Dylan J. Temples: Solution Set Seven

Quantum Field Theory I
Quantum Field Theory and the Standard Model - M. Schwartz
December 7, 2016

Contents

1 Amplitudes for spinor Yukawa processes.

Go to the "real" Yukawa theory, $\mathcal{L}_{int} = -ig\bar{\psi}\psi\phi$, where ψ is now massive Dirac fermion. Use Feynman rules to write down the amplitudes for the following processes:

A) $\psi(k_1) + \bar{\psi}(k_2) \to \psi(p_1) + \bar{\psi}(p_2)$

Let us begin by listing the Feynman rules for the Yukawa Fermion interaction, to which we will refer to for the remainder of this problem:

- Start at the end of fermion lines, and pick up a spinor with the appropriate momentum for each external fermion. A final state fermion gets a $\bar{u}(p)$, a final state anti-fermion gets a $v(p)$, an initial state fermion gets a $u(p)$, and an initial state anti-fermion gets a $\bar{v}(p)$.
- At each vertex, pick up a factor of $(-ig)$.
- At each vertex, conserve momentum to find momenta of propagators. External initial state particles have incoming momenta, and external final state particles have outgoing momenta. Internal fermion propagators have momentum flow determined by the direction of the $U(1)$ charge current.
- For each fermion propagator, pick up a factor of

$$\frac{i(\not{p} + m)}{p^2 - m_\psi^2 + i\epsilon} \tag{1}$$

while for each scalar propagator, pick up a factor of

$$\frac{i}{p^2 - m_\phi^2 + i\epsilon} \tag{2}$$

- Integrate over any undetermined (free) momenta in a loop.

Note that none of the processes we will explore here have identical fermions in the final state. Therefore the amplitudes simply add, there is no destructive interference.

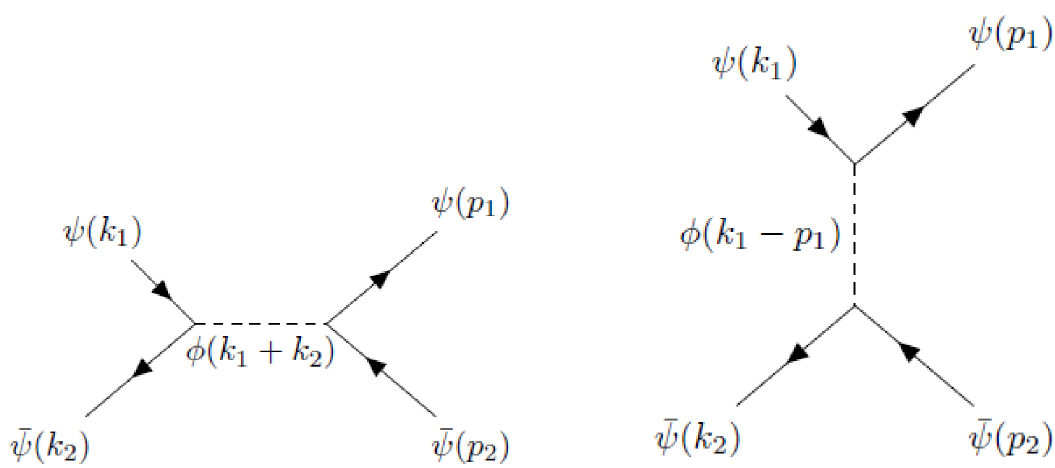

Figure 1: Feynman diagrams corresponding to the process $\psi(k_1) + \bar{\psi}(k_2) \to \psi(p_1) + \bar{\psi}(p_2)$.

Now we consider the process $\psi(k_1) + \bar{\psi}(k_2) \to \psi(p_1) + \bar{\psi}(p_2)$, which to leading-order, is a second order process. The diagrams for this process are shown in Figure 1. For the s-channel (Figure 1 left) process, following the product fermion lines, we have

$$\bar{u}(p_1)(-ig)v(p_2) \ , \tag{3}$$

for the scalar propagator, we have

$$\frac{i}{(k_1+k_2)^2 - m_\phi^2 + i\epsilon} = \frac{i}{(p_1+p_2)^2 - m_\phi^2 + i\epsilon} \tag{4}$$

and for the incoming fermion line, we have

$$\bar{v}(k_2)(-ig)u(k_1) \ . \tag{5}$$

Collecting factors yields the amplitude:

$$i\mathcal{A}_s = (-ig)^2 \left[\bar{u}(p_1)v(p_2)\right]\left[\bar{v}(k_2)u(k_1)\right] \frac{i}{(k_1+k_2)^2 - m_\phi^2 + i\epsilon} \ . \tag{6}$$

For the t-channel (Figure 1 right) , we have similarly:

$$i\mathcal{A}_t = (-ig)^2 \left[\bar{u}(p_1)u(k_1)\right]\left[\bar{v}(k_2)v(p_2)\right] \frac{i}{(k_1-p_1)^2 - m_\phi^2 + i\epsilon} \ . \tag{7}$$

The total amplitude is the sum of these two, because there are no identical particles in the final state.

