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Quantum Field Theory II
QFT and the Standard Model - M. Schwartz
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1 Renormalization of Scalar QED.

Consider QED coupled to a scalar field. Calculate the running of the electric charge in this theory. Show that the optical theorem is satisfied.

Determining the Counter-term Lagrangian.

In terms of the bare fields, denoted by tildes, the Lagrangian for scalar QED is

$$\mathcal{L}_0 = -\partial_\mu \tilde{\phi}^* \partial^\mu \tilde{\phi} - \tilde{m}^2 \tilde{\phi}^* \tilde{\phi} - \frac{1}{4}\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} + i\tilde{e}\left\{\tilde{\phi}\tilde{A}^\mu \partial_\mu \tilde{\phi}^* - \tilde{\phi}^* \tilde{A}^\mu \partial_\mu \tilde{\phi}\right\} + \tilde{e}^2 \tilde{\phi}^* \tilde{\phi} \tilde{A}_\mu \tilde{A}^\mu \ , \tag{1}$$

where $\tilde{F}_\mu \nu = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu$, $\tilde{m}$ is the renormalized scalar mass, and $\tilde{e}$ is the renormalized coupling. We then define the renormalized fields from the bare fields, starting with the photon: $\tilde{A}^\mu = \sqrt{Z_3} A^\mu$, so

$$\tilde{F}_{\mu\nu} = \sqrt{Z_3}\left[\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu\right] \ , \tag{2}$$

thus

$$-\frac{1}{4}\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} = -\frac{1}{4}Z_3 F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - (Z_3 - 1)\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \ . \tag{3}$$

We continue with the definition of the renormalized scalars fields

$$\tilde{\phi}^{(*)} = \sqrt{Z_2}\phi^{(*)} \ , \tag{4}$$

so

$$-\partial_\mu \tilde{\phi}^* \partial^\mu \tilde{\phi} = -Z_2 \partial_\mu \phi^* \partial^\mu \phi = -\partial_\mu \phi^* \partial^\mu \phi - (Z_2 - 1)\partial_\mu \phi^* \partial^\mu \phi \ . \tag{5}$$

For now, we ignore the bare constants $\tilde{e}$ and $\tilde{m}$, so that:

$$\tilde{\phi}\tilde{A}^\mu \partial_\mu \tilde{\phi}^* = Z_2\sqrt{Z_3}\phi A^\mu \partial_\mu \phi^* \tag{6}$$

$$\tilde{\phi}^* \tilde{\phi} \tilde{A}_\mu \tilde{A}^\mu = Z_2 Z_3 \phi^* \phi A_\mu A^\mu \ . \tag{7}$$

Now we define the bare constants as

$$\tilde{m} = \sqrt{\frac{1+\delta}{Z_2}}m \tag{8}$$

$$\tilde{e} = \frac{Z_1}{Z_2\sqrt{Z_3}}e \ . \tag{9}$$

Then the second term in the bare Lagrangian is

$$-\tilde{m}^2 \tilde{\phi}^* \tilde{\phi} = -\frac{1+\delta}{Z_2}m^2(Z_2\phi^*\phi) = -m^2\phi^*\phi - \delta m^2 \phi^* \phi \ , \tag{10}$$

and the fourth is

$$\begin{aligned} i\tilde{e}\left\{\tilde{\phi}\tilde{A}^\mu \partial_\mu \tilde{\phi}^* - \tilde{\phi}^* \tilde{A}^\mu \partial_\mu \tilde{\phi}\right\} &= i\frac{Z_1}{Z_2\sqrt{Z_3}}e(Z_2\sqrt{Z_3})\left\{\phi A^\mu \partial_\mu \phi^* - \phi^* A^\mu \partial_\mu \phi\right\} \\ &= ie\left\{\phi A^\mu \partial_\mu \phi^* - \phi^* A^\mu \partial_\mu \phi\right\} + ie(Z_1 - 1)\left\{\phi A^\mu \partial_\mu \phi^* - \phi^* A^\mu \partial_\mu \phi\right\} \end{aligned} \tag{11}$$

and the final term is

$$\tilde{e}^2\tilde{\phi}^*\tilde{\phi}\tilde{A}_\mu\tilde{A}^\mu = \frac{Z_1^2}{Z_2^2 Z_3}e^2(Z_2Z_3)\phi^*\phi A_\mu A^\mu \tag{12}$$

$$= \frac{Z_1^2}{Z_2}e^2\phi^*\phi A_\mu A^\mu = e^2\phi^*\phi A_\mu A^\mu + \left(\frac{Z_1^2}{Z_2} - 1\right)e^2\phi^*\phi A_\mu A^\mu \ . \tag{13}$$

Collecting these results, we have the QED Lagrangian:

$$\mathcal{L}_{\text{QED}} = -\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + ie\left\{\phi A^\mu\partial_\mu\phi^* - \phi^* A^\mu\partial_\mu\phi\right\} + e^2\phi^*\phi A_\mu A^\mu \ . \tag{14}$$

and the counter-term Lagrangian:

$$\mathcal{L}_{\text{ct}} = -(Z_2 - 1)\partial_\mu\phi^*\partial^\mu\phi - \delta m^2\phi^*\phi - (Z_3 - 1)\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + ie(Z_1 - 1)\left\{\phi A^\mu\partial_\mu\phi^* - \phi^* A^\mu\partial_\mu\phi\right\} + \left(\frac{Z_1^2}{Z_2} - 1\right)e^2\phi^*\phi A_\mu A^\mu \ , \tag{15}$$

and so

$$\mathcal{L}_0 = \mathcal{L}_{\text{QED}} + \mathcal{L}_{\text{ct}} \ . \tag{16}$$

Correction to the photon propagator.

