Dylan J. Temples: Solution Set Two

Quantum Field Theory II
QFT and the Standard Model - M. Schwartz
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Contents

1 Bhabha Scattering.

Compute the differential cross section for Bhabha scattering, $e^+e^- \to e^+e^-$ in the limit that the electron mass can be neglected. You should express your answer in terms of the Mandelstam invariants s, t, u.

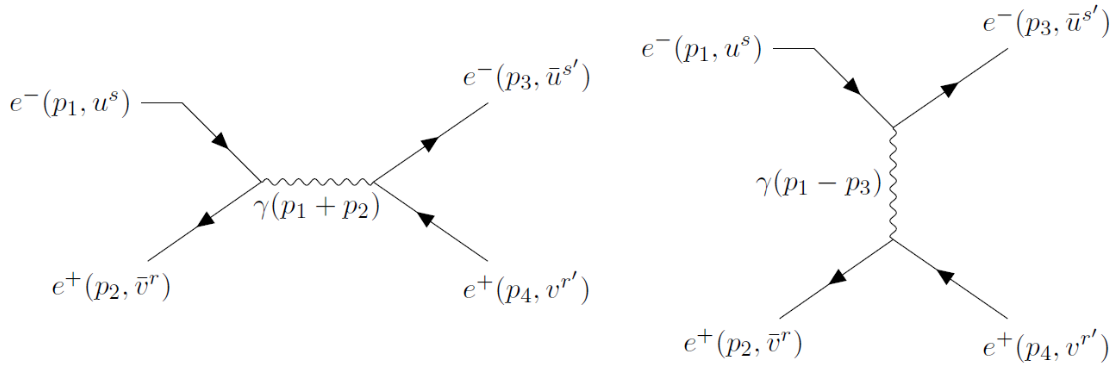

Figure 1: The tree-level Feynman diagrams describing the process $e^+e^- \to e^+e^-$. (Left) s-channel. (Right) t-channel.

The tree-level diagrams for the process $e^+e^- \to e^+e^-$ are shown in figure 1, the left diagram representing the s-channel and the right diagram representing the t-channel.

1.1 s-channel.

From the Feynman diagram, we can write the matrix element as

$$i\mathcal{M}_s = (-ie)[\bar{v}_2\gamma_\mu u_1]\frac{ig^{\mu\nu}}{q^2}(-ie)[\bar{u}_3\gamma_\nu v_4] = -\frac{ie^2}{q^2}[\bar{v}_2\gamma_\mu u_1][\bar{u}_3\gamma^\mu v_4] \ , \tag{1}$$

where the propagator momentum $q = p_1 + p_2 = \sqrt{s}$. We can take the modulus squared of this matrix element, noting that since $\mathcal{M}$ is a C-number taking the complex conjugate is equivalent to taking the hermitian conjugate:

$$|\mathcal{M}_s|^2 = \frac{e^4}{s^2}[\bar{v}_2\gamma_\mu u_1][\bar{u}_3\gamma^\mu v_4]\left([\bar{v}_2\gamma_\nu u_1][\bar{u}_3\gamma^\nu v_4]\right)^\dagger \tag{2}$$

$$= \frac{e^4}{s^2}[\bar{v}_2\gamma_\mu u_1][\bar{u}_3\gamma^\mu v_4][\bar{v}_2\gamma_\nu u_1]^\dagger[\bar{u}_3\gamma^\nu v_4]^\dagger \ , \tag{3}$$

since each quantity in square brackets is a C-number. Performing the conjugation yields

$$|\mathcal{M}_s|^2 = \frac{e^4}{s^2}[\bar{v}_2\gamma_\mu u_1][\bar{u}_3\gamma^\mu v_4][\bar{u}_1\gamma_\nu v_2][\bar{v}_4\gamma^\nu u_3] \tag{4}$$

$$= \frac{e^4}{s^2}[\bar{v}_2\gamma_\mu u_1][\bar{u}_1\gamma_\nu v_2][\bar{v}_4\gamma^\nu u_3][\bar{u}_3\gamma^\mu v_4] \ , \tag{5}$$

having noted:

$$[\bar{u}\gamma^\nu v]^\dagger = [u^\dagger\gamma^0\gamma^\nu v]^\dagger = v^\dagger\gamma^{\nu\dagger}\gamma^{0\dagger}u = v^\dagger(\gamma^0)^2\gamma^{\nu\dagger}\gamma^0 u = \bar{v}(\gamma^0\gamma^{\nu\dagger}\gamma^0)u = \bar{v}\gamma^\nu u \ , \tag{6}$$

where we've used the facts that γ^0 is hermitian, $\mathbb{1}_4 = (\gamma^0)^2$, and $\gamma^0\gamma^{\nu\dagger}\gamma^0 = \gamma^\nu$. Now, since a C-number is equal to its trace, we have

$$|\mathcal{M}_s|^2 = \frac{e^4}{s^2}\,\mathrm{Tr}[\bar{v}_2\gamma_\mu u_1\bar{u}_1\gamma_\nu v_2]\,\mathrm{Tr}[\bar{v}_4\gamma^\nu u_3\bar{u}_3\gamma^\mu v_4] \tag{7}$$

$$= \frac{e^4}{s^2}\,\mathrm{Tr}[v_2\bar{v}_2\gamma_\mu u_1\bar{u}_1\gamma_\nu]\,\mathrm{Tr}[v_4\bar{v}_4\gamma^\nu u_3\bar{u}_3\gamma^\mu]\ . \tag{8}$$

Now we must average over the initial spins and sum over the final spins:

$$|\bar{\mathcal{M}}_s|^2 = \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}| = \frac{1}{4}\sum_{r,s,r',s'}\frac{e^4}{s^2}\,\mathrm{Tr}[v_2^r\bar{v}_2^r\gamma_\mu u_1^s\bar{u}_1^s\gamma_\nu]\,\mathrm{Tr}[v_4^{r'}\bar{v}_4^{r'}\gamma^\nu u_3^{s'}\bar{u}_3^{s'}\gamma^\mu]\ , \tag{9}$$

let us move the sums into the traces:

$$|\bar{\mathcal{M}}_s|^2 = \frac{e^4}{4s^2}\,\mathrm{Tr}\left[\sum_r v_2^r\bar{v}_2^r\gamma_\mu\sum_s u_1^s\bar{u}_1^s\gamma_\nu\right]\mathrm{Tr}\left[\sum_{r'} v_4^{r'}\bar{v}_4^{r'}\gamma^\nu\sum_{s'} u_3^{s'}\bar{u}_3^{s'}\gamma^\mu\right]\ , \tag{10}$$

so using the completeness relations (in the $m_e \to 0$ limit) we have

$$|\bar{\mathcal{M}}_s|^2 = \frac{e^4}{4s^2}\,\mathrm{Tr}\left[\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu\right]\mathrm{Tr}\left[\not{p}_4\gamma^\nu\not{p}_3\gamma^\mu\right] = \frac{e^4}{4s^2}\,\mathrm{Tr}\left[\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu\right]\mathrm{Tr}\left[\not{p}_3\gamma^\mu\not{p}_4\gamma^\nu\right] \tag{11}$$

$$= \frac{e^4}{4s^2}p_2^\alpha p_1^\beta\,\mathrm{Tr}\left[\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\right]p_{3\alpha}p_{4\beta}\,\mathrm{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\right]\ . \tag{12}$$

Using the identities for traces of gamma matrices, we have

$$p_2^\alpha p_1^\beta\,\mathrm{Tr}\left[\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\right] = 4p_2^\alpha p_1^\beta\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu} + g_{\alpha\nu}g_{\mu\beta}\right) \tag{13}$$

$$= 4\left(p_{2\mu}p_{1\nu} - (p_2\cdot p_1)g_{\mu\nu} + p_{2\nu}p_{1\mu}\right) \tag{14}$$

$$p_{3\alpha}p_{4\beta}\,\mathrm{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\right] = 4p_{3\alpha}p_{4\beta}\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\mu\beta}\right) \tag{15}$$

$$= 4\left(p_3^\mu p_4^\nu - (p_3\cdot p_4)g^{\mu\nu} + p_3^\nu p_4^\mu\right)\ , \tag{16}$$

which we must take the product of, which is of the form (1+2+3)(A+B+C), and list the terms

$$1\mathrm{A}\ :\ \ (p_3\cdot p_2)(p_4\cdot p_1) \tag{17}$$

$$1\mathrm{B}\ :\ \ -(p_2\cdot p_1)(p_3\cdot p_4) \tag{18}$$

$$1\mathrm{C}\ :\ \ (p_4\cdot p_2)(p_3\cdot p_1) \tag{19}$$

$$2\mathrm{A}\ :\ \ -(p_3\cdot p_4)(p_2\cdot p_1) \tag{20}$$

$$2\mathrm{B}\ :\ \ 4(p_3\cdot p_4)(p_2\cdot p_1) \tag{21}$$

$$2\mathrm{C}\ :\ \ -(p_3\cdot p_4)(p_2\cdot p_1) \tag{22}$$

$$3\mathrm{A}\ :\ \ (p_3\cdot p_1)(p_4\cdot p_2) \tag{23}$$

$$3\mathrm{B}\ :\ \ -(p_2\cdot p_1)(p_3\cdot p_4) \tag{24}$$

$$3\mathrm{C}\ :\ \ (p_3\cdot p_2)(p_4\cdot p_1)\ , \tag{25}$$

which we must sum (including an overall factor of 4^2), note the factor of 4 in 2B comes from $g^{\mu\nu}g_{\mu\nu} = 4$. Let us note the relations:

$$1\mathrm{A} = 3\mathrm{C} \qquad 1\mathrm{C} = 3\mathrm{A} \qquad 2\mathrm{B} + 1\mathrm{B} + 2\mathrm{A} + 2\mathrm{C} + 3\mathrm{B} = 0\ , \tag{26}$$

so the sum is

$$|\bar{\mathcal{M}}_s|^2 = 4\frac{e^4}{s^2}\left[2(p_3 \cdot p_2)(p_4 \cdot p_1) + 2(p_4 \cdot p_2)(p_3 \cdot p_1)\right] \ . \tag{27}$$