B) $\psi(k_1) + \phi(k_2) \to \psi(p_1) + \phi(p_2)$

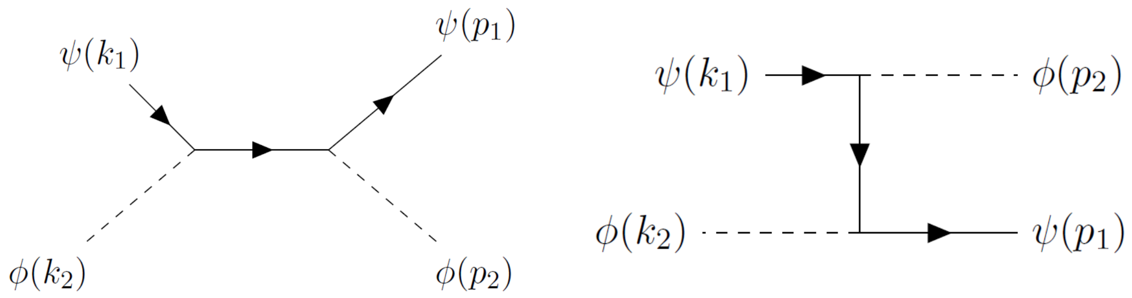

Figure 2: Feynman diagrams corresponding to the process $\psi(k_1) + \phi(k_2) \to \psi(p_1) + \phi(p_2)$.

Now we consider the process $\psi(k_1) + \phi(k_2) \to \psi(p_1) + \phi(p_2)$, which to leading-order, is a second order process. The diagrams for this process are shown in Figure 2. For the s-channel (Figure 2 left) process, we have:

$$\bar{u}(p_1)(-ig)\frac{i(\not{p}+m)}{p^2 - m_\psi^2 + i\epsilon}(-ig)u(k_1) \ . \tag{8}$$

To find p, we conserve momentum at both verteces to find

$$k_1 + k_2 = p \tag{9}$$
$$p = p_1 + p_2 \ , \tag{10}$$

so

$$i\mathcal{A}_s = \bar{u}(p_1)(-ig)\frac{i(\not{k}_1 + \not{k}_2 + m)}{(k_1+k_2)^2 - m_\psi^2 + i\epsilon}(-ig)u(k_1) \ . \tag{11}$$

For the u-channel (Figure 2 right) , we conserve momentum:

$$k_1 = p + p_2 \tag{12}$$
$$k_2 + p = p_1 \ , \tag{13}$$

so we have:

$$i\mathcal{A}_u = \bar{u}(p_1)(-ig)\frac{i(\not{k}_1 - \not{p}_2 + m)}{(k_1-p_2)^2 - m_\psi^2 + i\epsilon}(-ig)u(k_1) \ . \tag{14}$$

The total amplitude is the sum of these two, again because there are no identical particles in the final state.

C) $\psi(k_1) + \bar{\psi}(k_2) \to \phi(p_1) + \phi(p_2)$

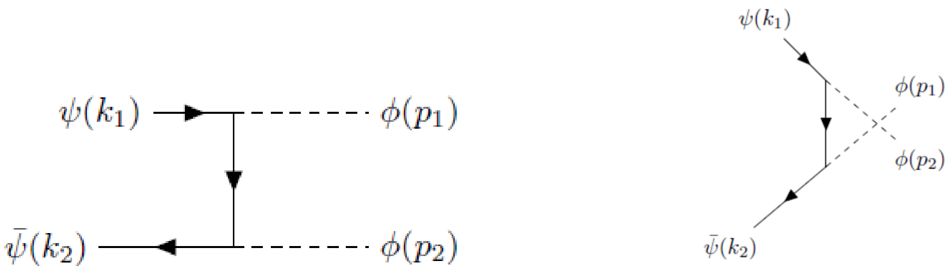

Figure 3: Feynman diagrams corresponding to the process $\psi(k_1) + \bar{\psi}(k_2) \to \phi(p_1) + \phi(p_2)$.

Now we consider the process $\psi(k_1) + \bar{\psi}(k_2) \to \phi(p_1) + \phi(p_2)$, which to leading-order, is a second order process. The diagrams for this process are shown in Figure 3. For the t-channel (Figure 3 left) process, we have conserve momentum:

$$k_1 = p_1 + p \tag{15}$$
$$k_2 + p = p_2 \ . \tag{16}$$

Therefore,

$$i\mathcal{A}_t = \bar{v}(k_2)(-ig)\frac{i(\not{k}_1 - \not{p}_1 + m)}{(k_1 - p_1)^2 - m_\psi^2 + i\epsilon}(-ig)u(k_1) \ . \tag{17}$$

For the u-channel (Figure 3 right) process, we have conserve momentum:

$$k_1 = p_2 + p \tag{18}$$
$$k_2 + p = p_1 \ . \tag{19}$$

Therefore,

$$i\mathcal{A}_u = \bar{v}(k_2)(-ig)\frac{i(\not{k}_1 - \not{p}_2 + m)}{(k_1 - p_2)^2 - m_\psi^2 + i\epsilon}(-ig)u(k_1) \ . \tag{20}$$

The total amplitude is the sum of these two, since scalar particles commute, there is no destructive interference.