We are interested in the running of the coupling constant in the scalar QED theory, so as seen for the spinor QED theory, we only must concern ourselves with the correction to the self-energy of the photon, *i.e.*, the counter-term containing Z_3. Quickly, we will review the Feynman rules for scalar QED. Let us work in the Feynman gauge, so the propagators are

$$\text{photon:} \quad \frac{-ig_{\mu\nu}}{p^2} \tag{17}$$

$$\text{scalar:} \quad \frac{i}{p^2 - m^2} \ , \tag{18}$$

and the vertex factors are shown in figure 1. This leads to two (excluding the counter-term) diagrams corresponding to leading-order corrections to the photon propagator, shown in figure 2.

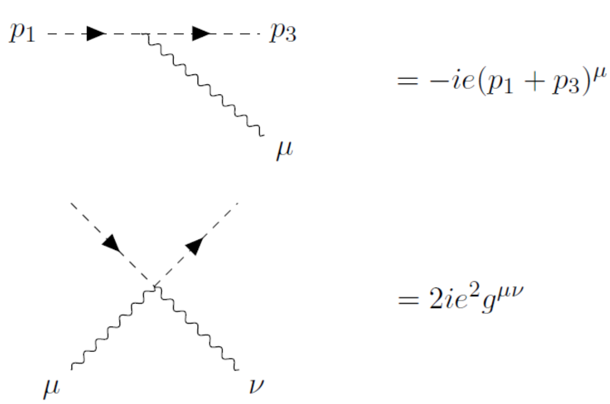

Figure 1: The allowed vertices in scalar QED.

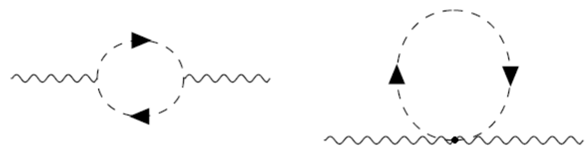

Figure 2: Corrections to the photon propagator in scalar QED.

From the first diagram, giving one scalar momentum k and the other $k+p$, we can write the correction to the photon propagator. Starting at the left vertex, we have

$$-i\hat{\Sigma}_1 = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left[-ie(2k^\mu + p^\mu)\right] \frac{i}{k^2 - m^2} \left[-ie(2k^\nu + p^\nu)\right] \frac{i}{(p+k)^2 - m^2} \ . \tag{19}$$

In the second diagram there is only one scalar propagator and one vertex, so

$$-i\hat{\Sigma}_2 = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left[2ie^2 g^{\mu\nu}\right] \frac{i}{k^2 - m^2} \ . \tag{20}$$

If we introduce a common denominator, we can write the total correction to the photon propagator as the sum of the two above expressions:

$$-i\hat{\Sigma}^{\mu\nu} = e^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\frac{1}{k^2 - m^2}\right) \left(\frac{1}{(p+k)^2 - m^2}\right) \left\{(2k^\mu + p^\mu)(2k^\nu + p^\nu) - 2g^{\mu\nu}\left((p+k)^2 - m^2\right)\right\} \ .$$

Let us note the so-called "Feynman trick":

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{\left\{xA + (1-x)B\right\}^2} \ , \tag{21}$$

so using $A = (k+p)^2 - m^2$, and $B = k^2 - m^2$, we can express the factors with denominators as

$$\left(\frac{1}{(p+k)^2 - m^2}\right) \left(\frac{1}{k^2 - m^2}\right) = \int_0^1 \frac{\mathrm{d}x}{\left\{x[(k+p)^2 - m^2] + (1-x)[k^2 - m^2]\right\}^2} \ , \tag{22}$$

expanding the denominator gives

$$\begin{aligned} x[(k+p)^2 - m^2] + (1-x)[k^2 - m^2] &= xp^2 + xk^2 + 2xp \cdot k - xm^2 + k^2 - m^2 - xk^2 + xm^2 \\ &= k^2 + xp^2 + 2xp \cdot k - m^2 \ . \end{aligned} \tag{23}$$

Now, we want to complete the square, so we shift the momenta that is integrated over: $k^\mu \to k^\mu - xp^\mu$, yielding:

$$k^2 + xp^2 + 2xp \cdot k - m^2 \to (k - xp)^2 + xp^2 + 2xp \cdot (k - xp) - m^2 \ , \tag{24}$$

which we can expand to find

$$k^2 + x^2p^2 - 2xk \cdot p + xp^2 + 2xk \cdot p - 2x^2p^2 - m^2 = k^2 - \Delta \ , \tag{25}$$

where

$$\Delta = m^2 - x(1-x)p^2 \ . \tag{26}$$

Using this, we can write the correction as (being sure to shift the ks in the numerator):

$$-i\hat{\Sigma}^{\mu\nu} = e^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(2k^\mu + [1-2x]p^\mu)(2k^\nu + [1-2x]p^\nu) - 2g^{\mu\nu}\left((k+[1-x]p)^2 - m^2\right)}{(k^2-\Delta)^2} \ .$$

