Temples: Solution Set Five

Quantum Field Theory I
Quantum Field Theory and the Standard Model - M. Schwartz
November 18, 2016

Contents

1 Invariance of Dirac Lagrangian.

A) Show that the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{1}$$

is invariant under C, P, and T separately. (Hints: i) recall that time-reversal operator T is anti-unitary, and ii) remember the identity $\sigma^2 \boldsymbol{\sigma} \sigma^2 = -\boldsymbol{\sigma}^T$.)

Writing out the Dirac Lagrnagian we have

$$\mathcal{L}_D = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \ , \tag{2}$$

so we need to investigate how each term transforms under C, P, and T individually[1].

Charge Conjugation.
Let us begin by denoting the charge conjugation operator as $\mathcal{C}$. Then, the Dirac spinors (four-component) transform as

$$\mathcal{C} : \psi \to \mathcal{C}\psi\mathcal{C} = -i\gamma^2\psi^* = -i(\bar{\psi}\gamma^0\gamma^2)^T \tag{3}$$

$$\mathcal{C} : \bar{\psi} \to \mathcal{C}\bar{\psi}\mathcal{C} = -i(\gamma^2\psi)^T\gamma^0 = -i(\gamma^0\gamma^2\psi)^T \ . \tag{4}$$

Consider the act of charge conjugation on the second term of the Lagrangian:

$$\mathcal{C} : \bar{\psi}\psi \to \mathcal{C}\bar{\psi}\psi\mathcal{C} = \left[-i(\gamma^0\gamma^2\psi)^T\right]\left[-i(\bar{\psi}\gamma^0\gamma^2)^T\right] = -(\gamma^0\gamma^2\psi)^T(\bar{\psi}\gamma^0\gamma^2)^T \ . \tag{5}$$

Let us note that the components of the product of two general matrices can be expressed as

$$(MN)_{ac} = M_{ab}N_{bc} \ , \tag{6}$$

where the repeated index $b = 1, 2$ is summed over, and thus

$$(MN)^T_{ac} = M^T_{ab}N^T_{bc} = M_{ba}N_{cb} = (MN)_{ca} \ . \tag{7}$$

Generalizing this logic, we can express Equation 5 as

$$\mathcal{C}\bar{\psi}\psi\mathcal{C} = -\gamma^0_{ab}\gamma^2_{bc}\psi_c\bar{\psi}_d\gamma^0_{de}\gamma^2_{ea} = \bar{\psi}_d\gamma^0_{ab}\gamma^2_{bc}\gamma^0_{de}\gamma^2_{ea}\psi_c \ , \tag{8}$$

note that we anticommuted the fermions to obtain a factor of -1. Now, since we are working in components, the ψ components are two-component spinors and the gamma matrix entries are either zero, the identity, or the second Pauli matrix, so we are free to move them around without consequence:

$$\mathcal{C}\bar{\psi}\psi\mathcal{C} = \bar{\psi}_d\gamma^0_{de}\gamma^2_{ea}\gamma^0_{ab}\gamma^2_{bc}\psi_c = \bar{\psi}\gamma^0\gamma^2\gamma^0\gamma^2\psi = -\bar{\psi}\gamma^0\gamma^2\gamma^2\gamma^0\psi \ , \tag{9}$$

by anticommuting the two-right most gamma matrices. We have that $(\gamma^0)^2 = \mathbb{1} = -(\gamma^2)^2$, so

$$\mathcal{C}\bar{\psi}\psi\mathcal{C} = \bar{\psi}\psi \ , \tag{10}$$

and thus the mass term of the Dirac Lagrangian is invariant under charge conjugation.

[1]The transformations of the spinors $\bar{\psi}$ and ψ under C, P, and T can be found in Peskin & Schroeder section 3.6.

Parity.
Denoting the parity operator as $\mathcal{P}$, the Dirac spinors (four-component) transform as

$$\mathcal{P} : \psi \to \mathcal{P}\psi(t,\mathbf{x})\mathcal{P} = \eta\gamma^0\psi(t,-\mathbf{x}) \tag{11}$$
$$\mathcal{P} : \bar{\psi} \to \mathcal{P}\bar{\psi}(t,\mathbf{x})\mathcal{P} = \eta^*\bar{\psi}(t,-\mathbf{x})\gamma^0 \ , \tag{12}$$

where η is a pure phase. The mass term in the Lagrangian transforms as

$$\mathcal{P} : \bar{\psi}\psi(t,\mathbf{x}) \to \mathcal{P}\bar{\psi}\psi(t,\mathbf{x})\mathcal{P} = |\eta|^2\bar{\psi}\gamma^0\gamma^0\psi = \bar{\psi}\psi(t,-\mathbf{x}) \ , \tag{13}$$

using $(\gamma^0)^2 = \mathbb{1}_4$ and $|\eta|^2 = 1$, and thus the mass term is invariant under parity.