D) $\phi(k) \to \psi(p_1) + \bar{\psi}(p_2)$

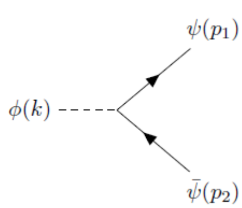

Figure 4: The leading order Feynman diagram corresponding to process $\phi(k) \to \psi(p_1) + \bar{\psi}(p_2)$.

Now we consider the process $\phi(k) \to \psi(p_1) + \bar{\psi}(p_2)$, which to leading-order, is a first order process. The diagram for this process is shown in Figure 3. There are no propagators, so

$$i\mathcal{A} = \bar{u}(p_1)(-ig)v(p_2) \ . \tag{21}$$

E) $\phi(k_1) + \phi(k_2) \to \phi(p_1) + \phi(p_2)$

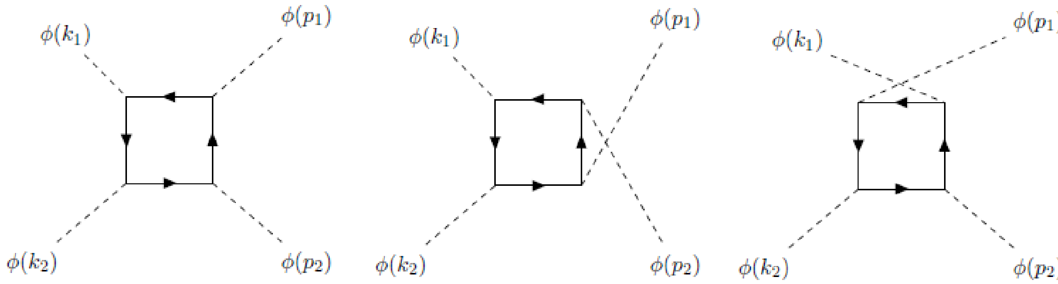

Figure 5: The leading order diagrams for the process $\phi(k_1) + \phi(k_2) \to \phi(p_1) + \phi(p_2)$.

Now we consider the process $\phi(k_1)+\phi(k_2) \to \phi(p_1)+\phi(p_2)$, which to leading-order, is a fourth order process. The (non-redundant) diagrams for this process are shown in Figure 2. There are no external fermions, so we acquire no spinors. We pick up a factor of $(-ig)^4$ for the four verteces. There are four fermion propagators so for each diagram the base amplitude is

$$i\mathcal{A} = (-ig)^4 \frac{i(\not{p} + m)}{(p)^2 - m_\psi^2 + i\epsilon} \frac{i(\not{p}' + m)}{(p')^2 - m_\psi^2 + i\epsilon} \frac{i(\not{p}'' + m)}{(p'')^2 - m_\psi^2 + i\epsilon} \frac{i(\not{p}''' + m)}{(p''')^2 - m_\psi^2 + i\epsilon} \ , \tag{22}$$

so the only exercise is to determine the four propagators' momenta: p, p', p'', p'''. Let us define these momenta such that p corresponds to the top leg of the box, p' to the left, p'' the bottom, and p''' the right. Now if we consider the left-most diagram, we have

$$k_1 + p = p' \tag{23}$$
$$p'' = k_2 + p' \tag{24}$$
$$p'' = p''' + p_2 \tag{25}$$
$$p''' = p + p_1 \ , \tag{26}$$

from conserving momentum at each vertex. Equating the second and fourth, solving for p' and inserting into the first gives an expression for p'''. Taking this expression and inserting it into the fourth yields and expression for p''. Using these, and the first equation, we have

$$p' = p + k_1 \tag{27}$$
$$p'' = p_1 + p_2 + p \tag{28}$$
$$p''' = p + p_1 \ , \tag{29}$$

where p is a free parameter, which we must integrate over. For the left-most diagram in Figure 5, we have

$$i\mathcal{A} = (-ig)^4 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(\not{p} + m)}{(p)^2 - m_\psi^2 + i\epsilon} \frac{i((\not{p} + \not{k}_1) + m)}{(p + k_1)^2 - m_\psi^2 + i\epsilon} \frac{i((\not{p}_1 + \not{p}_2 + \not{p}) + m)}{(p_1 + p_2 + p)^2 - m_\psi^2 + i\epsilon} \times \frac{i((\not{p} + \not{p}_1) + m)}{(p + p_1)^2 - m_\psi^2 + i\epsilon} \ . \tag{30}$$