We can expand the numerator as

$$N^{\mu\nu} \equiv 4k^\mu k^\nu + 2(1-2x)\left(k^\mu p^\nu + p^\mu k^\nu\right) + (1-2x)^2 p^\mu p^\nu - 2g^{\mu\nu}\left(k^2 + (1-x)^2p^2 + 2(1-x)k\cdot p - m^2\right) \ ,$$

but since we are integrating k over all space, any terms linear in k vanish under a symmetric integration, so the numerator is

$$N^{\mu\nu} = 4k^\mu k^\nu + (1-2x)^2 p^\mu p^\nu - 2g^{\mu\nu}\left(k^2 + (1-x)^2 p^2 - m^2\right) \ . \tag{27}$$

As we saw in class, due to symmetry arguments:

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2-\Delta)^2} = \frac{1}{d} g^{\mu\nu} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^2}{(k^2-\Delta)^2} \ , \tag{28}$$

so we can replace $k^\mu k^\nu$ with $(1/d)k^2 g^{\mu\nu}$ in the numerator, yielding:

$$N^{\mu\nu} = 2g^{\mu\nu}\left(\frac{2}{d} - 1\right)k^2 + (1-2x)^2 p^\mu p^\nu - 2g^{\mu\nu}\left((1-x)^2p^2 - m^2\right) \ . \tag{29}$$

Let us split the numerator, keeping the k dependence explicit:

$$N^{\mu\nu} = N_1^{\mu\nu} k^2 + N_2^{\mu\nu}(x) \ , \tag{30}$$

where we've made the definitions

$$N_1^{\mu\nu} = 2g^{\mu\nu}\left(\frac{2}{d} - 1\right) \tag{31}$$

$$N_2^{\mu\nu}(x) = (1-2x)^2 p^\mu p^\nu - 2g^{\mu\nu}\left((1-x)^2 p^2 - m^2\right) \ . \tag{32}$$

So the correction to the photon propagator can be written:

$$-i\hat{\Sigma}^{\mu\nu} = e^2 \left\{ \int_0^1 \mathrm{d}x N_1^{\mu\nu} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^2}{(k^2-\Delta)^2} + \int_0^1 \mathrm{d}x N_2^{\mu\nu}(x) \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2-\Delta)^2} \right\} \ . \tag{33}$$

Schwartz (Appendix B.3.2) gives expressions for integrals of the form:

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2-\Delta)^2} = \frac{i}{(4\pi)^{d/2}} \Gamma\left(\frac{4-d}{2}\right) \Delta^{-(2-\frac{d}{2})} \tag{34}$$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^2}{(k^2-\Delta)^2} = -\frac{d}{2} \frac{i}{(4\pi)^{d/2}} \Gamma\left(\frac{2-d}{2}\right) \Delta^{-(1-\frac{d}{2})} \ , \tag{35}$$

so we can use these to evaluate the momentum integrals in the correction:

$$-i\hat{\Sigma}^{\mu\nu} = -i\frac{e^2}{(4\pi)^{d/2}} \left\{ \frac{d}{2} \int_0^1 \mathrm{d}x N_1^{\mu\nu} \Gamma\left(\frac{2-d}{2}\right) \Delta^{-(1-\frac{d}{2})} - \int_0^1 \mathrm{d}x N_2^{\mu\nu}(x) \Gamma\left(\frac{4-d}{2}\right) \Delta^{-(2-\frac{d}{2})} \right\} \ . \tag{36}$$

We can use a property of the Gamma function $(\alpha-1)\Gamma(\alpha-1)=\Gamma(\alpha)$ to write:

$$\Gamma\left(\frac{2-d}{2}\right)=\Gamma\left(\frac{4-d}{2}-1\right)=\frac{2}{2-d}\Gamma\left(\frac{4-d}{2}\right)\ , \tag{37}$$

so we can factor out the gamma functions:

$$-i\hat{\Sigma}^{\mu\nu}=-i\frac{e^2}{(4\pi)^{d/2}}\Gamma\left(\frac{4-d}{2}\right)\left\{\frac{d}{2}\int_0^1 \mathrm{d}x N_1^{\mu\nu}\frac{2}{2-d}\Delta^{-(1-\frac{d}{2})}-\int_0^1 \mathrm{d}x N_2^{\mu\nu}(x)\Delta^{-(2-\frac{d}{2})}\right\}\ . \tag{38}$$