Now, let's investigate the Mandelstam invariants:

$$s = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 \to 2p_1 \cdot p_2 = 2p_3 \cdot p_4 \tag{28}$$
$$t = (p_3 - p_1)^2 = p_3^2 + p_1^2 - 2p_1 \cdot p_3 \to -2p_1 \cdot p_3 = -2p_4 \cdot p_2 \tag{29}$$
$$u = (p_4 - p_1)^2 = p_4^2 + p_1^2 - 2p_1 \cdot p_4 \to -2p_1 \cdot p_4 = -2p_2 \cdot p_3 \ , \tag{30}$$

since the square of a four-momentum reduces to the particle's mass, in this limit zero. The spin-averaged matrix element for the s-channel is then

$$|\bar{\mathcal{M}}_s|^2 = 2\frac{e^4}{s^2}\left[(-2)(p_3 \cdot p_2)(-2)(p_4 \cdot p_1) + (-2)(p_4 \cdot p_2)(-2)(p_3 \cdot p_1)\right] \tag{31}$$
$$= 2\frac{e^4}{s^2}\left[u^2 + t^2\right] = 2e^4\frac{t^2+u^2}{s^2} \ . \tag{32}$$

1.2 t-channel.

Let us do the same process as above for the t-channel:

$$i\mathcal{M}_t = (-ie)[\bar{u}_3\gamma_\mu u_1]\frac{ig^{\mu\nu}}{q^2}(-ie)[\bar{v}_2\gamma_\nu v_4] = -\frac{ie^2}{q^2}[\bar{u}_3\gamma_\mu u_1][\bar{v}_2\gamma^\mu v_4] \ , \tag{33}$$

where the propagator momentum $q = p_3 - p_1 = \sqrt{t}$. The modulus squared is

$$|\mathcal{M}_t|^2 = \frac{e^4}{t^2}[\bar{u}_3\gamma_\mu u_1][\bar{v}_2\gamma^\mu v_4][\bar{u}_3\gamma_\nu u_1]^\dagger[\bar{v}_2\gamma^\nu v_4]^\dagger \tag{34}$$
$$= \frac{e^4}{t^2}[\bar{u}_3\gamma_\mu u_1][\bar{v}_2\gamma^\mu v_4][\bar{u}_1\gamma_\nu u_3][\bar{v}_4\gamma^\nu v_2] \tag{35}$$
$$= \frac{e^4}{t^2}\,\mathrm{Tr}\left[\bar{u}_3\gamma_\mu u_1\bar{u}_1\gamma_\nu u_3\right]\mathrm{Tr}\left[\bar{v}_2\gamma^\mu v_4\bar{v}_4\gamma^\nu v_2\right] \tag{36}$$
$$= \frac{e^4}{t^2}\,\mathrm{Tr}\left[u_1\bar{u}_1\gamma_\nu u_3\bar{u}_3\gamma_\mu\right]\mathrm{Tr}\left[v_4\bar{v}_4\gamma^\nu v_2\bar{v}_2\gamma^\mu\right] \tag{37}$$

Now we must average over the initial spins and sum over the final spins:

$$|\bar{\mathcal{M}}_t|^2 = \frac{e^4}{4t^2}\,\mathrm{Tr}\left[\sum_s u_1^s\bar{u}_1^s\gamma_\nu\sum_{s'}u_3^{s'}\bar{u}_3^{s'}\gamma_\mu\right]\mathrm{Tr}\left[\sum_{r'}v_4^{r'}\bar{v}_4^{r'}\gamma^\nu\sum_r v_2^r\bar{v}_2^r\gamma^\mu\right] \ , \tag{38}$$

again we use the completeness relations:

$$|\bar{\mathcal{M}}_t|^2 = \frac{e^4}{4t^2}p_1^\alpha p_3^\beta\,\mathrm{Tr}\left[\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\mu\right]p_{4\alpha}p_{2\beta}\,\mathrm{Tr}\left[\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\mu\right] \ . \tag{39}$$

Using the identities for traces of gamma matrices, we have

$$p_1^\alpha p_3^\beta\,\mathrm{Tr}\left[\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\mu\right] = 4p_1^\alpha p_3^\beta\left(g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\beta}g_{\nu\mu} + g_{\alpha\mu}g_{\nu\beta}\right) \tag{40}$$
$$= 4\left(p_{1\nu}p_{3\mu} - (p_1 \cdot p_3)g_{\nu\mu} + p_{1\mu}p_{3\nu}\right) \tag{41}$$
$$p_{4\alpha}p_{2\beta}\,\mathrm{Tr}\left[\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\mu\right] = 4p_{4\alpha}p_{2\beta}\left(g^{\alpha\nu}g^{\beta\mu} - g^{\alpha\beta}g^{\nu\mu} + g^{\alpha\mu}g^{\nu\beta}\right) \tag{42}$$
$$= 4\left(p_4^\nu p_2^\mu - (p_4 \cdot p_2)g^{\nu\mu} + p_4^\mu p_2^\nu\right) \ , \tag{43}$$

which we take the product of, again using notation (1+2+3)(A+B+C), and list the terms

$$\begin{aligned}
1\text{A} &: \quad (p_4 \cdot p_1)(p_2 \cdot p_3) && (44)\\
1\text{B} &: \quad -(p_1 \cdot p_3)(p_4 \cdot p_2) && (45)\\
1\text{C} &: \quad (p_2 \cdot p_1)(p_4 \cdot p_3) && (46)\\
2\text{A} &: \quad -(p_4 \cdot p_2)(p_1 \cdot p_3) && (47)\\
2\text{B} &: \quad 4(p_4 \cdot p_2)(p_1 \cdot p_3) && (48)\\
2\text{C} &: \quad -(p_4 \cdot p_2)(p_1 \cdot p_3) && (49)\\
3\text{A} &: \quad (p_4 \cdot p_3)(p_2 \cdot p_1) && (50)\\
3\text{B} &: \quad -(p_1 \cdot p_3)(p_4 \cdot p_2) && (51)\\
3\text{C} &: \quad (p_4 \cdot p_1)(p_2 \cdot p_3) \ , && (52)
\end{aligned}$$

which we must sum (including an overall factor of 4^2), note the factor of 4 in 2B comes from $g^{\mu\nu}g_{\mu\nu} = 4$. Let us note the relations:

$$1\text{A} = 3\text{C} \qquad 1\text{C} = 3\text{A} \qquad 2\text{B} + 1\text{B} + 2\text{A} + 2\text{C} + 3\text{B} = 0 \ , \tag{53}$$

so the sum is

$$\begin{aligned}
|\bar{\mathcal{M}}_t|^2 &= 4\frac{e^4}{t^2}\left[2(p_4 \cdot p_1)(p_2 \cdot p_3) + 2(p_2 \cdot p_1)(p_4 \cdot p_3)\right] && (54)\\
&= 2\frac{e^4}{t^2}\left[(-2)(p_4 \cdot p_1)(-2)(p_2 \cdot p_3) + (2)(p_2 \cdot p_1)(2)(p_4 \cdot p_3)\right] && (55)\\
&= 2\frac{e^4}{t^2}\left[u^2 + s^2\right] = 2e^4\frac{s^2+u^2}{t^2} \ . && (56)
\end{aligned}$$

Note that we could have found this answer under interchange of momenta: $p_2^{(s)} \to -p_3^{(t)}$ and $p_3^{(s)} \to -p_2^{(t)}$, so the Mandelstam invariants become

$$\begin{aligned}
s = (p_1 + p_2)^2 &\to (p_1 - p_3)^2 = t && (57)\\
t = (p_3 - p_1)^2 &\to (-p_2 - p_1)^2 = s && (58)\\
u = (p_4 - p_1)^2 &\to (p_4 - p_1)^2 = u \ , && (59)
\end{aligned}$$

and one can see that making the substitutions $s \to t$ and $t \to s$ in equation 32 yields equation 56.