Time Reversal.
Denoting the time reversal operator as $\mathcal{T}$, the Dirac spinors (four-component) transform as

$$\mathcal{T} : \psi \to \mathcal{T}\psi(t,\mathbf{x})\mathcal{T} = -(\gamma^1\gamma^3)\psi(-t,\mathbf{x}) \tag{14}$$
$$\mathcal{T} : \bar{\psi} \to \mathcal{T}\bar{\psi}(t,\mathbf{x})\mathcal{T} = \bar{\psi}(-t,\mathbf{x})(\gamma^1\gamma^3) \ , \tag{15}$$

so the mass term transforms as

$$\mathcal{T} : \bar{\psi}\psi(t,\mathbf{x}) \to \mathcal{T}\bar{\psi}\psi(t,\mathbf{x})\mathcal{T} = -\bar{\psi}\gamma^1\gamma^3\gamma^1\gamma^3\psi = \bar{\psi}(\gamma^1)^2(\gamma^3)^2\psi \ , \tag{16}$$

using the anticommutator. Since $(\gamma^1)^2 = (\gamma^3)^2 = -\mathbb{1}_4$, we have

$$\mathcal{T} : \bar{\psi}\psi(t,\mathbf{x}) \to \mathcal{T}\bar{\psi}\psi(t,\mathbf{x})\mathcal{T} = \bar{\psi}\psi(-t,\mathbf{x}) \ , \tag{17}$$

and is thus invariant under time reversal.

The Kinetic Term.
Since the quantity $\gamma^\mu\partial_\mu$ behaves as a Lorentz scalar, we expect that the results from the mass terms hold for the kinetic terms, under C, P, and T individually. Therefore

$$\mathcal{C}\bar{\psi}\gamma^\mu\partial_\mu\psi\mathcal{C} = \bar{\psi}\gamma^\mu\partial_\mu\psi \tag{18}$$
$$\mathcal{P}\bar{\psi}\gamma^\mu\partial_\mu\psi\mathcal{P} = \bar{\psi}\gamma^\mu\partial_\mu\psi \tag{19}$$
$$\mathcal{T}\bar{\psi}\gamma^\mu\partial_\mu\psi\mathcal{T} = \bar{\psi}\gamma^\mu\partial_\mu\psi \ , \tag{20}$$

and thus the Dirac Lagrangian is invariant under C, P, and T individually - let's verify this. For charge conjugation we have

$$C\bar{\psi}\gamma^\mu\partial_\mu\psi\mathcal{C} = -i(\gamma^2\psi)^T\gamma^0\gamma^\mu\partial_\mu\left[-i\gamma^2\psi^*\right] \tag{21}$$
$$= -(\gamma^2\psi)^T\gamma^0\gamma^\mu\partial_\mu\gamma^2\psi^* = -\psi^T\gamma^2\gamma^0\gamma^\mu\gamma^2\partial_\mu\psi^* \tag{22}$$
$$= \psi^T\gamma^0\gamma^2\gamma^\mu\gamma^2\partial_\mu\psi^* \ . \tag{23}$$

We should now investigate the quantity $\gamma^2\gamma^\mu\gamma^2$:

$$\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^2\bar{\sigma}^\mu\sigma^2 \\ \sigma^2\sigma^\mu\sigma^2 & 0 \end{pmatrix} \ , \tag{24}$$

consider $\mu = 0$:

$$\gamma^2\gamma^0\gamma^2 = \begin{pmatrix} 0 & \sigma^2\mathbb{1}\sigma^2 \\ \sigma^2\mathbb{1}\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \gamma^0 \ , \tag{25}$$

because $(\sigma^\mu)^2 = \mathbb{1}$. Now consider the spatial components

$$\gamma^2\gamma^i\gamma^2 = \begin{pmatrix} 0 & -\sigma^2\sigma^i\sigma^2 \\ \sigma^2\sigma^i\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & (\sigma^i)^T \\ -(\sigma^i)^T & 0 \end{pmatrix} \ , \tag{26}$$

for $i = 1, 3$ we have that $(\sigma^i)^T = \sigma^i$ but for $i = 2$, we have $(\sigma^2)^T = -\sigma^2$, here we can also note that $(\gamma^2)^T = \gamma^2$. With these facts we see $\gamma^2\gamma^\mu\gamma^2 = \gamma^\mu$. Using this result, we have

$$C\bar{\psi}\gamma^\mu\partial_\mu\psi\mathcal{C} = \psi^T\gamma^0\gamma^\mu\partial_\mu\psi^* = \psi^T\partial_\mu\psi^*\gamma^0\gamma^\mu \ . \tag{27}$$