Furthermore, we can write:

$$\Delta^{-(1-\frac{d}{2})}=\Delta^{-(-1+2-\frac{d}{2})}=\Delta\Delta^{-(2-\frac{d}{2})}\ , \tag{39}$$

so

$$\hat{\Sigma}^{\mu\nu}=\frac{e^2}{(4\pi)^{d/2}}\Gamma\left(\frac{4-d}{2}\right)\int_0^1\frac{\mathrm{d}x}{\Delta^{\frac{4-d}{2}}}\left\{\frac{d}{2}N_1^{\mu\nu}\frac{2}{2-d}\Delta-N_2^{\mu\nu}(x)\right\}\ , \tag{40}$$

note we've divided both sides by $-i$. Let's insert the expression for $N_1^{\mu\nu}$:

$$\hat{\Sigma}^{\mu\nu}=\frac{e^2}{(4\pi)^{d/2}}\Gamma\left(\frac{4-d}{2}\right)\int_0^1\frac{\mathrm{d}x}{\Delta^{\frac{4-d}{2}}}\left\{\frac{d}{2}\left[2g^{\mu\nu}\left(\frac{2}{d}-1\right)\right]\frac{2}{4-d}\Delta-N_2^{\mu\nu}(x)\right\}\ , \tag{41}$$

note all the dimensional factors reduce to unity:

$$\hat{\Sigma}^{\mu\nu}=\frac{e^2}{(4\pi)^{d/2}}\Gamma\left(\frac{4-d}{2}\right)\int_0^1\frac{\mathrm{d}x}{\Delta^{\frac{4-d}{2}}}\left\{2g^{\mu\nu}\Delta-N_2^{\mu\nu}(x)\right\}\ . \tag{42}$$

Let's investigate the integrand, inserting $N_2^{\mu\nu}(x)$:

$$2g^{\mu\nu}\Delta-N_2^{\mu\nu}(x)=-(1-2x)^2p^\mu p^\nu+2g^{\mu\nu}\left(m^2-x(1-x)p^2+(1-x)^2p^2-m^2\right) \tag{43}$$

$$=-(1-2x)^2p^\mu p^\nu+2g^{\mu\nu}(1-x)(1-2x)p^2\ , \tag{44}$$

so we express the correction as

$$\hat{\Sigma}^{\mu\nu}=\frac{e^2}{(4\pi)^{d/2}}\Gamma\left(\frac{4-d}{2}\right)\int_0^1\frac{\mathrm{d}x}{\Delta^{\frac{4-d}{2}}}\left\{-(1-2x)^2p^\mu p^\nu+2g^{\mu\nu}(1-x)(1-2x)p^2\right\}\ . \tag{45}$$

We can work in a dimension $d=4-2\epsilon$, but we need to add a mass scale $\tilde{\mu}$ to the coupling constant such as to keep it dimensionless:

$$\hat{\Sigma}^{\mu\nu}=-\frac{e^2\tilde{\mu}^{2\epsilon}}{(4\pi)^2(4\pi)^{-\epsilon}}\Gamma(\epsilon)\int_0^1\mathrm{d}x\Delta^{-\epsilon}\left\{(1-2x)^2p^\mu p^\nu-2g^{\mu\nu}(1-x)(1-2x)p^2\right\}\ . \tag{46}$$

Following Srednicki, we can make the change of variable $x=y+\frac{1}{2}$, such that the limits of integration are evenly spaced about zero. With this, the integral simplifies to

$$\int_{-1/2}^{1/2}\mathrm{d}y\Delta^{-\epsilon}\left\{4y^2p^\mu p^\nu-2g^{\mu\nu}(2y^2-y)p^2\right\}\ , \tag{47}$$

now we can note that terms linear in y will vanish upon integration because they are odd:

$$4\left\{p^\mu p^\nu-g^{\mu\nu}p^2\right\}\int_{-1/2}^{1/2}\mathrm{d}y\Delta^{-\epsilon}y^2\ . \tag{48}$$

The remaining integral is

$$\int_{-1/2}^{1/2} \mathrm{d}y \Delta^{-\epsilon} y^2 = \int_{-1/2}^{1/2} \mathrm{d}y \frac{y^2}{\left\{m^2 - (\frac{1}{4} - y^2)p^2\right\}^\epsilon} \ , \tag{49}$$

and we obtain the expression for the photon propagator correction:

$$i\hat{\Sigma}^{\mu\nu} = \frac{4ie^2\tilde{\mu}^{2\epsilon}}{(4\pi)^2(4\pi)^{-\epsilon}} \Gamma(\epsilon) \left\{p^2 g^{\mu\nu} - p^\mu p^\nu\right\} \int_{-1/2}^{1/2} \mathrm{d}y \frac{y^2}{\left\{m^2 - (\frac{1}{4} - y^2)p^2\right\}^\epsilon} \ . \tag{50}$$

We can now take the limit of small ϵ, for the integrand this means:

$$i\hat{\Sigma}^{\mu\nu} = \frac{4ie^2\tilde{\mu}^{2\epsilon}}{(4\pi)^2(4\pi)^{-\epsilon}} \Gamma(\epsilon) \left\{p^2 g^{\mu\nu} - p^\mu p^\nu\right\} \int_{-1/2}^{1/2} y^2 \left(1 - \epsilon \log\left[m^2 - (\tfrac{1}{4} - y^2)p^2\right]\right) \mathrm{d}y \ . \tag{51}$$

Now we need to expand the Gamma function:

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) \ , \tag{52}$$

where γ_E is the Euler-Mascheroni constant. We will neglect terms of linear order or higher in ϵ because they vanish as $\epsilon \to 0$. Additionally, we can strip out the tensor structure to obtain

$$i\hat{\Sigma}(p^2) = \frac{4ie^2\tilde{\mu}^{2\epsilon}}{(4\pi)^2(4\pi)^{-\epsilon}} \left(\frac{1}{\epsilon} - \gamma_E\right) \int_{-1/2}^{1/2} y^2 \left(1 - \epsilon \log\left[m^2 - (\tfrac{1}{4} - y^2)p^2\right]\right) \mathrm{d}y \ . \tag{53}$$