1.3 Cross-terms.

in the calculation of the spin-averaged matrix elements earlier, we had forgotten about the cross-terms:

$$|\bar{\mathcal{M}}|^2 = |\bar{\mathcal{M}}_s|^2 + |\bar{\mathcal{M}}_t| + \bar{\mathcal{M}}_s\bar{\mathcal{M}}_t^* + \bar{\mathcal{M}}_s^*\bar{\mathcal{M}}_t \ , \tag{60}$$

which we can now calculate. The first is

$$\bar{\mathcal{M}}_s\bar{\mathcal{M}}_t^* = \frac{1}{4}\sum_{\text{spins}}\left[-\frac{ie^2}{s}[\bar{v}_2\gamma_\mu u_1][\bar{u}_3\gamma^\mu v_4]\right]\left[\frac{ie^2}{t}[\bar{u}_3\gamma_\nu u_1]^\dagger[\bar{v}_2\gamma^\nu v_4]^\dagger\right] \tag{61}$$

$$= \frac{e^4}{4st}\sum_{\text{spins}}[\bar{v}_2\gamma_\mu u_1][\bar{u}_1\gamma_\nu u_3][\bar{u}_3\gamma^\mu v_4][\bar{v}_4\gamma^\nu v_2] \tag{62}$$

$$= \frac{e^4}{4st}\sum_{r,s,r',s'}\text{Tr}[\bar{v}_2^r\gamma_\mu u_1^s\bar{u}_1^s\gamma_\nu u_3^{s'}\bar{u}_3^{s'}\gamma^\mu v_4^{r'}\bar{v}_4^{r'}\gamma^\nu v_2^r] \tag{63}$$

$$= \frac{e^4}{4st}\text{Tr}[\gamma_\mu\sum_s u_1^s\bar{u}_1^s\gamma_\nu\sum_{s'}u_3^{s'}\bar{u}_3^{s'}\gamma^\mu\sum_{r'}v_4^{r'}\bar{v}_4^{r'}\gamma^\nu\sum_r v_2^r\bar{v}_2^r] \tag{64}$$

$$= \frac{e^4}{4st}\text{Tr}[\gamma_\mu\not{p}_1\gamma_\nu\not{p}_3\gamma^\mu\not{p}_4\gamma^\nu\not{p}_2] = \frac{e^4}{4st}\text{Tr}[\not{p}_1\gamma_\nu\not{p}_3\gamma^\mu\not{p}_4\gamma^\nu\not{p}_2\gamma_\mu] \tag{65}$$

$$= \frac{e^4}{4st}g_{\mu\alpha}g_{\nu\beta}\,\text{Tr}[\not{p}_1\gamma^\beta\not{p}_3\gamma^\mu\not{p}_4\gamma^\nu\not{p}_2\gamma^\alpha] \ , \tag{66}$$

For the sake of calculations, let's cast this in a different form, using $p_1 \to p$, $p_2 \to k$, $p_3 \to p'$, and $p_4 \to k'$, we can then write the cross term as

$$\bar{\mathcal{M}}_s\bar{\mathcal{M}}_t^* = \frac{e^4}{4st}g_{\mu\alpha}g_{\nu\beta}\,\text{Tr}[\not{p}\gamma^\beta\not{p}'\gamma^\mu\not{k}'\gamma^\nu\not{k}\gamma^\alpha] = \frac{e^4}{4st}g_{\mu\alpha}g_{\nu\beta}\,\text{Tr}[\not{k}\gamma^\alpha\not{p}\gamma^\beta\not{p}'\gamma^\mu\not{k}'\gamma^\nu] \ , \tag{67}$$

now relabel our indeces (swapping β and μ):

$$\bar{\mathcal{M}}_s\bar{\mathcal{M}}_t^* = \frac{e^4}{4st}g_{\alpha\beta}g_{\mu\nu}\,\text{Tr}[\not{k}\gamma^\alpha\not{p}\gamma^\mu\not{p}'\gamma^\beta\not{k}'\gamma^\nu] \ . \tag{68}$$

Note the following relations which can be found in Appendix A.4 of Schwartz:

$$\gamma^\mu\gamma_\mu = 4 \tag{69}$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu \tag{70}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} \tag{71}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu \ , \tag{72}$$

so we can apply these:

$$\bar{\mathcal{M}}_s\bar{\mathcal{M}}_t^* = \frac{e^4}{4st}g_{\alpha\beta}\,\text{Tr}[\not{k}\gamma^\alpha\not{p}\gamma^\mu\not{p}'\gamma^\beta\not{k}'\gamma_\mu] = -\frac{e^4}{2st}g_{\alpha\beta}\,\text{Tr}[\not{k}\gamma^\alpha\not{p}\not{k}'\gamma^\beta\not{p}'] \tag{73}$$

$$= -\frac{e^4}{2st}\text{Tr}[\not{k}\gamma^\alpha\not{p}\not{k}'\gamma_\alpha\not{p}'] = -2\frac{e^4}{st}(p\cdot k')\,\text{Tr}[\not{k}\not{p}'] \tag{74}$$

$$= -8\frac{e^4}{st}(p\cdot k')(k\cdot p') \tag{75}$$

Let's recast this in terms of our original momenta definitions:

$$\bar{\mathcal{M}}_s\bar{\mathcal{M}}_t^* = -8\frac{e^4}{st}(p_1 \cdot p_4)(p_2 \cdot p_3) \ , \tag{76}$$

then in terms of the Mandelstam invariants:

$$\bar{\mathcal{M}}_s\bar{\mathcal{M}}_t^* = 2\frac{e^4}{st}u^2 \ . \tag{77}$$

Instead of calculating the other cross-term, I will claim

$$2\Re\{\bar{\mathcal{M}}_t\bar{\mathcal{M}}_u^*\} = \bar{\mathcal{M}}_t\bar{\mathcal{M}}_u^* + \bar{\mathcal{M}}_t^*\bar{\mathcal{M}}_u = 2\Re\{\bar{\mathcal{M}}_t^*\bar{\mathcal{M}}_u\} \ , \tag{78}$$

so

$$\bar{\mathcal{M}}_t\bar{\mathcal{M}}_u^* + \bar{\mathcal{M}}_t^*\bar{\mathcal{M}}_u = 4e^4\frac{u^2}{st} \ . \tag{79}$$

1.4 Differential cross-section.

The differential cross-section for this process is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}|\bar{\mathcal{M}}|^2 \ . \tag{80}$$

Let's calculate these for the s- and t-channel separately:

$$\frac{\mathrm{d}\sigma_s}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}2e^4\frac{t^2+u^2}{s^2} = \frac{1}{2s}\left(\frac{e^2}{4\pi}\right)^2\frac{t^2+u^2}{s^2} = \frac{\alpha^2}{2s}\frac{t^2+u^2}{s^2} \tag{81}$$

$$\frac{\mathrm{d}\sigma_t}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}2e^4\frac{s^2+u^2}{t^2} = \frac{\alpha^2}{2s}\frac{s^2+u^2}{t^2} \ . \tag{82}$$

The contribution from the cross-terms is:

$$\frac{\mathrm{d}\sigma_{st}}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}2\Re\{\bar{\mathcal{M}}_s\bar{\mathcal{M}}_t^*\} = \frac{1}{4(4\pi)^2 s}4e^4\frac{u^2}{st} = \left(\frac{e^2}{4\pi}\right)^2\frac{1}{s}\frac{u}{st} = \frac{\alpha^2}{s}\frac{u}{st} \tag{83}$$

Thus, the total-differential cross-section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{2s}\frac{t^2+u^2}{s^2} + \frac{\alpha^2}{2s}\frac{s^2+u^2}{t^2} + \frac{\alpha^2}{s}\frac{u^2}{st} \tag{84}$$

$$= \frac{\alpha^2}{2s}\left[\frac{t^2+u^2}{s^2} + \frac{s^2+u^2}{t^2} + \frac{2u^2}{st}\right] \tag{85}$$

$$= \frac{\alpha^2}{2s}\left[\left(\frac{t}{s}\right)^2 + \left(\frac{t}{s}\right)^2 + u^2\left(\frac{1}{s^2} + \frac{1}{t^2} + \frac{2}{st}\right)\right] \ , \tag{86}$$

which can be written

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{2s}\left[\left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2 + u^2\left(\frac{1}{s} + \frac{1}{t}\right)^2\right] \ . \tag{87}$$

If we integrate over the azimuth, we acquire a factor of 2π, yielding

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\pi\alpha^2}{s}\left[\left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2 + u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2\right] , \tag{88}$$

which is in agreement with Peskin (problem 5.2). Let's investigate the angular dependence of the differential cross section. In the center of mass frame, we have

$$p_1 = (E_{cm}/2, \mathbf{p}) \qquad p_3 = (E_{cm}/2, \mathbf{k}) \tag{89}$$

$$p_2 = (E_{cm}/2, -\mathbf{p}) \qquad p_4 = (E_{cm}/2, -\mathbf{k}) , \tag{90}$$

where $|\mathbf{p}| = |\mathbf{k}| = E_{cm}/2$. Using this, the Mandelstam invariants can be written

$$s = 2p_1 \cdot p_2 = E_{cm}^2 \tag{91}$$

$$t = -2p_1 \cdot p_3 = -2(p_1^0 p_3^0 - (\mathbf{p}\cdot\mathbf{k})) \tag{92}$$

$$= -2\frac{E_{cm}^2}{4}(1-\cos\theta) \tag{93}$$

$$u = -2p_1 \cdot p_4 = -2(p_1^0 p_4^0 - (\mathbf{p}\cdot -\mathbf{k})) \tag{94}$$

$$= -2\frac{E_{cm}^2}{4}(1+\cos\theta) . \tag{95}$$

Inserting these into the differential-cross section yields

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\pi\alpha^2}{E_{cm}^2}\Bigg[\left(\frac{-2\frac{E_{cm}^2}{4}(1-\cos\theta)}{E_{cm}^2}\right)^2 + \left(\frac{E_{cm}^2}{-2\frac{E_{cm}^2}{4}(1-\cos\theta)}\right)^2 + \left(-2\frac{E_{cm}^2}{4}(1+\cos\theta)\right)^2\left(\frac{1}{E_{cm}^2} + \frac{1}{-2\frac{E_{cm}^2}{4}(1-\cos\theta)}\right)^2\Bigg] ,$$

canceling some factors:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\pi\alpha^2}{E_{cm}^2}\left\{\frac{(1-\cos\theta)^2}{4} + \frac{4}{(1-\cos\theta)^2} + \frac{(1+\cos\theta)^2}{4}\left(1 - \frac{2}{1-\cos\theta}\right)^2\right\} .$$

Letting our old pal Mathematica handle the algebra, we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\pi\alpha^2}{E_{cm}^2}\left\{\frac{\left(\cos^2\theta+3\right)^2}{2(\cos\theta-1)^2}\right\} . \tag{96}$$

2 QED with Massive Photon.

Consider QED with an electron coupled to the usual massless photon. Add another massive spin-1 field to this theory.