Putting this in component notation, we have

$$C\bar{\psi}\gamma^\mu\partial_\mu\psi\mathcal{C} = \psi_a(\partial_\mu\psi^*)_b(\gamma^0\gamma^\mu)_{ab} \ . \tag{28}$$

Schwartz equation 3.14 allows us to do integration by parts neglecting the boundary terms:

$$A\partial_\mu B = -(\partial_\mu A)B \ , \tag{29}$$

so the previous equation can be written

$$C\bar{\psi}\gamma^\mu\partial_\mu\psi\mathcal{C} = -(\partial_\mu\psi)_a\psi_b^*(\gamma^0\gamma^\mu)_{ab} = -(\partial_\mu\psi)\psi^\dagger\gamma^0\gamma^\mu = -(\partial_\mu\psi)\bar{\psi}\gamma^\mu = \bar{\psi}\partial_\mu\psi\gamma^\mu \tag{30}$$

$$= \bar{\psi}\gamma^\mu\partial_\mu\psi = \bar{\psi}\partial\!\!\!/\psi \ , \tag{31}$$

and thus is invariant under charge conjugation. This proves the entire Dirac Lagrangian is invariant under charge conjugation.

Under parity, we have

$$\mathcal{P}\bar{\psi}\gamma^\mu\partial_\mu\psi\mathcal{P} = \eta^*\bar{\psi}\gamma^0\gamma^\mu\partial_\mu\eta^*\bar{\psi}\gamma^0 = |\eta|^2\bar{\psi}\gamma^0\gamma^\mu\partial_\mu\bar{\psi}\gamma^0 = \bar{\psi}\gamma^0\gamma^\mu\gamma^0\partial_\mu\bar{\psi} \ . \tag{32}$$

Consider the spatial components:

$$\mathcal{P}\bar{\psi}\gamma^i\partial_i\psi\mathcal{P} = \bar{\psi}\gamma^0\gamma^i\gamma^0(-\partial_i)\bar{\psi} = -\bar{\psi}\gamma^0\gamma^0\gamma^i(-\partial_i)\bar{\psi} = \bar{\psi}\gamma^i\partial_i\bar{\psi} \ , \tag{33}$$

and the temporal component:

$$\mathcal{P}\bar{\psi}\gamma^0\partial_0\psi\mathcal{P} = \bar{\psi}\gamma^0\gamma^0\gamma^0\partial_0\bar{\psi} = \bar{\psi}\gamma^0\partial_0\bar{\psi} \ , \tag{34}$$

so $\mathcal{P}\bar{\psi}\gamma^\mu\partial_\mu\psi\mathcal{P} = \bar{\psi}\gamma^\mu\partial_\mu\psi = \bar{\psi}\partial\!\!\!/\psi$, and is thus invariant under parity. This proves the Dirac Lagrangian is invariant under parity.

Under time reversal, we have

$$\mathcal{T}\bar{\psi}\gamma^\mu\partial_\mu\psi\mathcal{T} = -\bar{\psi}\gamma^1\gamma^3(\gamma^\mu)^*\partial_\mu\gamma^1\gamma^3\psi = -\bar{\psi}\gamma^1\gamma^3(\gamma^\mu)^*\gamma^1\gamma^3\partial_\mu\psi = \bar{\psi}\gamma^1\gamma^3(\gamma^\mu)^*\gamma^3\gamma^1\partial_\mu\psi \ , \tag{35}$$

note that the complex conjugate of the gamma matrices only affects γ^2, it picks up a negative. For the temporal component:

$$\mathcal{T}\bar{\psi}\gamma^0\partial_0\psi\mathcal{T} = \bar{\psi}\gamma^1\gamma^3\gamma^0\gamma^3\gamma^1\partial_0\psi = \bar{\psi}\gamma^1\gamma^3\gamma^3\gamma^1(-1)^2\gamma^0\partial_0\psi \tag{36}$$

$$= \bar{\psi}\gamma^1(-\mathbb{1})\gamma^1\gamma^0\partial_0\psi = (-1)^2\bar{\psi}\gamma^0\partial_0\psi = \bar{\psi}\gamma^0\partial_0\psi \ , \tag{37}$$

and the second spatial component:

$$\mathcal{T}\bar{\psi}\gamma^2\partial_2\psi\mathcal{T} = \bar{\psi}\gamma^2\partial_2\psi \ , \tag{38}$$

using an identical process (we picked up a negtive from the complex conjugate, and a negative sign from the derivative), since $\{\gamma^0, \gamma^{(1,3)}\} = \{\gamma^2, \gamma^{(1,3)}\} = 0$. This is more complicated for $i = 1, 3$:

$$\mathcal{T}\bar{\psi}\gamma^3\partial_3\psi\mathcal{T} = \bar{\psi}\gamma^1\gamma^3\gamma^3\gamma^3\gamma^1(-\partial_3)\psi = -\bar{\psi}\gamma^1\gamma^3\gamma^1(-\partial_3)\psi = \bar{\psi}\gamma^1\gamma^1\gamma^3(-\partial_3)\psi = \bar{\psi}\gamma^3\partial_3\psi \tag{39}$$

and for $i = 1$ use the same process but in the first step you pick up two factors of -1 from anticommuting both pairs of $\gamma^1\gamma^3$. Collecting the results, we find

$$\mathcal{T}\bar{\psi}\gamma^\mu\partial_\mu\psi\mathcal{T} = \bar{\psi}\partial\!\!\!/\psi \ , \tag{40}$$

proving the kinetic term, and therefore the Dirac Lagrangian is invariant under time reversal.