For the center diagram, conservation of momentum yields

$$k_1 + p = p' \tag{31}$$
$$p'' = k_2 + p' \tag{32}$$
$$p'' = p''' + p_1 \tag{33}$$
$$p''' = p + p_2 \ , \tag{34}$$

which give the relations

$$p' = k_1 + p \tag{35}$$
$$p'' = k_1 + k_2 + p \tag{36}$$
$$p''' = p + p_2 \ , \tag{37}$$

where p is a free parameter, which we must integrate over, yielding

$$i\mathcal{A} = (-ig)^4 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(\not{p} + m)}{(p)^2 - m_\psi^2 + i\epsilon} \frac{i((\not{k}_1 + \not{p}) + m)}{(k_1 + p)^2 - m_\psi^2 + i\epsilon} \frac{i((\not{k}_1 + \not{k}_2 + \not{p}) + m)}{(k_1 + k_2 + p)^2 - m_\psi^2 + i\epsilon} \times \frac{i((\not{p}_2 - \not{p}) + m)}{(p_2 - p)^2 - m_\psi^2 + i\epsilon} \ . \tag{38}$$

For the right-most diagram, conservation of momentum yields

$$k_1 + p = p''' \tag{39}$$
$$p = p' + p_1 \tag{40}$$
$$p' + k_2 = p'' \tag{41}$$
$$p'' = p_2 + p''' \ , \tag{42}$$

Solving the fourth for p' and equating it with the third yields an expression for p''. Inserting this expression into the third gives an expression for p'. The first equation gives the expression for p'''. These give the results

$$p' = p - p_1 \tag{43}$$
$$p'' = p_2 + k_1 + p \tag{44}$$
$$p''' = k_1 + p \ , \tag{45}$$

where p is a free parameter, which we must integrate over, yielding

$$i\mathcal{A} = (-ig)^4 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(\not{p} + m)}{(p)^2 - m_\psi^2 + i\epsilon} \frac{i((\not{p} - \not{p}_1) + m)}{(p - p_1)^2 - m_\psi^2 + i\epsilon} \frac{i((\not{p}_2 + \not{k}_1 + \not{p}) + m)}{(p_2 + k_1 + p)^2 - m_\psi^2 + i\epsilon} \times \frac{i((\not{k}_1 + \not{p}) + m)}{(k_1 + p)^2 - m_\psi^2 + i\epsilon} \ . \tag{46}$$

The total amplitude is the sum of these three.

2 Complex scalar field.

Consider a complex scalar field ϕ coupling to the muon (μ^-) field μ and the neutrino (ν) field ν_L through the following interaction

$$\mathcal{L}_I = C_F(\partial_\mu \phi)(\bar{\mu}\gamma^\mu \nu_L) + (\text{h.c.}), \tag{47}$$

where we have assumed the neutrino is massless and exists only in the left-handed component $\nu_L = (1/2)(1-\gamma_5)\nu$. The complex scalar ϕ has a mass m_ϕ and the muon has a mass m_μ. Compute the differential cross-section $\mathrm{d}\sigma/\mathrm{d}\Omega$ for the scattering process $\bar{\nu} + \nu \to \mu^+ + \mu^-$ to leading order in C_F in the centre-of-mass frame.

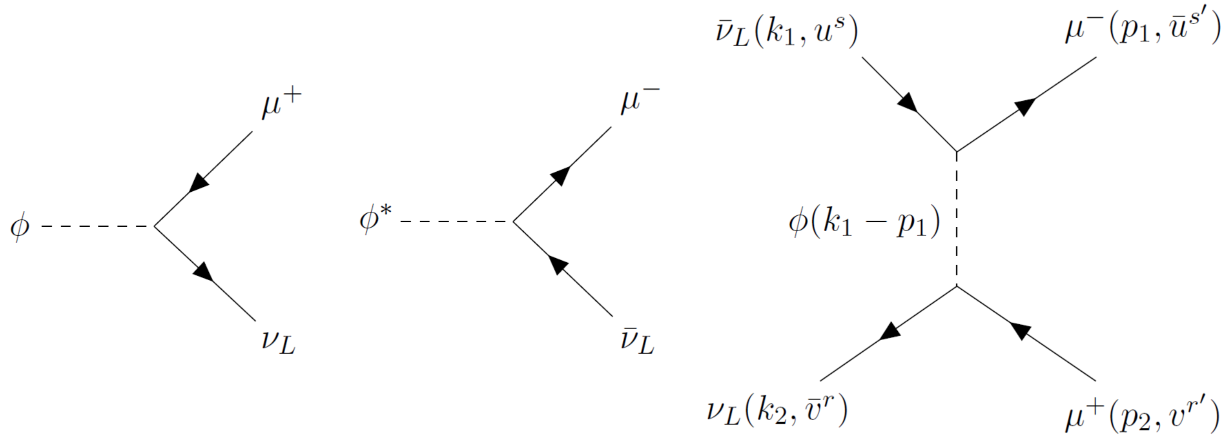

Figure 6: (Left, center) The vertex diagrams allowed by the interaction Lagrangian. (Right) The only allowed diagram for the process $\bar{\nu} + \nu \to \mu^+ + \mu^-$, momenta and spinors of each particle are noted.