(a) What is the most general Lagrangian consistent with QED gauge invariance (hint: you should have the usual QED Lagrangian, the free Lagrangian for the massive spin-1 field, and one additional term)?

The QED Lagrangian is

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \bar{\psi}[i\not{\mathcal{D}} - m]\psi \ , \tag{97}$$

where the $\mathcal{D}_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative, and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{98}$$

The free Lagrangian for the massive spin-1 field B^μ is

$$\mathcal{L}_B = -\frac{1}{4}H^{\mu\nu}H_{\mu\nu} + \frac{1}{2}m_B^2 B_\mu B^\mu \ , \tag{99}$$

where

$$H_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \ . \tag{100}$$

Note we can write the kinetic terms as

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A_\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A_\mu \ , \tag{101}$$

and after integrating by parts, we have

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}A_\mu \left\{\Box g^{\mu\nu} - \partial^\mu\partial^\nu\right\} A_\nu \equiv \frac{1}{2}A_\mu M^{\mu\nu} A_\nu \ , \tag{102}$$

and similarly for the B field. We now need to couple the massive spin-1 field B^μ to the QED Lagrangian, *i.e.*, add an interaction term. The most straightforward interaction we can write down is

$$\mathcal{L}_{\text{int}} = -g\bar{\psi}\gamma^\mu\psi B_\mu \ , \tag{103}$$

but if we enforce that the fermion is uncharged under $U(1)_B$, then $g = 0$. So we instead look towards a kinetic mixing term:

$$\mathcal{L}_{\text{int}} = -\frac{\epsilon}{4}F^{\mu\nu}H_{\mu\nu} - \frac{\epsilon}{4}H^{\mu\nu}F_{\mu\nu} = \frac{\epsilon}{2}A_\mu M^{\mu\nu}B_\nu + \frac{\epsilon}{2}B_\mu M^{\mu\nu}A_\nu \ . \tag{104}$$

Ignoring the interaction between the fermion and the standard model photon, and fixing our gauge allows us to write:

$$\mathcal{L}_0 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}H^{\mu\nu}H_{\mu\nu} + \frac{1}{2}m^2 B_\mu B^\mu = \frac{1}{2}\left(A_\mu B_\nu\right)\begin{pmatrix} M^{\mu\nu} & 0 \\ 0 & M^{\mu\nu} + \epsilon m_B^2 g^{\mu\nu}\end{pmatrix}\begin{pmatrix} A_\nu \\ B_\mu \end{pmatrix} \ .$$

If we include the kinetic mixing term, we have

$$\mathcal{L} = \frac{1}{2}\left(A_\mu B_\mu\right)\begin{pmatrix} M^{\mu\nu} & \epsilon M^{\mu\nu} \\ \epsilon M^{\mu\nu} & M^{\mu\nu} + m_B^2 g^{\mu\nu}\end{pmatrix}\begin{pmatrix} A_\nu \\ B_\nu \end{pmatrix} \ , \tag{105}$$

which is no longer diagonalized, so we must do that. If we concern ourselves with only the kinetic terms of the bosonic fields:

$$\mathcal{L}_T = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}H^{\mu\nu}H_{\mu\nu} - \frac{\epsilon}{2}F^{\mu\nu}H_{\mu\nu} \tag{106}$$

$$= \frac{1}{2}A_\mu M^{\mu\nu}A_\nu + \frac{1}{2}B_\mu M^{\mu\nu}B_\nu + \epsilon A_\mu M^{\mu\nu}B_\nu \ , \tag{107}$$

we can express this as

$$\mathcal{L}_T = \frac{1}{2}\begin{pmatrix} A_\mu & B_\mu \end{pmatrix}\begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}\begin{pmatrix} M^{\mu\nu}A_\nu \\ M^{\mu\nu}B_\nu \end{pmatrix} \ , \tag{108}$$

so we simply need to diagonalize the matrix

$$N = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix} \ , \tag{109}$$

which has eigenvalues $\lambda = 1 \pm \epsilon$, and a transition matrix:

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \ , \tag{110}$$

where the first column corresponds to the eigenvector for $\lambda = 1 - \epsilon$, and the second for $\lambda = 1 + \epsilon$. We can scale each column by a factor $1/\sqrt{\lambda}$, where λ is the eigenvector for the column:'

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} -1/\sqrt{1-\epsilon} & 1/\sqrt{1+\epsilon} \\ 1/\sqrt{1-\epsilon} & 1/\sqrt{1+\epsilon} \end{pmatrix} \ , \tag{111}$$

such that the mixing matrix in this rotated basis is now diagonal

$$T^{\mathrm{T}}\begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix} T = \mathbb{1}_4 \ . \tag{112}$$

The fields can be written as this transformation on another set of fields:

$$\begin{pmatrix} A^\mu \\ B^\mu \end{pmatrix} = T\begin{pmatrix} \tilde{A}^\mu \\ \tilde{B}^\mu \end{pmatrix} \ , \tag{113}$$

so in terms of the original fields, we have

$$\begin{pmatrix} \tilde{A}^\mu \\ \tilde{B}^\mu \end{pmatrix} = T^{-1}\begin{pmatrix} A^\mu \\ B^\mu \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sqrt{1-\epsilon} & \sqrt{1+\epsilon} \\ \sqrt{1-\epsilon} & \sqrt{1+\epsilon} \end{pmatrix}\begin{pmatrix} A^\mu \\ B^\mu \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{1-\epsilon}(-A^\mu + B^\mu) \\ \sqrt{1+\epsilon}(A^\mu + B^\mu) \end{pmatrix} \ . \tag{114}$$

Now, in the original kinetic Lagrangian (equation 108), we make the substitution $\{A^\mu, B^\mu\} \to \{\tilde{A}^\mu, \tilde{B}^\mu\}$ to verify it is indeed diagonal:

$$\mathcal{L}_T = \frac{1}{2}\left\{\tilde{A}_\mu M^{\mu\nu}\tilde{A}_\nu + \tilde{B}_\mu M^{\mu\nu}\tilde{B}_\nu + \epsilon\tilde{A}_\mu M^{\mu\nu}\tilde{B}_\nu + \epsilon\tilde{B}_\mu M^{\mu\nu}\tilde{A}_\nu\right\} \ , \tag{115}$$

let's investigate these terms:

$$\tilde{A}_\mu M^{\mu\nu} \tilde{A}_\nu = \frac{(1-\epsilon)}{2}(-A_\mu + B_\mu)(-M^{\mu\nu}A_\nu + M^{\mu\nu}B_\nu) \tag{116}$$

$$= \frac{(1-\epsilon)}{2}\left\{A_\mu M^{\mu\nu}A_\nu + B_\mu M^{\mu\nu}B_\nu - A_\mu M^{\mu\nu}B_\nu - B_\mu M^{\mu\nu}A_\nu\right\} \tag{117}$$

$$\tilde{B}_\mu M^{\mu\nu} \tilde{B}_\nu = \frac{(1+\epsilon)}{2}(A_\mu + B_\mu)(M^{\mu\nu}A_\nu + M^{\mu\nu}B_\nu) \tag{118}$$

$$= \frac{(1-\epsilon)}{2}\left\{A_\mu M^{\mu\nu}A_\nu + B_\mu M^{\mu\nu}B_\nu + A_\mu M^{\mu\nu}B_\nu + B_\mu M^{\mu\nu}A_\nu\right\} \ , \tag{119}$$

note that if we sum these, we obtain

$$\tilde{A}_\mu M^{\mu\nu} \tilde{A}_\nu + \tilde{B}_\mu M^{\mu\nu} \tilde{B}_\nu = A_\mu M^{\mu\nu}A_\nu + B_\mu M^{\mu\nu}B_\nu + \epsilon(A_\mu M^{\mu\nu}B_\nu + B_\mu M^{\mu\nu}A_\nu) \ . \tag{120}$$

The remaining two terms are

$$\epsilon\tilde{A}_\mu M^{\mu\nu} \tilde{B}_\nu = \frac{\epsilon}{2}\sqrt{1-\epsilon}\sqrt{1+\epsilon}(-A_\mu + B_\mu)(M^{\mu\nu}A_\mu + M^{\mu\nu}B_\mu) \tag{121}$$

$$= \frac{\epsilon}{2}\sqrt{1-\epsilon}\sqrt{1+\epsilon}\left\{-A_\mu M^{\mu\nu}A_\nu + B_\mu M^{\mu\nu}B_\nu - A_\mu M^{\mu\nu}B_\nu + B_\mu M^{\mu\nu}A_\nu\right\} \tag{122}$$

$$\epsilon\tilde{B}_\mu M^{\mu\nu} \tilde{A}_\nu = \frac{\epsilon}{2}\sqrt{1-\epsilon}\sqrt{1+\epsilon}(A_\mu + B_\mu)(-M^{\mu\nu}A_\mu + M^{\mu\nu}B_\mu) \tag{123}$$

$$= \frac{\epsilon}{2}\sqrt{1-\epsilon}\sqrt{1+\epsilon}\left\{-A_\mu M^{\mu\nu}A_\nu + B_\mu M^{\mu\nu}B_\nu - A_\mu M^{\mu\nu}B_\nu + B_\mu M^{\mu\nu}A_\nu\right\} \ , \tag{124}$$

which sum to

$$\epsilon\tilde{A}_\mu M^{\mu\nu} \tilde{B}_\nu + \epsilon\tilde{B}_\mu M^{\mu\nu} \tilde{A}_\nu = \epsilon\sqrt{1-\epsilon^2}(-A_\mu M^{\mu\nu}A_\nu + B_\mu M^{\mu\nu}B_\nu) \ . \tag{125}$$