We can redefine our mass scale to be

$$\tilde{\mu}^{2\epsilon} = \mu^{2\epsilon}(4\pi)^{-\epsilon} \left\{\exp(\gamma_E)\right\}^{-\epsilon} = (4\pi)^{-\epsilon} \left(\frac{e^{-\gamma_E}}{\mu^2}\right)^{-\epsilon} \ , \tag{54}$$

noting that the factors of 4π cancel, we can now expand for small ϵ:

$$\tilde{\mu}^{2\epsilon} = (4\pi)^{-\epsilon} \left\{1 - \epsilon \log\left[\frac{e^{-\gamma_E}}{\mu^2}\right]\right\} = (4\pi)^{-\epsilon} \left(1 + \epsilon\gamma_E + \epsilon \log(\mu^2)\right) \ . \tag{55}$$

Inserting this into the propagator correction yields

$$i\hat{\Sigma}(p^2) = \frac{4ie^2}{(4\pi)^2} \left(1 + \epsilon\gamma_E + \epsilon \log(\mu^2)\right) \left(\frac{1}{\epsilon} - \gamma_E\right) \int_{-1/2}^{1/2} y^2 \left(1 - \epsilon \log\left[\Delta(y)\right]\right) \mathrm{d}y \ , \tag{56}$$

where

$$\Delta(y) = m^2 - (\tfrac{1}{4} - y^2)p^2 \ . \tag{57}$$

We can now multiply out all of the factors with multiple terms:

$$\begin{aligned} &\left(1 + \epsilon\gamma_E + \epsilon \log(\mu^2)\right) \left(\frac{1}{\epsilon} - \gamma_E\right) \left(1 - \epsilon \log\left[\Delta(y)\right]\right) \\ &\quad = \gamma^2\epsilon^2 \log\Delta - \epsilon\gamma^2 + \gamma\epsilon^2 \log\Delta \log\mu^2 - \gamma\epsilon \log\mu^2 - \epsilon \log\Delta \log\mu^2 - \log\Delta + \log\mu^2 + \frac{1}{\epsilon} \end{aligned} \tag{58}$$

but we can drop terms that are linear or higher order in ϵ, yielding:

$$i\hat{\Sigma}(p^2) = \frac{4ie^2}{(4\pi)^2}\int_{-1/2}^{1/2} y^2 \left(\frac{1}{\epsilon} - \log\left[\frac{\Delta}{\mu^2}\right]\right) \mathrm{d}y \ , \tag{59}$$

now using $\alpha = e^2/4\pi$, and dividing both sides by i, we have

$$\hat{\Sigma}(p^2) = \frac{\alpha}{\pi}\int_{-1/2}^{1/2} y^2 \left(\frac{1}{\epsilon} - \log\left[\frac{\Delta}{\mu^2}\right]\right) \mathrm{d}y \ , \tag{60}$$

which we will note is dimensionless. We calculated the correction ignoring the counter-term, so the full correction is

$$\Pi^{\mu\nu} = \left\{p^\mu p^\nu - g^{\mu\nu}p^2\right\}\left[\hat{\Sigma}(p^2) - i(Z_3 - 1)\right] \ , \tag{61}$$

so we can incorporate the divergent $1/\epsilon$ into our renormailzation factor Z_3. Thus the renormalized two-point function (after stripping the tensor structure) is

$$\Pi^R = -\frac{\alpha}{\pi}\int_{-1/2}^{1/2} y^2 \log\left[\frac{\Delta}{\mu^2}\right] \mathrm{d}y \ . \tag{62}$$

We can icorporate the propagator correction into the coupling constant instead, resulting in an effective electric charge:

$$e^2_{\text{eff}} = \frac{e^2}{1 - \Pi^R(s)} \ , \tag{63}$$

where the argument s is the Mandelstam invariant. Since Δ has dependence on the propagator momentum squared, this can be cast in terms of s.

Optical Theorem

The optical theorem tells us that the imaginary part of the correction we had just calculated is equivalent to the decay rate of a off-shell photon (Peskin eq. 7.5):

$$\Im\left[\gamma^* \to \gamma^*\right] = \Im\left[\Pi^R(s)\right]\Big|_{s=M^2_{\gamma^*}} = M_{\gamma^*}\Gamma\left[\gamma^* \to \phi^*\phi\right] \ , \tag{64}$$

where M_{γ^*} is the mass of the off-shell photon. Let's investigate the renormalized two-point function (where I've re-inserted the implicit $-i\epsilon$):

$$\Pi^R\Big|_{s=M^2_{\gamma^*}} = -\frac{\alpha}{\pi}\int_{-1/2}^{1/2} y^2 \log\left[\frac{m^2 - (\frac{1}{4} - y^2)(M_{\gamma^*})^2 - i\epsilon}{\mu^2}\right] \mathrm{d}y \ , \tag{65}$$