This process is the scattering of a fermion/anti-fermion pair, which has an amplitude found in problem 1A, however the Feynman rules are different. First we should note this interaction breaks the $U(1)$ symmetry of the complex scalar propagator, and it therefore carries no $U(1)$ charge. Additionally, the interaction of the neutrino and muon couples to the vector channel of the complex scalar, so at each vertex we pick up a factor

$$iC_F\gamma^\mu q_\mu P_L \tag{48}$$

where the left-projection operator comes from the fact we are only interested in the left-handed interaction, and q is the momentum of the scalar propagator (from the vector derivative). We have defined the left- and right- hand projection operators as

$$P_L = (1/2)(1-\gamma_5) \tag{49}$$
$$P_R = (1/2)(1+\gamma_5)\ , \tag{50}$$

which satisfy $\gamma^\mu P_L = P_R\gamma^\mu$ and vice-versa, also: $P_L^2 = P_L$ and $P_R^2 = P_R$.

Let us first note that the only allowable channel for this process to occur is the t channel. Since there is no vertex coupling two muons or two neutrinos to a scalar the s channel is forbidden. Additionally, the final and initial states are both comprised of distinguishable particles, so the u

channel process cannot occur either. Consider the only allowable process (t-channel) shown by the left diagram in Figure 6, starting with the top fermion line:

$$(iC_F)\bar{u}^{s'}(p_1)\gamma^\mu q_\mu P_L u^s(k_1) \ , \tag{51}$$

and the bottom line:

$$(iC_F)\bar{v}^{r}(k_2)\gamma^\nu q_\nu P_L v^{r'}(p_2) \ . \tag{52}$$

In this case, the scalar propagator is

$$\frac{i}{q^2 - m_\phi^2 + i\epsilon} \tag{53}$$

Note the factor of p^2 in the numerator of the propagator. Since the vector fermion current couples to the derivative of the scalar, we acquire a factor of p, but this is a second-order process, so in reality we acquire two factors. Conserving momentum at each vertex yields: $k_1 - p_1 = p = p_2 - k_2$, so tossing in the scalar propagator gives the result:

$$i\mathcal{A}_t = (iC_F)^2 \left[\bar{u}^{s'}(p_1)\gamma^\mu q_\mu P_L u^s(k_1)\right] \frac{i}{q^2 - m_\phi^2 + i\epsilon} \left[\bar{v}^{r}(k_2)\gamma^\nu q_\nu P_L v^{r'}(p_2)\right] \ . \tag{54}$$

It is useful to note that each term (propagator, and both fermion lines) in Equation 54 is a C number, as expected for an amplitude. For this reason, we can take the modulus squared of each of these factors independently.

First Fermion Line

Consider the first fermion line:

$$\left|\bar{u}^{s'}(p_1)\gamma^\mu q_\mu P_L u^s(k_1)\right|^2 = \bar{u}^{s'}\gamma^\mu q_\mu P_L u^s u^{s\dagger} P_L q_\sigma \gamma^{\sigma\dagger}\gamma^{0\dagger} u^{s'} \tag{55}$$

$$= \bar{u}^{s'}\gamma^\mu q_\mu P_L u^s u^{s\dagger} P_L q_\sigma (\gamma^0)^2 \gamma^{\sigma\dagger}\gamma^{0} u^{s'} \tag{56}$$

using the fact $(\gamma^0)^2 = \mathbb{1}$ and $\gamma^{0\dagger} = \gamma^0$, and that the projection operators are Hermitian. Noting $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$ and $\gamma^\mu P_L = P_R\gamma^\mu$, so moving a projection operator past two Dirac matrices leaves it unchanged and we can see

$$\left|\bar{u}^{s'}(p_1)\gamma^\mu q_\mu P_L u^s(k_1)\right|^2 = \bar{u}^{s'}\gamma^\mu q_\mu P_L u^s u^{s\dagger} P_L \gamma^0 q_\sigma \gamma^{\sigma} u^{s'} \tag{57}$$

$$= \bar{u}^{s'}\gamma^\mu q_\mu P_L u^s u^{s\dagger} \gamma^0 q_\sigma \gamma^{\sigma} P_L u^{s'} \tag{58}$$

$$= \bar{u}^{s'}\gamma^\mu q_\mu u^s u^{s\dagger} \gamma^0 q_\sigma \gamma^{\sigma} P_L u^{s'} \ , \tag{59}$$

this quantity remains a C-number, so we can take the trace:

$$\left|\bar{u}^{s'}(p_1)\gamma^\mu q_\mu P_L u^s(k_1)\right|^2 = \mathrm{Tr}\left[\bar{u}^{s'}\gamma^\mu q_\mu u^s u^{s\dagger}\gamma^0 q_\sigma\gamma^\sigma P_L u^{s'}\right] = q_\sigma q_\mu \,\mathrm{Tr}\left[\gamma^\mu u^s u^{s\dagger}\gamma^0\gamma^\sigma P_L u^{s'}\bar{u}^{s'}\right] \ , \tag{60}$$