B) Replace the derivative ∂^μ by the covariant derivative $D^\mu = \partial^\mu - igA^\mu$ and show that, if electromagnetism is invariant under charge conjugation, the photon A^μ must be odd under charge conjugation

$$CA^\mu(x)C = -A^\mu(x) \ . \tag{41}$$

The electromagnetic Lagrangian is

$$\mathcal{L} = \bar{\psi}(i\gamma_\mu\partial^\mu + g\gamma_\mu A^\mu - m)\psi = \mathcal{L}_D + g\bar{\psi}\gamma_\mu A^\mu\psi \ , \tag{42}$$

so under charge conjugation we have:

$$C\mathcal{L}C = C\mathcal{L}_DC + gC\bar{\psi}\gamma_\mu A^\mu\psi C = \mathcal{L}_D + gC\bar{\psi}\gamma_\mu A^\mu\psi C \ , \tag{43}$$

because we have shown the Dirac Lagrangian is invariant under charge conjugation. For the electromagnetic Lagrangian to be invariant under charge conjugation, we enforce

$$\mathcal{L} = \mathcal{L}_D + gC\bar{\psi}\gamma_\mu A^\mu\psi C \ , \tag{44}$$

but comparing this to Equation 42 leaves us the following condition for the above equation to be true:

$$\bar{\psi}\gamma_\mu A^\mu\psi = C\bar{\psi}\gamma_\mu A^\mu\psi C = C\bar{\psi}\gamma_\mu\psi A^\mu C \ . \tag{45}$$

Since $CC = \mathbb{1}$, we have

$$\bar{\psi}\gamma_\mu A^\mu\psi = (C\bar{\psi}\gamma_\mu\psi C)(CA^\mu C) \ , \tag{46}$$

for now let us assume that the vector Dirac bilinear is odd under charge conjugation. In this case,

$$\bar{\psi}\gamma_\mu A^\mu\psi = (-\bar{\psi}\gamma_\mu\psi)(CA^\mu C) = -\bar{\psi}\gamma_\mu(CA^\mu C)\psi \ , \tag{47}$$

and for the above statement to be true it must be that

$$CA^{\mu}(x)C = -A^{\mu}(x)\ . \tag{48}$$

so proving that the vector Dirac bilinear is odd under charge conjugation proves the photon A_μ is also odd under charge conjugation. Consider the charge conjugation of the vector Dirac bilinear:

$$C\bar{\psi}\gamma_\mu\psi C = \left[-i(\gamma^2\psi)^T\gamma^0\right]\gamma_\mu\left[-i\gamma^2\psi^*\right] = -(\gamma^2\psi)^T\gamma^0\gamma_\mu\gamma^2\psi^* = -\psi^T\gamma^2\gamma^0\gamma_\mu\gamma^2\psi^* \tag{49}$$

$$= \psi^T\gamma^0\gamma^2\gamma_\mu\gamma^2\psi^* = \psi^T\gamma^0\psi^*\gamma^2\gamma_\mu\gamma^2 = -\bar{\psi}\psi\gamma^2\gamma_\mu\gamma^2\ , \tag{50}$$

using Schwartz equations 11.48 and 11.49. Moving the ψ back to the right and using the result from before $\gamma^2\gamma^\mu\gamma^2 = \gamma^\mu$, we have

$$C\bar{\psi}\gamma_\mu\psi C = -\bar{\psi}\gamma_\mu\psi\ , \tag{51}$$

and is in fact odd under charge conjugation.

2 Feynman propagator as Green's function.

One way to define the Feynman propagator is as follows:

$$D_F(x-y) = \theta(x^0 - y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0 - x^0)\langle 0|\phi(y)\phi(x)|0\rangle \ . \tag{52}$$

Show, by direct differentiation, that $D_F(x-y)$ is a Green's function of the Klein-Gordon operator:

$$(\partial_\mu \partial^\mu + m^2) D_F(x-y) = -i\delta^{(4)}(x-y) \ , \tag{53}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$.