So the kinetic-diagonalized Lagrangian is

$$\mathcal{L} = -\frac{\epsilon}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\epsilon}{4}H^{\mu\nu}H_{\mu\nu} + \frac{1}{2}m_B^2\tilde{B}_\mu\tilde{B}^\mu + \bar{\psi}[i\gamma^\mu\left(\partial_\mu + ie\tilde{A}_\mu\right) - m]\psi \ , \tag{126}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{127}$$

$$H_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \ , \tag{128}$$

note these are the un-transformed fields. We now need to investigate the mass terms under mixing:

$$\tilde{B}_\mu\tilde{B}^\mu = \frac{1+\epsilon}{2}(A^\mu + B^\mu)(A_\mu + B_\mu) = \frac{1+\epsilon}{2}(A^\mu A_\mu + B^\mu B_\mu + A^\mu B_\mu + B^\mu A_\mu) \ , \tag{129}$$

or in matrix form

$$\tilde{B}_\mu\tilde{B}^\mu = \frac{1+\epsilon}{2}\begin{pmatrix} A^\mu & B^\mu \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} A_\mu \\ B_\mu \end{pmatrix} \ . \tag{130}$$

The eigenvalues of the matrix are 2 and 0 , so the transformation matrix is

$$T = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \ , \tag{131}$$

so to diagonalize the mass term, we simply scale the B field, this does not have an effect on the previous diagonalization. Luckily, we recover a massless photon. The final form for the Lagrangian is

$$\mathcal{L} = -\frac{\epsilon}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\epsilon}{4}H^{\mu\nu}H_{\mu\nu} + \frac{1}{2}m_B^2 B_\mu B^\mu + \bar{\psi}[i\gamma^\mu\left(\partial_\mu - ieA_\mu + ieB_\mu\right) - m]\psi \ , \tag{132}$$

so the B field only couples to the fermion field, through a term generated by the kinetic mixing:

$$-e\bar{\psi}\gamma^\mu B_\mu\psi \ , \tag{133}$$

which couples the massive photon to the fermion fields. Note this form is identical to standard QED.

(b) What decay modes does the massive spin-1 particle A' have? Calculate its lifetime in terms of the Lagrangian parameters.

Since the massive spin-1 field only couples to the fermion field, the only possible decay mode is $B \to e^+e^-$. Let us give the massive spin-1 particle momentum k and the electron and positron momenta p_1 and p_2 respectively. In the rest frame of the B particle, we have

$$k = (m_B, 0) \qquad p_1 = (E, \mathbf{p}) \qquad p_2 = (E, -\mathbf{p}) \ , \tag{134}$$

and we can define the axes such that $\mathbf{p} = |\mathbf{p}|\hat{\mathbf{z}}$. The Feynman rules for interactions with the massive spin-1 should be identical to that of "standard" QED, but with a mass term in the propagator. The tree-level diagram for this process, is just a single vertex, so the matrix element is

$$\mathcal{M} = g\bar{u}_1\gamma^\mu v_2\epsilon^*_\mu \ , \tag{135}$$

where ϵ_μ is the polarization vector for the final state B particle. We can square the matrix element:

$$\begin{aligned} |\mathcal{M}|^2 &= g^2[\bar{u}_1\gamma^\mu v_2\epsilon^*_\mu][\bar{u}_1\gamma^\nu v_2\epsilon^*_\nu]^\dagger = g^2[\bar{u}_1\gamma^\mu v_2\epsilon^*_\mu][\epsilon_\nu\bar{v}_2\gamma^\nu u_1] & (136)\\ &= g^2\,\mathrm{Tr}\{\bar{u}_1\gamma^\mu v_2\bar{v}_2\gamma^\nu u_1\}\epsilon_\nu\epsilon^*_\mu = g^2\,\mathrm{Tr}\{\gamma^\mu v_2\bar{v}_2\gamma^\nu u_1\bar{u}_1\}\epsilon_\nu\epsilon^*_\mu \ . & (137)\end{aligned}$$

Now we need to average over the initial polarization states of the B boson, and sum over the final spin states. Since the B boson is massive, the third polarization (longitudinal) is allowed, thus

$$\begin{aligned} |\bar{\mathcal{M}}|^2 &= \frac{1}{3}\sum_{\lambda,r,s} g^2\,\mathrm{Tr}\{\gamma^\mu v_2^r\bar{v}_2^r\gamma^\nu u_1^s\bar{u}_1^s\}\epsilon^\lambda_\nu\epsilon^{*\lambda}_\mu = \frac{1}{3}g^2\,\mathrm{Tr}\{\gamma^\mu \not{p}_2\gamma^\nu\not{p}_1\}\sum_\lambda \epsilon^\lambda_\nu\epsilon^{*\lambda}_\mu & (138)\\ &= \frac{1}{3}g^2 p_{1\alpha}p_{2\beta}\,\mathrm{Tr}\{\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\alpha\}\sum_\lambda \epsilon^\lambda_\nu\epsilon^{*\lambda}_\mu = \frac{g^2}{3}p_{1\alpha}p_{2\beta}\,\mathrm{Tr}\{\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\}\sum_\lambda \epsilon^\lambda_\nu\epsilon^{*\lambda}_\mu \ , & (139)\end{aligned}$$

where we have taken the massless electron limt. The trace evaluates to

$$p_{1\alpha}p_{2\beta}\,\mathrm{Tr}\{\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\} = 4(p_1^\mu p_2^\nu - (p_1\cdot p_2)g^{\mu\nu} + p_1^\nu p_2^\mu) \ , \tag{140}$$

using the results from the previous problem. We now need to evaluate the polarization sum, but for any set of orthonormal vectors, one obtains[1]

$$\sum_\lambda \epsilon^\lambda_\nu\epsilon^{*\lambda}_\mu = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m_B^2} \ , \tag{141}$$

so the polarization-averaged matrix element is

$$\begin{aligned} |\bar{\mathcal{M}}|^2 &= \frac{g^2}{3}4(p_1^\mu p_2^\nu - (p_1\cdot p_2)g^{\mu\nu} + p_1^\nu p_2^\mu)\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{m_B^2}\right)\\ &= \frac{4g^2}{3}\left[-(p_1\cdot p_2) + 4(p_1\cdot p_2) - (p_1\cdot p_2) + \frac{(p_1\cdot k)(p_2\cdot k)}{m_B^2} - \frac{k\cdot k}{m_B^2}(p_1\cdot p_2) + \frac{(p_1\cdot k)(p_2\cdot k)}{m_B^2}\right]\\ &= \frac{4g^2}{3}\left[2(p_1\cdot p_2) + 2\frac{(p_1\cdot k)(p_2\cdot k)}{m_B^2} - (p_1\cdot p_2)\right] = \frac{4g^2}{3}\left[(p_1\cdot p_2) + 2\frac{(p_1\cdot k)(p_2\cdot k)}{m_B^2}\right] \ , \end{aligned}$$

since we are in the rest frame of the B boson, $k\cdot k = m_B^2$. Looking back at the definitions for our momenta, we see that $k\cdot p_1 = k\cdot p_2 = Em_B$, so

$$|\bar{\mathcal{M}}|^2 = \frac{4g^2}{3}\left[(p_1\cdot p_2) + 2\frac{(m_BE)(m_BE)}{m_B^2}\right] = \frac{4g^2}{3}\left[(p_1\cdot p_2) + 2E^2\right] \ , \tag{142}$$

but since the electrons share the energy of the B boson equally, we have that $E = m_B/2$:

$$|\bar{\mathcal{M}}|^2 = \frac{4g^2}{3}\left[(p_1\cdot p_2) + \frac{1}{2}m_B^2\right] \ . \tag{143}$$

[1] http://www.staff.science.uu.nl/ wit00103/ftip/Ch04.pdf equation 4.21.

Now we simply need the overlap of the electrons' four-momenta:

$$p_1 \cdot p_2 = E^2 - (\mathbf{p} \cdot -\mathbf{p}) = E^2 + |\mathbf{p}|^2 = E^2 + E^2 - m_e^2 = 2E^2 - m_e^2 = \frac{m_B^2}{2} - m_e^2 \ , \tag{144}$$

using the Lorentz condition. If we investigate the limit $m_e \ll m_B$, then we neglect the final term, and the matrix element becomes

$$|\bar{\mathcal{M}}|^2 = \frac{4g^2}{3}\left[\frac{1}{2}m_B^2 + \frac{1}{2}m_B^2\right] = \frac{4g^2}{3}m_B^2 \ , \tag{145}$$

which seems to be incorrect because it is a dimensionful quantity, which matrix elements squared should not be. We will see that in the end, we get the correct dimensions. Now we need the decay rate, which by Peskin equation A.57 is

$$\mathrm{d}\Gamma = \frac{1}{2m_B}\left(\frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1}\right)\left(\frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_2}\right)|\bar{\mathcal{M}}|^2 (2\pi)^4 \delta^{(4)}(k - p_1 - p_2) \ , \tag{146}$$

which simplifies to (Peskin A.58)

$$\Gamma = \int \frac{1}{2m_B}|\bar{\mathcal{M}}|^2 \frac{\mathrm{d}\Omega}{4\pi}\frac{1}{8\pi}\left(\frac{2|\mathbf{p}|}{E_{cm}}\right) = \int \frac{|\mathbf{p}|}{m_B E_{cm}}|\bar{\mathcal{M}}|^2 \frac{\mathrm{d}\Omega}{2(4\pi)^2} \ . \tag{147}$$