after cyclic permutation and factoring out the momentum transfer vectors. To find the amplitude we will need to sum over all possible spin states:

$$\sum_{spins}\left|\bar{u}^{s'}(p_1)\gamma^\mu q_\mu P_L u^s(k_1)\right|^2 = \sum_{s,s'} q_\sigma q_\mu \,\mathrm{Tr}\left[\gamma^\mu u^s \bar{u}^s \gamma^\sigma P_L u^{s'}\bar{u}^{s'}\right] \ , \tag{61}$$

and if we note the completeness relations for Dirac spinors

$$\sum_s u^s(p)\bar{u}^s(p) = \gamma^\mu p_\mu + m = \not{p} + m \tag{62}$$

$$\sum_s v^s(p)\bar{v}^s(p) = \not{p} - m \ , \tag{63}$$

we can move the sum into the trace and find

$$\sum_{spins} \left|\bar{u}^{s'}(p_1)\gamma^\mu q_\mu P_L u^s(k_1)\right|^2 = \frac{q_\sigma q_\mu}{2} \operatorname{Tr}\left[\gamma^\mu \not{k}_1 \gamma^\sigma (1-\gamma_5)(\not{p}_1 + m_\mu)\right] \tag{64}$$

$$= \frac{q_\sigma q_\mu}{2} \operatorname{Tr}\left[\gamma^\mu \not{k}_1 \gamma^\sigma \not{p}_1\right] + m_\mu \operatorname{Tr}\left[\gamma^\mu \not{k}_1 \gamma^\sigma\right] - \operatorname{Tr}\left[\gamma^\mu \not{k}_1 \gamma^\sigma \gamma_5 \not{p}_1\right] - m_\mu \operatorname{Tr}\left[\gamma^\mu \not{k}_1 \gamma^\sigma \gamma_5\right] \ . \tag{65}$$

We can again factor the momenta vectors out, and note that traces of an odd number of Dirac matrices (γ_5 counts as four γ matrices) vanish:

$$\begin{aligned}
&= \frac{q_\sigma q_\mu}{2}\left\{k_{1\alpha}p_{1\beta} \operatorname{Tr}\left[\gamma^\mu\gamma^\alpha\gamma^\sigma\gamma^\beta\right] + k_{1\delta}p_{1\epsilon} \operatorname{Tr}\left[\gamma^\mu\gamma^\delta\gamma^\sigma\gamma^\epsilon\gamma_5\right]\right\} \\
&= \frac{q_\sigma q_\mu}{2}\left\{4k_{1\alpha}p_{1\beta}(g^{\mu\alpha}g^{\sigma\beta} - g^{\mu\sigma}g^{\alpha\beta} + g^{\mu\beta}g^{\alpha\sigma}) + k_{1\delta}p_{1\epsilon}(-4)i\epsilon^{\mu\delta\sigma\epsilon}\right\} \\
&= 2\left\{(q\cdot k_1)(q\cdot p_1) - (q\cdot q)(k_1\cdot p_1) + (q\cdot p_1)(q\cdot k_1) - iq_\sigma q_\mu k_{1\delta}p_{1\epsilon}\epsilon^{\mu\delta\sigma\epsilon}\right\} \ ,
\end{aligned}$$

bringing us to the simplest expression for the first fermion line:

$$\sum_{spins} \left|\bar{u}^{s'}(p_1)\gamma^\mu q_\mu P_L u^s(k_1)\right|^2 = 2\left\{2(q\cdot k_1)(q\cdot p_1) - q^2(k_1\cdot p_1) - iq_\sigma q_\mu k_{1\delta}p_{1\epsilon}\epsilon^{\mu\delta\sigma\epsilon}\right\} \ . \tag{66}$$

Second Fermion Line

Consider the second (anti-)fermion line:

$$\left|\bar{v}^r(k_2)\gamma^\nu q_\nu P_L v^{r'}(p_2)\right|^2 = \bar{v}^r\gamma^\mu q_\mu P_L v^{r'} v^{r'\dagger} P_L q_\nu \gamma^{\nu\dagger}\gamma^0 v^r \tag{67}$$

$$= \bar{v}^r\gamma^\mu q_\mu P_L v^{r'} v^{r'\dagger} P_L q_\nu (\gamma^0)^2 \gamma^{\nu\dagger}\gamma^0 v^r \tag{68}$$

$$= \bar{v}^r\gamma^\mu q_\mu P_L v^{r'} v^{r'\dagger} q_\nu \gamma^0 \gamma^\nu P_L v^r \tag{69}$$

$$= \bar{v}^r\gamma^\mu q_\mu v^{r'} v^{r'\dagger} q_\nu \gamma^0 \gamma^\nu P_L P_L v^r \tag{70}$$

$$= \bar{v}^r\gamma^\mu q_\mu v^{r'} \bar{v}^{r'} q_\nu \gamma^\nu P_L v^r \ , \tag{71}$$