We should begin by noting that $\theta(x)$ in the above expression is the Heaviside function. The Feynman propagator can also be expressed as

$$D_F(x-y) = \langle 0|\mathscr{T}\{\phi(x)\phi(y)\}|0\rangle \ , \tag{54}$$

where $\mathscr{T}$ is the time-ordered product operator (operators at later times are to the left of operators which occur earlier). The four-dimensional derivative of the propagator has terms:

$$\frac{\partial}{\partial x^0}\langle 0|\mathscr{T}\{\phi(x)\phi(y)\}|0\rangle \quad \Rightarrow \quad \frac{\partial^2}{\partial (x^0)^2}\langle 0|\mathscr{T}\{\phi(x)\phi(y)\}|0\rangle \tag{55}$$

$$\boldsymbol{\nabla}\cdot\langle 0|\mathscr{T}\{\phi(x)\phi(y)\}|0\rangle \quad \Rightarrow \quad \nabla^2\langle 0|\mathscr{T}\{\phi(x)\phi(y)\}|0\rangle \ . \tag{56}$$

We will begin with the time derivative:

$$\frac{\partial}{\partial x^0} D_F(x-y) = \langle 0|\frac{\partial}{\partial x^0}\mathscr{T}\{\phi(x)\phi(y)\}|0\rangle \ , \tag{57}$$

the time-derivative of a time-ordered product is given by the lemma[2]:

$$\frac{\partial}{\partial t}\left(\mathscr{T}\left\{\hat{A}(t)\hat{B}(t_0)\right\}\right) \ = \ \mathscr{T}\left\{\left(\frac{\partial \hat{A}(t)}{\partial t}\right)\hat{B}(t_0)\right\} \ + \ \delta(t-t_0)\times\left[\hat{A}(t),\hat{B}(t_0)\right] \ . \tag{58}$$

Using this, Equation 57 is

$$\frac{\partial}{\partial x^0} D_F(x-y) = \langle 0|\mathscr{T}\left\{\left(\frac{\partial \phi(x)}{\partial x^0}\right)\phi(y)\right\} \ + \ \delta(x^0-y^0)\times\left[\phi(x),\phi(y)\right]|0\rangle \ , \tag{59}$$

the second term, due to the Delta function, can only be nonzero for $x^0 = y^0$, if this is the case:

$$\left[\phi(x^0,\mathbf{x}),\phi(x^0,\mathbf{y})\right] = 0 \ , \tag{60}$$

so

$$\frac{\partial}{\partial x^0} D_F(x-y) = \langle 0|\mathscr{T}\left\{\left(\frac{\partial \phi(x)}{\partial x^0}\right)\phi(y)\right\}|0\rangle \ . \tag{61}$$

We now take a second time derivative:

$$\frac{\partial^2}{\partial (x^0)^2} D_F(x-y) = \langle 0|\frac{\partial}{\partial x^0}\mathscr{T}\left\{\left(\frac{\partial \phi(x)}{\partial x^0}\right)\phi(y)\right\}|0\rangle \tag{62}$$

$$= \langle 0|\mathscr{T}\left\{(\partial_0)^2\phi(x)\phi(y)\right\} \ + \ \delta(x^0-y^0)\times\left[\partial_0\phi(x),\phi(y)\right]|0\rangle \ , \tag{63}$$

[2]See Kaplunovsky, "Feynman Propagator of a Scalar Field", equations 6- 9 for a derivation.

using the lemma again. Noting the canonical momentum to ϕ, we have

$$\frac{\partial^2}{\partial(x^0)^2} D_F(x-y) = \langle 0|\mathscr{T}\left\{(\partial_0)^2\phi(x)\phi(y)\right\} \ + \ \delta(x^0-y^0)\times\left[\pi(x^0,\mathbf{x}),\phi(x^0,\mathbf{y})\right]|0\rangle \ , \tag{64}$$

but

$$\left[\pi(x^0,\mathbf{x}),\phi(x^0,\mathbf{y})\right] = -\left[\phi(x^0,\mathbf{y}),\pi(x^0,\mathbf{x})\right] = -i\delta^{(3)}(\mathbf{y}-\mathbf{x}) \ , \tag{65}$$

yielding

$$\frac{\partial^2}{\partial(x^0)^2} D_F(x-y) = \langle 0|\mathscr{T}\left\{(\partial_0)^2\phi(x)\phi(y)\right\} \ - i\delta(x^0-y^0)\times\delta^{(3)}(\mathbf{x}-\mathbf{y})]|0\rangle \tag{66}$$

$$= \langle 0|\mathscr{T}\left\{(\partial_0)^2\phi(x)\phi(y)\right\} \ - i\delta^{(4)}(x-y)]|0\rangle \ . \tag{67}$$