The center of mass energy is simply m_B, and by the Lorentz condition, we have for a final-state electron:

$$E^2 - |\mathbf{p}|^2 = m_e^2 \quad \Rightarrow \quad \frac{m_B^2}{4} - m_e^2 = |\mathbf{p}|^2 \ , \tag{148}$$

so

$$\Gamma = \int \frac{\sqrt{\frac{m_B^2}{4} - m_e^2}}{m_B^2}|\bar{\mathcal{M}}|^2 \frac{\mathrm{d}\Omega}{2(4\pi)^2} = \int \frac{|\bar{\mathcal{M}}|^2}{2m_B}\frac{\mathrm{d}\Omega}{2(4\pi)^2} \tag{149}$$

after neglecting the electron mass. Finally, we have

$$\Gamma = \int \frac{1}{m_B}|\bar{\mathcal{M}}|^2 \left(\frac{\mathrm{d}\Omega}{4^3\pi^2}\right) = \int \frac{\mathrm{d}\Omega}{4m_B}\frac{|\bar{\mathcal{M}}|^2}{(4\pi)^2} = \int \frac{\mathrm{d}\Omega}{4m_B}\frac{1}{(4\pi)^2}\frac{4g^2}{3}m_B^2 \tag{150}$$

$$= m_B\left(\frac{g}{4\pi}\right)^2 \int \frac{\mathrm{d}\Omega}{3} = \frac{g^2}{4\pi}\frac{m_B}{3} \ , \tag{151}$$

which has dimensions of mass, so we ended up on the right track. Given this decay rate, the lifetime is

$$\tau = 1/\Gamma = \frac{12\pi}{g^2 m_B} \ , \tag{152}$$

which has mass dimension -1, which is equivalent to time. If we insert the explicit form of g from part a, we see

$$\tau = \frac{3(4\pi)}{e^2 m_B} = \frac{3}{\alpha m_B} \ . \tag{153}$$

3 QED with Fermion Coupling to Gluon.

Consider a theory with massless fermions f. We will have f couple also to a gluon g, a massless spin-1 field that mediates the strong force. Calculate the differential lifetime for the process of an off-shell (massive) photon decay $\gamma^* \to f\bar{f}g$ in terms of the variables

$$x_i = \frac{2k_i \cdot q}{q^2},$$

where k_i for $i = 1, 2$, are the momenta of the final-state fermions f and $\bar{f}$, and q is the four-momentum of the off-shell photon. To handle the effects of the color quantum numbers of the gluon and f, calculate as if g was a photon, and multiply your result at the end by $C_F N_c$, where $C_F = 4/3$ and $N_c = 3$. Attempt to integrate your result over x_1 and x_2 to derive the total cross section. Explain your result.

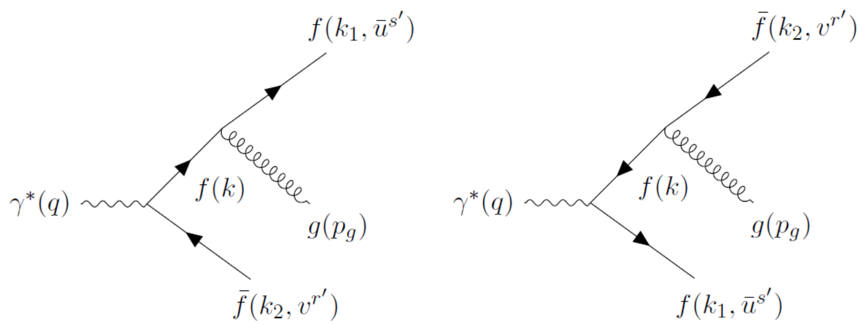

Figure 2: The tree-level Feynman diagram describing the process $\gamma^* \to f\bar{f}g$.

We will use the Feynman rules for QED to evaluate the lifetime of the massive photon.

3.1 Diagram 1.

Consider the first diagram, with the fermion emitting the gluon. The matrix element is

$$\mathcal{M} = \bar{u}_1(eQ_f\gamma^\mu)\frac{\not{k}}{k^2}(e\gamma^\nu)v_2(\epsilon_g^*)_\mu(\epsilon_\gamma)_\nu \tag{154}$$

$$= e^2 Q_f \frac{k_\alpha}{k^2}\left[\bar{u}_1\gamma^\mu\gamma^\alpha\gamma^\nu v_2\right](\epsilon_g^*)_\mu(\epsilon_\gamma)_\nu \tag{155}$$

$$\mathcal{M}^* = e^2 Q_f \frac{k_\alpha}{k^2}\left[\bar{v}_2\gamma^\nu\gamma^\alpha\gamma^\mu u_1\right](\epsilon_g)_\mu(\epsilon_\gamma^*)_\nu \ , \tag{156}$$

where the fermion propagator momentum (for the first diagram) is

$$k = q - k_2 = k_1 + p_g \ . \tag{157}$$

Let's investigate the Ward identity, which is satisfied for un-physical (read: off-shell) particles:

$$\mathcal{M}^{\mu\nu} = e^2 Q_f \frac{k_\alpha}{k^2}\left[\bar{u}_1\gamma^\mu\gamma^\alpha\gamma^\nu v_2\right] \quad \Rightarrow \quad \mathcal{M}^\nu = e^2 Q_f \frac{k_\alpha}{k^2}\left[\bar{u}_1\gamma^\mu\gamma^\alpha\gamma^\nu v_2\right](\epsilon_g^*)_\mu \ , \tag{158}$$

and the Ward identity states $\mathcal{M}^\nu q_\nu = 0$, so we have

$$0 = e^2 Q_f \frac{1}{k^2}\left[\bar{u}_1\gamma^\mu\not{k}\not{q}v_2\right](\epsilon_g^*)_\mu \ , \tag{159}$$

let's divide out the constants and multiply on the left by u_1 and on the right by $\bar{v}_2$:

$$0 = \left[u_1 \bar{u}_1 \gamma^\mu \not{k} \not{q} v_2 \bar{v}_2\right] (\epsilon_g^*)_\mu \ , \tag{160}$$

now we want to sum over the final state fermion spins, and apply the completeness relations:

$$0 = \left[\not{k}_1 \gamma^\mu \not{k} \not{q} \not{k}_2\right] (\epsilon_g^*)_\mu \ , \tag{161}$$

so the Ward identity implies the above quantity must be zero. Taking the modulus squared of the matrix element yields

$$|\mathcal{M}|^2 = \frac{(e^2 Q_f)^2}{k^4} \left[\bar{u}_1 \gamma^\mu \not{k} \gamma^\nu v_2\right] \left[\bar{v}_2 \gamma^\beta \not{k} \gamma^\alpha u_1\right] (\epsilon_g)_\alpha (\epsilon_g^*)_\mu (\epsilon_\gamma)_\nu (\epsilon_\gamma^*)_\beta \tag{162}$$

$$= \frac{(e^2 Q_f)^2}{k^4} \mathrm{Tr} \left[u_1 \bar{u}_1 \gamma^\mu \not{k} \gamma^\nu v_2 \bar{v}_2 \gamma^\beta \not{k} \gamma^\alpha\right] (\epsilon_g)_\alpha (\epsilon_g^*)_\mu (\epsilon_\gamma)_\nu (\epsilon_\gamma^*)_\beta \ , \tag{163}$$

now we sum over the fermion spins:

$$|\bar{\mathcal{M}}|^2 = \frac{(e^2 Q_f)^2}{k^4} \mathrm{Tr} \left[\not{k}_1 \gamma^\mu \not{k} \gamma^\nu \not{k}_2 \gamma^\beta \not{k} \gamma^\alpha\right] (\epsilon_g)_\alpha (\epsilon_g^*)_\mu (\epsilon_\gamma)_\nu (\epsilon_\gamma^*)_\beta \ . \tag{164}$$

Now we can average over the polarization states of the incoming off-shell photon (acquiring a factor of 1/3 from the average):

$$|\bar{\mathcal{M}}|^2 \to \frac{(e^2 Q_f)^2}{3k^4} \mathrm{Tr} \left[\not{k}_1 \gamma^\mu \not{k} \gamma^\nu \not{k}_2 \gamma^\beta \not{k} \gamma^\alpha\right] (\epsilon_g)_\alpha (\epsilon_g^*)_\mu \sum_{\gamma^*} (\epsilon_\gamma)_\nu (\epsilon_\gamma^*)_\beta \ , \tag{165}$$

using the completeness relations. We now need to evaluate the polarization sums, but for any set of orthonormal vectors, one obtains[2], for a massive particle

$$\sum_\lambda \epsilon_\nu^\lambda \epsilon_\mu^{*\lambda} = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2} \ , \tag{166}$$

while for a masless particle with momentum k:

$$\sum_\lambda \epsilon_\nu^\lambda \epsilon_\mu^{*\lambda} = -g_{\mu\nu} \ . \tag{167}$$

Using this, the spin-averaged matrix element squared is

$$|\bar{\mathcal{M}}|^2 = \frac{(e^2 Q_f)^2}{3k^4} \mathrm{Tr} \left[\not{k}_1 \gamma^\mu \not{k} \gamma^\nu \not{k}_2 \gamma^\beta \not{k} \gamma^\alpha\right] (\epsilon_g)_\alpha (\epsilon_g^*)_\mu \left(-g_{\beta\nu} + \frac{q_\beta q_\nu}{m^2}\right) \ . \tag{168}$$