where on the second line we've inserted an identity, on the third line we've used $\gamma^0\gamma\alpha\dagger = \gamma^0 = \gamma^\alpha$, and moved (the right-most) P_L past two gamma matrices, so it remains P_L, on the fourth again moved (the left-most) P_L through two gamma matrices, and on the fourth we've contracted the adjoint of a spinor and γ^0 to get a barred spinor. We can now take the trace, perform two cyclic permutations and factor out the momenta vectors:

$$\left|\bar{v}^r(k_2)\gamma^\nu q_\nu P_L v^{r'}(p_2)\right|^2 = q_\mu q_\nu \operatorname{Tr}\left[v^{r'}\bar{v}^{r'}\gamma^\nu P_L v^r \bar{v}^r \gamma^\mu\right] \ , \tag{72}$$

and will need to sum over all spins:

$$\sum_{spins} \left|\bar{v}^r(k_2)\gamma^\nu q_\nu P_L v^{r'}(p_2)\right|^2 = \sum_{r,r'} q_\mu q_\nu \operatorname{Tr}\left[v^{r'}\bar{v}^{r'}\gamma^\nu P_L v^r \bar{v}^r \gamma^\mu\right] \ . \tag{73}$$

We can move the sum into the trace, and replace the spinor products with their completeness relations:

$$\sum_{spins} \left| \bar{v}^r(k_2)\gamma^\nu q_\nu P_L v^{r'}(p_2) \right|^2 = \frac{q_\mu q_\nu}{2} \operatorname{Tr}\left[(\not{p}_2 - m_\mu)\gamma^\nu(1-\gamma_5)\not{k}_2\gamma^\mu \right] , \tag{74}$$

and inserting an explicit expression for the projection operator. Carrying out the multiplication yields:

$$\sum_{spins} \left| \bar{v}^r(k_2)\gamma^\nu q_\nu P_L v^{r'}(p_2) \right|^2 = \frac{q_\mu q_\nu}{2} \operatorname{Tr}\left[(\not{p}_2\gamma^\nu - \not{p}_2\gamma^\nu\gamma_5 - m_\mu\gamma^\nu + m_\mu\gamma^\nu\gamma_5)\not{k}_2\gamma^\mu \right] \tag{75}$$

$$= \frac{q_\mu q_\nu}{2} \left\{ \operatorname{Tr}\left[\not{p}_2\gamma^\nu\not{k}_2\gamma^\mu \right] - \operatorname{Tr}\left[\not{p}_2\gamma^\nu\gamma_5\not{k}_2\gamma^\mu \right] - m_\mu \operatorname{Tr}\left[\gamma^\nu\not{k}_2\gamma^\mu \right] + m_\mu \operatorname{Tr}\left[\gamma^\nu\gamma_5\not{k}_2\gamma^\mu \right] \right\} \tag{76}$$

$$= \frac{q_\mu q_\nu}{2} \left\{ p_{2\alpha}k_{2\beta} \operatorname{Tr}\left[\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\mu \right] - p_{2\delta}k_{2\epsilon} \operatorname{Tr}\left[\gamma^\delta\gamma^\nu\gamma_5\gamma^\epsilon\gamma^\mu \right] \right\} \tag{77}$$

after noting traces of an odd number of Dirac matrices vanish (as well as an odd number times γ_5). We can permute the trace until γ_5 is at the end, then use the identities for traces of Dirac matrices (see *e.g.*, Peskin A.3) to find:

$$= \frac{q_\mu q_\nu}{2} \left\{ 4p_{2\alpha}k_{2\beta}(g^{\alpha\nu}g^{\beta\mu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\mu}g^{\nu\beta}) + 4i\epsilon^{\epsilon\mu\delta\nu}p_{2\delta}k_{2\epsilon} \right\} \tag{78}$$

$$= 2\left\{ (p_2\cdot q)(k_2\cdot q) - (p_2\cdot k_2)(q\cdot q) + (p_2\cdot q)(k_2\cdot q) + ip_{2\delta}k_{2\epsilon}q_\mu q_\nu\epsilon^{\epsilon\mu\delta\nu} \right\} \tag{79}$$

$$= 2\left\{ 2(p_2\cdot q)(k_2\cdot q) - q^2(p_2\cdot k_2) + ip_{2\delta}k_{2\epsilon}q_\mu q_\nu\epsilon^{\epsilon\mu\delta\nu} \right\} , \tag{80}$$

which is the simplest form of the anti-fermion line.

Propagator and Constants

Finally we need the modulus squared of the propagator and the coupling factors from the verteces:

$$\left| (iC_F)^2 \frac{i}{q^2 - m_\phi^2 + i\epsilon} \right|^2 = C_F^4 \frac{1}{(q^2 - m_\phi^2)^2} \tag{81}$$

where we've taken $\epsilon \to 0$. We then sum over the all the spin states so we acquire a factor of 2^4.