Moving on to the spatial part, we have

$$\nabla_x^2 D_F(x-y) = \langle 0|\nabla^2\mathscr{T}\left\{\phi(x)\phi(y)\right\}|0\rangle = \langle 0|\mathscr{T}\left\{(\nabla^2\phi(x))\phi(y)\right\}|0\rangle \ , \tag{68}$$

and now we can write the Klein-Gordon operator acting on the Feynman propagator:

$$(\partial_\mu\partial^\mu + m^2)D_F(x-y) = \langle 0|\left[(\partial_0)_x^2 - \nabla_x^2 + m^2\right]\mathscr{T}\left\{\phi(x)\phi(y)\right\}|0\rangle \tag{69}$$

$$= \langle 0|(\partial_0)_x^2\mathscr{T}\left\{\phi(x)\phi(y)\right\} - \nabla_x^2\mathscr{T}\left\{\phi(x)\phi(y)\right\} + m^2\mathscr{T}\left\{\phi(x)\phi(y)\right\}|0\rangle \ , \tag{70}$$

inserting our previous results to the right-hand side yields:

$$\langle 0|\mathscr{T}\left\{(\partial_0)^2\phi(x)\phi(y)\right\} \ - i\delta^{(4)}(x-y) - \mathscr{T}\left\{(\nabla^2\phi(x))\phi(y)\right\} + \mathscr{T}\left\{m^2\phi(x)\phi(y)\right\}|0\rangle \tag{71}$$

$$\langle 0|\mathscr{T}\left\{\left[(\partial_0)_x^2 - \nabla_x^2 + m^2\right]\phi(x)\phi(y)\right\} - i\delta^{(4)}(x-y)|0\rangle \tag{72}$$

The Klein-Gordon operator (for coordinate x) is acting on the field $\phi(x)$, which satisfies the Klein-Gordon equation:

$$\left[(\partial_0)^2 - \nabla^2 + m^2\right]\phi(x) = 0, \tag{73}$$

thus the first term in the previous equation is zero:

$$(\partial_\mu\partial^\mu + m^2)D_F(x-y) = \langle 0| - i\delta^{(4)}(x-y)|0\rangle = -i\delta^{(4)}(x-y)\,\langle 0|0\rangle = -i\delta^{(4)}(x-y) \ , \tag{74}$$

and therefore the Feynman propagator is a Green's function.

3 Schwartz 6.1.

Calculate the Feynman propagator in position space. To get the pole structure correct, you may find it helpful to use Schwinger parameters (see Appendix B). Take the $m \to 0$ limit of your result to find

$$\langle 0|T\{\phi_0(x_1)\phi_0(x_2)\}|0\rangle = -\frac{1}{4\pi^2}\frac{1}{(x_1-x_2)^2 - i\epsilon} \ . \tag{75}$$

We can express the Feynman propagator as an integral over momentum-space:

$$D_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)} \ . \tag{76}$$

Let's factorize the exponential:

$$e^{-ip\cdot(x-y)} = e^{-ip^0(x_0-y_0)}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \ , \tag{77}$$

which lets us write the propagator as

$$D_F(x-y) = \int \frac{\mathrm{d}p^0}{2\pi}e^{-ip^0(x_0-y_0)}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{i}{p^2-m^2+i\epsilon}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \ . \tag{78}$$

Let us now define $z^0 = x^0 - y^0$, $\mathbf{z} = \mathbf{x} - \mathbf{y}$, and $z = |\mathbf{z}|$, and switch the integration from Cartesian to spherical coordinates:

$$D_F(z) = \int \frac{\mathrm{d}p^0}{2\pi}e^{-ip^0z_0}\int_{p=0}^{\infty}\int_{\vartheta=-1}^{1}\frac{2\pi p^2\mathrm{d}\vartheta\mathrm{d}p}{(2\pi)^3}\frac{i}{p^2-m^2+i\epsilon}e^{ipz\vartheta} \ , \tag{79}$$

where we have integrated the azimuthal angle out (gaining a factor of 2π) and used the integration variable $\vartheta \equiv \cos\theta$ instead of the polar angle θ. Doing some rearrangement we have the expression:

$$D_F(z) = \int \frac{\mathrm{d}p^0}{2\pi}e^{-ip^0z_0}\int_{p=0}^{\infty}\frac{p^2\mathrm{d}p}{(2\pi)^2}\frac{i}{p^2-m^2+i\epsilon}\int_{\vartheta=-1}^{1}\mathrm{d}\vartheta e^{ipz\vartheta} \tag{80}$$

$$= \mathcal{F}(z_0)\int_{p=0}^{\infty}\frac{p^2\mathrm{d}p}{(2\pi)^2}\frac{i}{p^2-m^2+i\epsilon}\left(\frac{2}{pz}\sin(pz)\right) \tag{81}$$

$$= \mathcal{F}(z_0)\frac{2}{z}\frac{i}{(2\pi)^2}\int_{p=0}^{\infty}\mathrm{d}p\frac{p\sin(pz)}{p^2-m^2+i\epsilon} \ , \tag{82}$$

where we've defined

$$\mathcal{F}(z_0) = \int_0^{\infty}\frac{\mathrm{d}p^0}{2\pi}e^{-ip^0z_0} \ . \tag{83}$$