where we are interested in the imaginary part. Due to the bounds of integration the quantity $\frac{1}{4} - y^2 \leq 0$, so for $M_{\gamma^*} > 2m$, the argument of the logarithm is positive and the integrand is real (in the limit $\epsilon \to 0$). We can split the logarithm into the difference of the logarithms of the numerator and denominator, and then drop the term proportional to $\log(\mu^2)$ since it is purely real. Furthermore, we can apply the identity:

$$\log(-A - i\epsilon) = \log A - i\pi \ , \tag{66}$$

such that when the argument of the log is negative, we have

$$\Pi^R\Big|_{s=M_{\gamma^*}^2} = -\frac{\alpha}{\pi}\int_{-1/2}^{1/2} y^2 \left\{\log\left[\frac{-m^2+(\frac{1}{4}-y^2)(M_{\gamma^*})^2}{\mu^2}\right] - i\pi\right\}\Theta\left[(\tfrac{1}{4}-y^2)(M_{\gamma^*})^2 - m^2\right]\mathrm{d}y \ ,$$

and thus

$$\Im\left[\Pi^R(s)\right]\Big|_{s=M_{\gamma^*}^2} = \alpha\int_{-1/2}^{1/2}\Theta\left[(\tfrac{1}{4}-y^2)(M_{\gamma^*})^2 - m^2\right]y^2\mathrm{d}y \ . \tag{67}$$

Since the integrand is even, this is equivalent to

$$\Im\left[\Pi^R(s)\right]\Big|_{s=M_{\gamma^*}^2} = 2\alpha\int_{0}^{1/2}\Theta\left[(\tfrac{1}{4}-y^2)(M_{\gamma^*})^2 - m^2\right]y^2\mathrm{d}y \ . \tag{68}$$

The Heaviside function is satisfied when

$$y < \sqrt{\frac{1}{4}-\frac{m^2}{M_{\gamma^*}^2}} \ , \tag{69}$$

so

$$\begin{aligned}\Im\left[\Pi^R(s)\right]\Big|_{s=M_{\gamma^*}^2} &= 2\alpha\int_0^{\sqrt{\frac{1}{4}-\frac{m^2}{M_{\gamma^*}^2}}} y^2\mathrm{d}y = 2\alpha\left[\frac{1}{24}\left(1-\frac{4m^2}{M^2}\right)^{3/2}\right] \\ &= \frac{\alpha}{12}\left(1-\frac{4m^2}{M_{\gamma^*}^2}\right)^{3/2} \ .\end{aligned} \tag{70}$$

Note that for this quantity to be real (*i.e.*, the imaginary part is truly imaginary), we require $M > 2m$, which is intuitive.

We now need to compute the decay width for a massive photon decaying to two scalars: $\gamma^*(p) \to \phi(k_1) + \phi^*(k_2)$. This process, at tree-level, is just a vertex factor contracted with the external polarization of the incoming photon, the matrix element is simply

$$i\mathcal{M} = -ie(k_1-k_2)^\mu\epsilon_\mu \ , \tag{71}$$

and thus

$$|\mathcal{M}|^2 = e^2(k_1-k_2)^\mu(k_1-k_2)^\nu\epsilon_\nu^*\epsilon_\mu \ , \tag{72}$$

which we need to average over the initial polarizations and sum over the final state spins. Since the photon is massive, there are three polarization states, so

$$|\bar{\mathcal{M}}|^2 = \frac{e^2}{3}(k_1-k_2)^\mu(k_1-k_2)^\nu\sum_{\text{pols}}\epsilon_\mu\epsilon_\nu^* \tag{73}$$

$$= \frac{e^2}{3}(k_1-k_2)^\mu(k_1-k_2)^\nu\left(-g_{\nu\mu}+\frac{p^\nu p^\mu}{M_{\gamma^*}^2}\right) \ , \tag{74}$$

using the completeness relations for polarizations of a massive vector boson. Contracting indeces yields

$$|\bar{\mathcal{M}}|^2 = \frac{e^2}{3}\left(-(k_1^2+k_2^2-2k_1\cdot k_2)+\frac{1}{M_{\gamma^*}^2}(k_1\cdot p-k_2\cdot p)^2\right) \tag{75}$$

$$= \frac{e^2}{3}\left(\frac{1}{M_{\gamma^*}^2}(k_1\cdot p-k_2\cdot p)^2-2(m^2-k_1\cdot k_2)\right) , \tag{76}$$

which we can evaluate in the center-of-mass frame. In this case:

$$p=(M_{\gamma^*},0) \qquad k_1=(E,\mathbf{k}) \qquad k_2=(E,-\mathbf{k}) , \tag{77}$$

with $E=M_{\gamma^*}/2$, $|\mathbf{k}|^2=E^2-m^2$, $k_1^2=k_2^2=m^2$, for simplicity, we will drop the subscript on the photon mass. The relevant scalar products are

$$k_1\cdot k_2 = E^2+|\mathbf{k}|^2 = \frac{1}{4}M^2+|\mathbf{k}|^2 \tag{78}$$

$$p\cdot k_i = M\frac{E}{2} = \frac{1}{4}M^2 , \tag{79}$$

so

$$|\bar{\mathcal{M}}|^2 = \frac{e^2}{3}\left(\frac{1}{M^2}\left(\frac{1}{4}M^2-\frac{1}{4}M^2\right)^2-2\left(m^2-\frac{1}{4}M^2-|\mathbf{k}|^2\right)\right) . \tag{80}$$