Gathering the three moduli squared summed over all spin states, we have:

$$\sum_{spins} |i\mathcal{A}|^2 = 2^2\left\{ 2(q\cdot k_1)(q\cdot p_1) - q^2(k_1\cdot p_1) - iq_\sigma q_\mu k_{1\delta}p_{1\epsilon}\epsilon^{\mu\delta\sigma\epsilon} \right\}$$

$$\times \left(C_F^4 \frac{2^4}{(q^2 - m_\phi^2)^2} \right) \left\{ 2(p_2\cdot q)(k_2\cdot q) - q^2(p_2\cdot k_2) + ip_{2\delta}k_{2\epsilon}q_\mu q_\nu\epsilon^{\epsilon\mu\delta\nu} \right\} , \tag{82}$$

where $q = k_1 - p_1$ is the propagator momentum. We will note that

$$q^2 = (k_1 - p_1)\cdot(k_1 - p_1) = k_1^2 + p_1^2 - 2(p_1\cdot k_1) \tag{83}$$

$$= k_2^2 + p_2^2 - 2(p_2\cdot k_2) \tag{84}$$

In the centre-of-mass frame, the three-momenta of the neutrinos have the same magnitude but in opposite directions, so

$$\begin{cases} k_1 &= (|\mathbf{k}|, \mathbf{k}) \\ k_2 &= (|\mathbf{k}|, -\mathbf{k}) \end{cases} \quad \Rightarrow \quad k_1 + k_2 = (2|\mathbf{k}|, 0) \equiv (E_{tot}, 0) \ , \tag{85}$$

because they are massless. Additionally, in the final state:

$$\begin{cases} p_1 &= (\sqrt{|\mathbf{p}|^2 + m_\mu^2}, \mathbf{p}) \\ p_2 &= (\sqrt{|\mathbf{p}|^2 + m_\mu^2}, -\mathbf{p}) \end{cases} \quad \Rightarrow \quad p_1 + p_2 = (2\sqrt{|\mathbf{p}|^2 + m_\mu^2}, 0) = (E_{tot}, 0) \ , \tag{86}$$

yielding the energy conservation condition:

$$|\mathbf{p}|^2 + m_\mu^2 = \frac{E_{tot}^2}{4} \ . \tag{87}$$

The differential cross-section of a two-body scattering process is given by

$$\mathrm{d}\sigma = \sum_{\{q\}} \frac{1}{4E_1E_2|\mathbf{v}_1 - \mathbf{v}_2|} |\mathcal{A}|^2 \mathrm{d}\tilde{\pi}_2 \ , \tag{88}$$

where $\{q\}$ is the set of all possible quantum numbers in the final state, E_i and $\mathbf{v}_i$ are the energy and velocities of the incoming particles, and $\mathrm{d}\tilde{\pi}_2$ is the two-body phase-space factor:

$$\mathrm{d}\tilde{\pi}_2 = (2\pi)^4 \delta^{(4)}(p_i - p_f) \frac{1}{16\pi^2} \frac{|\tilde{\mathbf{k}}_1|}{E_{tot}} \mathrm{d}\Omega \ , \tag{89}$$

where p_i and p_f are the four-momenta of the initial and final states, and $\tilde{\mathbf{k}}_1$ is the momentum of one of the outgoing particles which satisfies the energy conservation condition. We have already performed the sum over spin states when calculating the modulus squared of the amplitude ($\Sigma|\mathcal{A}|^2$), so in the specific case at hand, we have

$$\mathrm{d}\sigma = \frac{1}{8E^2} \Sigma|\mathcal{A}|^2 \mathrm{d}\tilde{\pi}_2 \tag{90}$$

$$\mathrm{d}\tilde{\pi}_2 = (2\pi)^4 \delta^{(4)}((k_1 + k_2) - (p_1 + p_2)) \frac{1}{16\pi^2} \frac{|\tilde{\mathbf{p}}|}{2E} \mathrm{d}\Omega \tag{91}$$

$$= (2\pi)^4 \delta \left(E_{tot} - 2\sqrt{|\mathbf{p}|^2 + m_\mu^2} \right) \delta^{(3)}(0) \frac{1}{16\pi^2} \frac{\sqrt{\frac{E_{tot}^2}{4} - m_\mu^2}}{2E} \mathrm{d}\Omega \ , \tag{92}$$

where E is the energy of one neutrino in the centre-of-mass frame. Note since the neutrinos are moving at the speed of light: $|\mathbf{v}_1 - \mathbf{v}_2| = 2$. Collecting these results, we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\pi^2}{16E^3} \Sigma|\mathcal{A}|^2 \sqrt{\frac{E_{tot}^2}{4} - m_\mu^2} \delta \left(E_{tot} - 2\sqrt{|\mathbf{p}|^2 + m_\mu^2} \right) \delta^{(3)}(0) \ , \tag{93}$$

with the sum over spin states of the modulus squared of the amplitude $\Sigma|\mathcal{A}|^2$ is given by Equation 82.