We can factor the denominator:

$$D_F(z) = \mathcal{F}(z_0)\frac{2}{z}\frac{i}{(2\pi)^2}\int_{p=0}^{\infty}\mathrm{d}p\frac{p\sin(pz)}{[p-(m-i\epsilon)][p-(-m+i\epsilon)]} \ , \tag{84}$$

and we see the pole structure of the integrand. If we integrate this in the complex plane by selecting a contour along the real axis closed by an infinite arc in the upper half plane. The arc vanishes because the integrand approaches zero at complex infinity. Therefore, by Cauchy's integral formula, we have

$$2\pi i \sum_{p_0} a_1(p_0) = \int_{p=0}^{\infty} \mathrm{d}p \frac{p\sin(pz)}{[p-(m-i\epsilon)][p-(-m+i\epsilon)]} \ , \tag{85}$$

where $\sum_{p_0} a_1(p_0)$ is the sum of all residues of the poles enclosed by the contour. With the selected contour, we only enclose the pole at $p_0 = -m + i\epsilon$, so

$$a_1(p_0 = m+i\epsilon) = \lim_{p_0 \to m+i\epsilon} (p-p_0) \frac{p\sin(pz)}{[p-(m-i\epsilon)][p-(-m+i\epsilon)]} \tag{86}$$

$$= \frac{(-m+i\epsilon)\sin(z[-m+i\epsilon])}{(-m+i\epsilon)-(m-i\epsilon)} = \frac{(-m+i\epsilon)\sin(z[-m+i\epsilon])}{2(-m+i\epsilon)} \tag{87}$$

$$= \frac{1}{2}\sin(z[-m+i\epsilon]) \ . \tag{88}$$

Inserting this result, we find

$$D_F(z) = \mathcal{F}(z_0)\frac{2}{z}\frac{i}{(2\pi)^2}\frac{1}{2}\sin(z[-m+i\epsilon]) = \mathcal{F}(x_0-y_0)\frac{i}{|\mathbf{x}-\mathbf{y}|}\frac{\sin(|\mathbf{x}-\mathbf{y}|[-m+i\epsilon])}{(2\pi)^2} \ , \tag{89}$$

taking the $m \to 0$ limit yields:

$$D_F(x-y) = \mathcal{F}(x_0-y_0)\frac{i}{|\mathbf{x}-\mathbf{y}|}\frac{\sin(i\epsilon|\mathbf{x}-\mathbf{y}|])}{(2\pi)^2} \ . \tag{90}$$

Alternatively, from Equation 82, we can use Scwhinger parameters:

$$D_F(z) = \mathcal{F}(z_0)\frac{2}{z}\frac{1}{(2\pi)^2}\int_{p=0}^{\infty} \mathrm{d}p \left[\frac{i}{p^2-m^2+i\epsilon}\right] p\sin(pz) \tag{91}$$

$$= \mathcal{F}(z_0)\frac{2}{z}\frac{1}{(2\pi)^2}\int_{p=0}^{\infty} \mathrm{d}p \left[\int_0^{\infty} \mathrm{d}\alpha e^{i(p^2-m^2+i\epsilon)\alpha}\right] p\sin(pz) \tag{92}$$

$$= \int_0^{\infty} \mathrm{d}\alpha \mathcal{F}(z_0)\frac{2}{z}\frac{1}{(2\pi)^2}\int_{p=0}^{\infty} \mathrm{d}p e^{i(p^2-m^2+i\epsilon)\alpha} p\sin(pz) \ . \tag{93}$$

This is hideous, let's start from the beginning using Schwinger parameters:

$$D_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\left[\int_0^{\infty} \mathrm{d}\alpha e^{i(p^2-m^2+i\epsilon)\alpha}\right] e^{-ip\cdot(x-y)} \tag{94}$$

$$= \int_0^{\infty} \mathrm{d}\alpha \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{i(p^2-m^2+i\epsilon)\alpha} e^{-ip\cdot(x-y)} \ , \tag{95}$$

here we make the definition of the four-vector $z = x - y$:

$$D_F(x-y) = \int_0^{\infty} \mathrm{d}\alpha \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-i\alpha(m^2-i\epsilon)} e^{ip^2\alpha} e^{-ip\cdot z} \tag{96}$$

$$= \int_0^{\infty} \mathrm{d}\alpha \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-i\alpha(m^2-i\epsilon)} e^{i\alpha(p^2-\frac{p\cdot z}{\alpha})} \ . \tag{97}$$