Now let's investigate the second term, which is proportional to

$$\mathrm{Tr} \left[\not{k}_1 \gamma^\mu \not{k} \not{q} \not{k}_2 \not{q} \not{k} \gamma^\alpha\right] (\epsilon_g^*)_\mu = \mathrm{Tr} \left[\not{k}_1 \gamma^\mu \not{k} \not{q} \not{k}_2 (\epsilon_g^*)_\mu \not{q} \not{k} \gamma^\alpha\right] = \mathrm{Tr} \left[0 (\not{q} \not{k} \gamma^\alpha)\right] = 0 \ , \tag{169}$$

so the second sum in the polarization does not contribute due to the Ward identity. Now,

$$|\bar{\mathcal{M}}|^2 = \frac{(e^2 Q_f)^2}{3k^4} \mathrm{Tr} \left[\not{k}_1 \gamma^\mu \not{k} \gamma^\nu \not{k}_2 \gamma^\beta \not{k} \gamma^\alpha\right] (\epsilon_g)_\alpha (\epsilon_g^*)_\mu \left(-g_{\beta\nu}\right) \ . \tag{170}$$

[2] http://www.staff.science.uu.nl/ wit00103/ftip/Ch04.pdf equation 4.21.

but we still have to sum over the gluon polarizations:

$$|\bar{\mathcal{M}}|^2 = -\frac{(e^2Q_f)^2}{3k^4}\,\mathrm{Tr}\left[\not{k}_1\gamma^\mu\not{k}\gamma^\nu\not{k}_2\gamma^\beta\not{k}\gamma^\alpha\right] g_{\beta\nu}\sum_q(\epsilon_g)_\alpha(\epsilon_g^*)_\mu$$

$$= -\frac{(e^2Q_f)^2}{3k^4}\,\mathrm{Tr}\left[\not{k}_1\gamma^\mu\not{k}\gamma^\nu\not{k}_2\gamma^\beta\not{k}\gamma^\alpha\right] g_{\beta\nu}(-g_{\alpha\mu})\ . \quad (171)$$

let's swap the index labels α and ν:

$$|\bar{\mathcal{M}}|^2 = \frac{(e^2Q_f)^2}{3k^4}\,\mathrm{Tr}\left[\not{k}_1\gamma^\mu\not{k}\gamma^\alpha\not{k}_2\gamma^\beta\not{k}\gamma^\nu\right] g_{\alpha\beta}g_{\mu\nu}\ . \quad (172)$$

Now we can contract with the $\mu\nu$ metric:

$$|\bar{\mathcal{M}}|^2 = \frac{(e^2Q_f)^2}{3k^4}\,\mathrm{Tr}\left[\not{k}_1\gamma^\mu\not{k}\gamma^\alpha\not{k}_2\gamma^\beta\not{k}\gamma_\mu\right](g_{\alpha\beta})$$

$$= -2\frac{(e^2Q_f)^2}{3k^4}\,\mathrm{Tr}\left[\not{k}_1\not{k}\gamma^\beta\not{k}_2\gamma^\alpha\not{k}\right] g_{\alpha\beta}\ , \quad (173)$$

and contract with the remaining metric:

$$|\bar{\mathcal{M}}|^2 = -2\frac{(e^2Q_f)^2}{3k^4}\,\mathrm{Tr}\left[\not{k}_1\not{k}\gamma^\beta\not{k}_2\gamma_\beta\not{k}\right] = 4\frac{(e^2Q_f)^2}{3k^4}\,\mathrm{Tr}\left[\not{k}_1\not{k}\not{k}_2\not{k}\right]\ . \quad (174)$$

The trace of four slashed vectors is

$$\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}] = a^\mu b^\nu c^\rho d^\sigma\,\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4\left((a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\right)\ , \quad (175)$$

so now let's evaluate the trace:

$$\mathcal{T}_1 = \mathrm{Tr}\left[\not{k}_1\not{k}\not{k}_2\not{k}\right] = 4[(k_1\cdot k)(k_2\cdot k) - (k_1\cdot k_2)(k^2) + (k_1\cdot k)(k_2\cdot k)]$$

$$= 4[2(k_1\cdot k)(k_2\cdot k) - (k_1\cdot k_2)(k^2)]\ . \quad (176)$$

Using the definition of the fermion propagator momentum $k = q - k_2$, we have the quantities

$$k_1\cdot k = k_1\cdot q - k_1\cdot k_2 = q^2\left(\frac{x_1}{2} - \frac{k_1\cdot k_2}{q^2}\right) \quad (177)$$

$$k_2\cdot k = k_2\cdot q = \frac{q^2}{2}x_2 \quad (178)$$

$$q\cdot k = q^2 - q\cdot k_2 = q^2\left(1 - \frac{x_2}{2}\right) \quad (179)$$

$$k^2 = (q-k_2)^2 = q^2 + k_2^2 - 2q\cdot k_2 = q^2 - 2q\cdot k_2 = q^2(1-x_2)\ . \quad (180)$$

Let's look at the terms from the trace:

$$8(k_1\cdot k)(k_2\cdot k) = 4q^4\left(\frac{x_1}{2} - \frac{k_1\cdot k_2}{q^2}\right)x_2 \quad (181)$$

$$-4(k_1\cdot k_2)(k^2) = -4q^4(1-x_2)\frac{(k_1\cdot k_2)}{q^2}\ , \quad (182)$$

which sum to

$$\mathcal{T}_1 = 4q^4\left\{\frac{x_1x_2}{2} - \frac{(k_1\cdot k_2)}{q^2}\right\} \quad \Rightarrow \quad |\bar{\mathcal{M}}|^2 = 4\frac{(e^2Q_f)^2}{3k^4}4q^4\left\{\frac{x_1x_2}{2} - \frac{(k_1\cdot k_2)}{q^2}\right\}\ . \quad (183)$$

Now, inserting the fermion propagator's momentum:

$$|\bar{\mathcal{M}}|^2 = \frac{16}{3}\frac{(e^2Q_f)^2}{(1-x_2)^2}\left\{\frac{x_1x_2}{2} - \frac{(k_1\cdot k_2)}{q^2}\right\}\ . \quad (184)$$

3.2 Diagram 2.

The second diagram (with the antifermion emitting the gluon) has a matrix element

$$\mathcal{M} = e^2 Q_f \frac{k_\alpha}{k^2} \left[\bar{u}_1 \gamma^\mu \gamma^\alpha \gamma^\nu v_2\right] (\epsilon_g^*)_\nu (\epsilon_\gamma)_\mu \tag{185}$$

$$\mathcal{M}^* = e^2 Q_f \frac{k_\alpha}{k^2} \left[\bar{v}_2 \gamma^\nu \gamma^\alpha \gamma^\mu u_1\right] (\epsilon_g)_\nu (\epsilon_\gamma^*)_\mu \ , \tag{186}$$

which is identical to the first (except the expression for k) under swapping the Lorentz indeces on the polarization vectors. Therefore

$$|\bar{\mathcal{M}}|^2 = \frac{(e^2 Q_f)^2}{3k^4} \operatorname{Tr}\left[\not{k}_1 \gamma^\mu \not{k} \gamma^\nu \not{k}_2 \gamma^\beta \not{k} \gamma^\alpha\right] (g_{\beta\nu}) \left(g_{\alpha\mu} - \frac{q_\alpha q_\mu}{m^2}\right) \ , \tag{187}$$

but again we will argue that, by the Ward identity, the second term from the photon polarization sum does not contribute. Therefore, after contracting the $\beta\nu$ metric,

$$\begin{aligned} |\bar{\mathcal{M}}|^2 &= -2\frac{(e^2 Q_f)^2}{3k^4} \operatorname{Tr}\left[\not{k}_1 \gamma^\mu \not{k} \not{k}_2 \not{k} \gamma^\alpha\right] (g_{\alpha\mu}) = -2\frac{(e^2 Q_f)^2}{3k^4} \operatorname{Tr}\left[\not{k}_1 \gamma^\mu \not{k} \not{k}_2 \not{k} \gamma_\mu\right] \\ &= 4\frac{(e^2 Q_f)^2}{3k^4} \operatorname{Tr}\left[\not{k}_1 \not{k} \not{k}_2 \not{k}\right] \ , \end{aligned} \tag{188}$$

and we see the trace is:

$$\mathcal{T}_1 = \operatorname{Tr}\left[\not{k}_1 \not{k} \not{k}_2 \not{k}\right] = 4[2(k_1 \cdot k)(k_2 \cdot k) - (k_1 \cdot k_2)(k^2)] \ . \tag{189}$$

Using the definition of the fermion propagator momentum $k = q - k_1$, we have the quantities

$$k_1 \cdot k = k_1 \cdot q = \frac{q^2}{2} x_1 \tag{190}$$

$$k_2 \cdot k = k_2 \cdot q - k_2 \cdot k_1 = \frac{q^2}{2} x_2 - (k_2 \cdot k_1) = q^2 \left(\frac{x_2}{2} - \frac{k_1 \cdot k_2}{q^2}\right) \tag{191}$$

$$q \cdot k = q^2 - q \cdot k_1 = q^2 \left(1 - \frac{x_1}{2}\right) \tag{192}$$

$$k^2 = (q - k_1)^2 = q^2 + k_1^2 - 2q \cdot k_2 = q^2 - 2q \cdot k_1 = q^2(1 - x_1) \ . \tag{193}$$

So the terms from the trace are

$$8(k_1 \cdot k)(k_2 \cdot k) = 8\frac{q^4}{2} x_1 \left(\frac{x_2}{2} - \frac{k_1 \cdot k_2}{q^2}\right) \tag{194}$$

$$-4(k_1 \cdot k_2)(k^2) = -4(k_1 \cdot k_2) q^2 (1 - x_1) = -4q^4 (1 - x_1) \frac{k_1 \cdot k_2}{q^2} \ , \tag{195}$$

which sum to

$$\mathcal{T}_1 = 4q^4 \left\{\frac{x_1 x_2}{2} - \frac{k_1 \cdot k_2}{q^2}\right\} \quad \Rightarrow \quad |\bar{\mathcal{M}}|^2 = 4\frac{(e^2 Q_f)^2}{3k^4} 4q^4 \left\{\frac{x_1 x_2}{2} - \frac{k_1 \cdot k_2}{q^2}\right\} \ . \tag{196}$$