From the Lorentz condition, we have

$$m^2=k_1^2=E^2-|\mathbf{k}|^2 \quad\Rightarrow\quad |\mathbf{k}|^2=E^2-m^2=\frac{1}{4}M^2-m^2=\frac{1}{4}M^2\left(1-\frac{4m^2}{M^2}\right) , \tag{81}$$

so

$$|\bar{\mathcal{M}}|^2 = \frac{2e^2}{3}\left(\frac{1}{2}M^2-2m^2\right)=\frac{e^2}{3}\left(M^2-4m^2\right)=\frac{e^2}{3}M^2\left(1-\frac{4m^2}{M}\right) . \tag{82}$$

From this, we calculate the rate (Peskin Appendix A.5):

$$\Gamma=\int\frac{1}{2M}|\bar{\mathcal{M}}|^2\frac{d\Omega}{4\pi}\frac{1}{8\pi}\left(\frac{2|\mathbf{k}|}{E_{cm}}\right)=\int\frac{|\mathbf{k}|}{M^2}|\bar{\mathcal{M}}|^2\frac{d\Omega}{2(4\pi)^2} . \tag{83}$$

Inserting the matrix element yields

$$\Gamma=\int\frac{e^2}{3}|\mathbf{k}|\left(1-\frac{4m^2}{M}\right)\frac{d\Omega}{2(4\pi)^2}=\int\frac{e^2}{3}\frac{M}{2}\left(1-\frac{4m^2}{M^2}\right)^{1/2}\left(1-\frac{4m^2}{M}\right)\frac{d\Omega}{2(4\pi)^2} \tag{84}$$

$$=\frac{e^2}{4\pi}\frac{M}{12}\left(1-\frac{4m^2}{M^2}\right)^{3/2}\int\frac{d\Omega}{4\pi}=\frac{\alpha}{12}\left(1-\frac{4m^2}{M^2}\right)^{3/2}M . \tag{85}$$

Using the optical theorem, we have

$$\Im\left[\Pi^R(s)\right]\Big|_{s=M_{\gamma^*}^2}=M_{\gamma^*}\Gamma\left[\gamma^*\to\phi^*\phi\right]=\frac{\alpha}{12}\left(1-\frac{4m^2}{M_{\gamma^*}^2}\right)^{3/2}M_{\gamma^*}^2 , \tag{86}$$

however, the definition we used for Π^R differs from that of Schwartz. Our renormalized Π^R is dimensionless, but in Schwartz section 24.1, the argument of the imaginary part in the optical theorem has mass dimension 2, so we need really need:

$$\Im\left[s\Pi^R(s)\right]\Big|_{s=M_{\gamma^*}^2} = \frac{\alpha}{12}\left(1-\frac{4m^2}{M_{\gamma^*}^2}\right)^{3/2} M_{\gamma^*}^2 \tag{87}$$

$$M_{\gamma^*}^2\frac{\alpha}{12}\left(1-\frac{4m^2}{M_{\gamma^*}^2}\right)^{3/2} = \frac{\alpha}{12}\left(1-\frac{4m^2}{M_{\gamma^*}^2}\right)^{3/2} M_{\gamma^*}^2\ , \tag{88}$$

which is clearly true, and thus the optical theorem is satisfied.

2 Photon-Photon Scattering.

In the low energy limit $s \ll m_e^2$, derive the scattering amplitude for $\gamma\gamma \to \gamma\gamma$.

2.1 Quantum Electrodynamics.

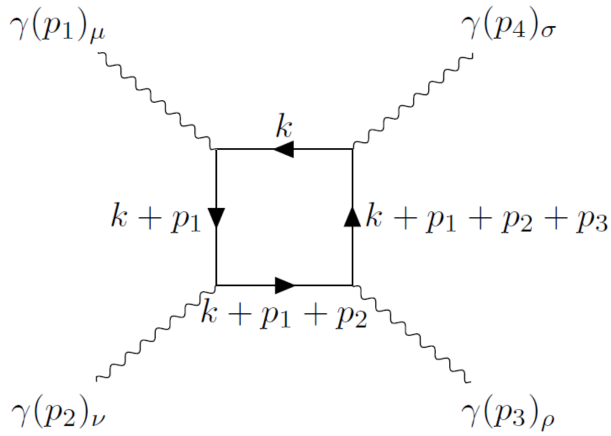

Figure 3: One of the Feynman diagrams describing the process $\gamma\gamma \to \gamma\gamma$. The remaining diagrams are obtained via: interchanging momenta p_3 and p_4, interchanging momenta p_2 and p_3, then these three with the direction of U(1) charge flow reversed - totaling six diagrams. Note we have defined the external momenta such that they all point inwards.