Completing the square in the expoenential:

$$p^2 - \frac{p\cdot z}{\alpha} = (p-\frac{z}{2\alpha})^2 - \frac{z^2}{4\alpha^2} \ , \tag{98}$$

yields the expression for the Feynman propagator:

$$D_F(x-y) = \int_0^\infty \mathrm{d}\alpha \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-i\alpha(m^2-i\epsilon)} e^{i\alpha(p-\frac{z}{2\alpha})^2} e^{-(i\alpha)\frac{z^2}{4\alpha^2}} \tag{99}$$

$$= \int_0^\infty \mathrm{d}\alpha e^{-i\frac{z^2}{4\alpha}} \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-i\alpha(m^2-i\epsilon)} e^{i\alpha(p-\frac{z}{2\alpha})^2} \ . \tag{100}$$

Now we will define $k = p - z/2\alpha$, such that $\mathrm{d}^4k = \mathrm{d}^4k$:

$$D_F(x-y) = \int_0^\infty \mathrm{d}\alpha e^{-i\frac{z^2}{4\alpha}} e^{-i\alpha(m^2-i\epsilon)} \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{i\alpha k^2} \ . \tag{101}$$

Since this is an isotropic four-dimensional integral (*i.e.*, the integrand has no angular dependence), we can write this as

$$D_F(x-y) = \int_0^\infty \mathrm{d}\alpha e^{-i\frac{z^2}{4\alpha}} e^{-i\alpha(m^2-i\epsilon)} \int \frac{2\pi^2 k^3 \mathrm{d}k}{(2\pi)^4} e^{i\alpha k^2} \tag{102}$$

$$= \int_0^\infty \mathrm{d}\alpha e^{-iz^2/4\alpha} e^{-i\alpha(m^2-i\epsilon)} \frac{2\pi^2}{(2\pi)^4} \int k^3 \mathrm{d}k e^{i\alpha k^2} \ . \tag{103}$$

Now consider only the momentum integral:

$$\int k^3 \mathrm{d}k e^{i\alpha k^2} \ , \tag{104}$$

we can make a substitution $u = -i\alpha k^2$ (such that $\mathrm{d}u = -2i\alpha k \mathrm{d}k$):

$$\int k^3 \left(\frac{\mathrm{d}u}{-2i\alpha k}\right) e^{-u} = \int k^2 \left(\frac{\mathrm{d}u}{-2i\alpha}\right) e^{-u} = \int \frac{u}{-i\alpha}\left(\frac{\mathrm{d}u}{-2i\alpha}\right) e^{-u} = -\frac{1}{2\alpha^2}\int u e^{-u}\mathrm{d}u \tag{105}$$

$$= -\frac{1}{2\alpha^2} \ , \tag{106}$$

using identities for the Gamma funtion. Therefore the Feynman propagator can be written

$$D_F(x-y) = \int_0^\infty \mathrm{d}\alpha e^{-iz^2/4\alpha} e^{-i\alpha(m^2-i\epsilon)} \frac{2\pi^2}{(2\pi)^4}\left(-\frac{1}{2\alpha^2}\right) \tag{107}$$

$$= -\frac{1}{2^4\pi^2}\int_0^\infty \frac{\mathrm{d}\alpha}{\alpha^2} e^{-iz^2/4\alpha} e^{-i\alpha(m^2-i\epsilon)} \ . \tag{108}$$

We can make another substitution $w = 1/\alpha$, so $\mathrm{d}w = -1/\alpha^2 \mathrm{d}\alpha$:

$$D_F(x-y) = \frac{1}{2^4\pi^2}\int_0^\infty \mathrm{d}w e^{-\frac{i}{4}z^2 w} e^{-i(m^2-i\epsilon)/w} \ . \tag{109}$$

Here we note that this integral is of the form of Bessel functions of the second kind:

$$D_F(x-y) = \frac{1}{2^4\pi^2}\int_0^\infty \mathrm{d}w e^{-\frac{i}{4}z^2 w} e^{-i(m^2-i\epsilon)/w} = \frac{im}{4\pi^2\sqrt{-z^2+i\epsilon}} K_1\left(im\sqrt{-z^2+i\epsilon}\right) \ . \tag{110}$$

If we expand the Bessel function into a Laurent series and keep only the leading order:

$$K_1\left(im\sqrt{-z^2+i\epsilon}\right) \to \frac{1}{im\sqrt{-z^2+i\epsilon}} \ , \tag{111}$$

inserting this to our result, we find:

$$D_F(x-y) = \frac{im}{4\pi^2\sqrt{-z^2+i\epsilon}}\frac{1}{im\sqrt{-z^2+i\epsilon}} = \frac{1}{4\pi^2}\frac{1}{-z^2+i\epsilon} = -\frac{1}{4\pi^2}\frac{1}{(x-y)^2-i\epsilon} \ . \tag{112}$$