Finally, inserting the value for the fermion propagator's momentum, we have

$$|\bar{\mathcal{M}}|^2 = \frac{16}{3} \frac{(e^2 Q_f)^2}{(1 - x_1)^2} \left\{\frac{x_1 x_2}{2} - \frac{k_1 \cdot k_2}{q^2}\right\} \ . \tag{197}$$

3.3 Cross Term.

We are now interested in the quantity $2\Re\{\mathcal{M}_1\mathcal{M}_2^*\}$, so we need

$$\mathcal{M}_1 = e^2 Q_f \frac{k_\alpha}{k^2} \left[\bar{u}_1 \gamma^\mu \gamma^\alpha \gamma^\nu v_2\right] (\epsilon_g^*)_\mu (\epsilon_\gamma)_\nu \tag{198}$$

$$\mathcal{M}_2^* = e^2 Q_f \frac{p_\beta}{p^2} \left[\bar{v}_2 \gamma^\sigma \gamma^\beta \gamma^\rho u_1\right] (\epsilon_g)_\sigma (\epsilon_\gamma^*)_\rho \tag{199}$$

with

$$k \equiv q - k_2 \tag{200}$$

$$p \equiv q - k_1 \ , \tag{201}$$

so let's note the quantities

$$\begin{aligned} k^2 &= q^2(1-x_2) & p^2 &= q^2(1-x_1) \\ k\cdot k_1 &= q^2\left(\frac{x_1}{2} - \frac{k_1\cdot k_2}{q^2}\right) & p\cdot k_1 &= \frac{q^2}{2}x_1 \\ k\cdot k_2 = k_2\cdot q &= \frac{q^2}{2}x_2 & p\cdot k_2 &= q^2\left(\frac{x_2}{2} - \frac{k_1\cdot k_2}{q^2}\right) \\ k\cdot q &= q^2\left(1-\frac{x_2}{2}\right) & p\cdot q &= q^2\left(1-\frac{x_1}{2}\right) \ , \end{aligned}$$

and

$$\begin{aligned} k\cdot p &= (q-k_2)\cdot(q-k_1) = q^2 + k_1\cdot k_2 - q\cdot k_1 - q\cdot k_2 \\ &= q^2\left\{1 + \frac{k_1\cdot k_2}{q^2} - \frac{x_1}{2} - \frac{x_2}{2}\right\} \ . \end{aligned} \tag{202}$$

The product of the matrix elements is

$$\mathscr{M}^2 \equiv \mathcal{M}_1\mathcal{M}_2^* = \frac{(e^2Q_f)^2}{k^2p^2}\left[\bar{u}_1\gamma^\mu \not{k} \gamma^\nu v_2\right]\left[\bar{v}_2\gamma^\sigma \not{p} \gamma^\rho u_1\right](\epsilon_g)_\sigma(\epsilon_g^*)_\mu(\epsilon_\gamma)_\nu(\epsilon_\gamma^*)_\rho \ , \tag{203}$$

note the terms in square brackets are C-numbers, so we can take their trace. We then sum over final state fermion spins and gluon's polarization, and average over the off-shell photon's polarizations. We have shown before that the Ward identity is satisfied, so the second term in the photon's polarization completeness relation does not contribute, and thus

$$\bar{\mathscr{M}}^2 = \frac{(e^2Q_f)^2}{3k^2p^2}\,\mathrm{Tr}\left[\not{k}_1\gamma^\mu\not{k}\gamma^\nu\not{k}_2\gamma^\sigma\not{p}\gamma^\rho\right] g_{\sigma\mu}g_{\nu\rho} \ . \tag{204}$$

Contracting with the first metric removes the γ^μ and γ^σ matrices, reverses the order of the matrices in between them, and contributes a factor of -2, so

$$\bar{\mathscr{M}}^2 = \frac{(e^2Q_f)^2}{3k^2p^2}\,\mathrm{Tr}\left[\not{k}_1\not{k}_2\gamma^\nu\not{k}\not{p}\gamma^\rho\right] g_{\nu\rho} = \frac{(e^2Q_f)^2}{3k^2p^2}\,\mathrm{Tr}\left[\gamma^\rho\not{k}_1\not{k}_2\gamma^\nu\not{k}\not{p}\right] g_{\nu\rho} \ . \tag{205}$$

Contracting with the second yields $\gamma^\rho\not{k}_1\not{k}_2\gamma^\nu g_{\nu\rho} = 4g^{\alpha\beta}k_{1\alpha}k_{2\beta}$, so

$$\bar{\mathscr{M}}^2 = 4\frac{(e^2Q_f)^2}{3k^2p^2}4(k_1\cdot k_2)\,\mathrm{Tr}\left[\not{k}\not{p}\right] = 4^3\frac{(e^2Q_f)^2}{3k^2p^2}(k_1\cdot k_2)(k\cdot p) \ . \tag{206}$$

Inserting the expressions for k^2 and p^2, we have

$$\bar{\mathscr{M}}^2 = \frac{4^3}{3}\frac{(e^2Q_f)^2}{(1-x_1)(1-x_2)}\frac{(k_1\cdot k_2)}{q^2}\left\{1 + \frac{k_1\cdot k_2}{q^2} - \frac{x_1}{2} - \frac{x_2}{2}\right\} \ . \tag{207}$$

3.4 Off-Shell Photon Lifetime.

Collecting the three terms from the previous sections, we have

$$|\mathcal{M}|^2 = \frac{2^4}{3}\frac{(e^2Q_f)^2}{(1-x_2)^2}\left\{\frac{x_1x_2}{2} - \frac{(k_1\cdot k_2)}{q^2}\right\} + \frac{2^4}{3}\frac{(e^2Q_f)^2}{(1-x_1)^2}\left\{\frac{x_1x_2}{2} - \frac{k_1\cdot k_2}{q^2}\right\} + \frac{2^7}{3}\frac{(e^2Q_f)^2}{(1-x_1)(1-x_2)}\frac{(k_1\cdot k_2)}{q^2}\left\{1 + \frac{k_1\cdot k_2}{q^2} - \frac{x_1}{2} - \frac{x_2}{2}\right\} . \tag{208}$$

Let us note something about the kinematics:, we have (for the first diagram) $q = k_1 + k$, but $k = k_2 + p_g$, so we have $q = k_1 + k_2 + p_g$ (which is just overall energy-momentum conservation), if we take the scalar product of both sides with $2q$, we have

$$2q^2 = 2q\cdot k_2 + 2q\cdot k_1 + 2q\cdot p_g \quad \Rightarrow \quad 2 = x_1 + x_2 + \frac{2p_g\cdot q}{q^2} , \tag{209}$$

which leads to the definition of x_3, and the fact that $x_1+x_2+x_3 = 2$. Now consider the final factor in curly braces:

$$\left\{1 + \frac{k_1\cdot k_2}{q^2} - \frac{x_1}{2} - \frac{x_2}{2}\right\} = \left\{\frac{2-x_1-x_2}{2} + \frac{k_1\cdot k_2}{q^2}\right\} = \left\{\frac{x_3}{2} + \frac{k_1\cdot k_2}{q^2}\right\} , \tag{210}$$

and the denominator of the same term is

$$(1-x_1)(1-x_2) = 1 - x_1 - x_2 + x_1x_2 = (x_3-1) + x_1x_2. \tag{211}$$

Consider the quantity

$$(k_1+k_2)^2 = 2k_1\cdot k_2 , \tag{212}$$

since the fermions are massless, but from momentum conservation, we have $k_1 + k_2 = q - p_g$, so

$$2k_1\cdot k_2 = (q-p_g)^2 = q^2 - 2q\cdot p_g \quad \Rightarrow \quad \frac{k_1\cdot k_2}{q^2} = \frac{1-x_3}{2} . \tag{213}$$

We can then write the total matrix element squared as

$$|\mathcal{M}|^2 = \frac{2^4e^4Q_f^2}{3}\left\{\frac{1}{(1-x_2)^2}\left\{\frac{x_1x_2}{2} - \frac{1-x_3}{2}\right\} + \frac{1}{(1-x_1)^2}\left\{\frac{x_1x_2}{2} - \frac{1-x_3}{2}\right\} + \frac{(1-x_3)}{(1-x_1)(1-x_2)}\right\} ,$$

letting Mathematica handle the algebra results in

$$|\mathcal{M}|^2 = \frac{2^4e^4Q_f^2}{3}\left\{\frac{x_1^2+x_2^2}{2(1-x_1)(1-x_2)}\right\} . \tag{214}$$

If we now include the prefactor $C_FN_c = 4$ for the color consideration, we have

$$|\mathcal{M}|^2 = \frac{2^5e^4Q_f^2}{3}\left\{\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}\right\} . \tag{215}$$

We now calculate the differential width using Peskin equation A.57:

$$d\Gamma = \frac{1}{2m_{\gamma^*}}\left(\frac{d^3k_1}{(2\pi)^3 2E_1}\right)\left(\frac{d^3k_2}{(2\pi)^3 2E_2}\right)\left(\frac{d^3p_g}{(2\pi)^3 2E_g}\right)|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(q-k_1-k_2-p_g) , \tag{216}$$

which we must integrate over dx_i, with each $x_i \in (0,1)$. We will quote the result from evaluating the Lorentz-invariant phase space from Schwartz equation 20.44. The differential width of the massive photon is (up to factors of 2 and π):

$$\Gamma \propto e^4Q_f^2q^2\int_0^1 dx_1\int_{1-x_1}^1 dx_2\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)} , \tag{217}$$

which we note diverges as the $x_i \to 1$. This is an infrared divergence.