For the first diagram, we can write the matrix element:

$$i\mathcal{M}_1 = -(\epsilon_1)_\mu(\epsilon_2)_\nu \left\{ \int \frac{\mathrm{d}^d k}{(2\pi)^d}(-ie\gamma^\mu) i\frac{\not{k} + \not{p}_1 + m}{(k+p_1)^2 - m^2}(-ie\gamma^\nu) i\frac{\not{k} + \not{p}_1 + \not{p}_2 + m}{(k+p_1+p_2)^2 - m^2} \right.$$
$$\left. \times (-ie\gamma^\rho) i\frac{\not{k} + \not{p}_1 + \not{p}_2 + \not{p}_3 + m}{(k+p_1+p_2+p_3)^2 - m^2}(-ie\gamma^\sigma) i\frac{\not{k} + m}{k^2 - m^2} \right\} (\epsilon_3^*)_\rho(\epsilon_4^*)_\sigma \ , \quad (89)$$

where the overall minus sign comes from the closed fermion loop. Let us strip off the exterior polarization vectors to obtain

$$\mathcal{M}_1^{\mu\nu\sigma\rho} = ie^4 \int \frac{\mathrm{d}^d k}{(2\pi)^d}\gamma^\mu \frac{\not{k} + \not{p}_1 + m}{(k+p_1)^2 - m^2}\gamma^\nu \frac{\not{k} + \not{p}_1 + \not{p}_2 + m}{(k+p_1+p_2)^2 - m^2}$$
$$\times \gamma^\rho \frac{\not{k} + \not{p}_1 + \not{p}_2 + \not{p}_3 + m}{(k+p_1+p_2+p_3)^2 - m^2}\gamma^\sigma \frac{\not{k} + m}{k^2 - m^2} \ . \quad (90)$$

If we write out the matrix indeces, we see the numerator is a trace, so we can define the numerator as

$$N^{\mu\nu\sigma\rho} = \mathrm{Tr}\left\{ (\not{k} + m)\gamma^\mu(\not{k} + \not{p}_1 + m)\gamma^\nu(\not{k} + \not{p}_1 + \not{p}_2 + m)\gamma^\sigma(\not{k} + \not{p}_1 + \not{p}_2 + \not{p}_3 + m)\gamma^\rho \right\} \ , \quad (91)$$

and we'll define the denominator as

$$\frac{1}{D} = \frac{1}{k^2 - m^2}\frac{1}{(k+p_1)^2 - m^2}\frac{1}{(k+p_1+p_2)^2 - m^2}\frac{1}{(k+p_1+p_2+p_3)^2 - m^2} \ . \quad (92)$$

Using Feynman parameterization, we have:

$$\frac{1}{\alpha\beta\gamma\delta} = 3! \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \int_0^y \mathrm{d}z \frac{1}{(\alpha + x(\beta-\alpha) + y(\gamma-\beta) + z(\delta-\gamma))^4} \ . \tag{93}$$

From the definition of D, labeling each factor in order, the denominator from Feynman parameterization becomes

$$D = \left(k^2 + z(2(k+p_1+p_2)+p_3)\cdot p_3 + 2x(k\cdot p_1) + 2y(k\cdot p_2) - m^2 + xp_1^2 + 2y(p_1\cdot p_2) + yp_2^2\right)^4 \ .$$

To complete the Feynman trick, we need to shift the loop momenta so we can complete the square:

$$k \to k - xp_1 - yp_2 - zp_3 \ , \tag{94}$$

which makes the denominator:

$$D = \left(k^2 - m^2 - x(x-1)p_1^2 - 2(x-1)(yp_2 + zp_3)\cdot p_1 - y(y-1)p_2^2 - 2z(y-1)(p_2\cdot p_3) - z(z-1)p_3^2\right)^4 \ .$$

we can then define Δ such that $D = (k^2 - \Delta)^4$ so

$$\Delta = m^2 + x(x-1)p_1^2 + 2(x-1)(yp_2 + zp_3)\cdot p_1 + y(y-1)p_2^2 + 2z(y-1)(p_2\cdot p_3) + z(z-1)p_3^2 \ .$$

We can then write

$$\mathcal{M}_1^{\mu\nu\sigma\rho} = ie^4 \int \frac{\mathrm{d}^dk}{(2\pi)^d} 3! \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \int_0^y \mathrm{d}z \frac{N^{\mu\nu\sigma\rho}}{(k^2-\Delta)^4} = i3!(4\pi\alpha)^2 \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \int_0^y \mathrm{d}z \int \frac{\mathrm{d}^dk}{(2\pi)^d} \frac{N^{\mu\nu\sigma\rho}}{(k^2-\Delta)^4} \ , \tag{95}$$

if we then contract in the external polarization vectors, we have

$$\mathcal{M}_1 = i3!(4\pi\alpha)^2 \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \int_0^y \mathrm{d}z \int \frac{\mathrm{d}^dk}{(2\pi)^d} \frac{N^{\mu\nu\sigma\rho}(\epsilon_1)_\mu(\epsilon_2)_\nu(\epsilon_3^*)_\rho(\epsilon_4^*)_\sigma}{(k^2-\Delta)^4} \ , \tag{96}$$

but we have not yet made the shift of the loop momentum in the numerator. Let's go ahead and do that:

$$\not{k} + m \to \not{k} - x\not{p}_1 - y\not{p}_2 - z\not{p}_3 + m \equiv \tau_1 \tag{97}$$

$$\not{k} + \not{p}_1 + m \to \not{k} + (1-x)\not{p}_1 - y\not{p}_2 - z\not{p}_3 + m \equiv \tau_2 \tag{98}$$

$$\not{k} + \not{p}_1 + \not{p}_2 + m \to \not{k} + (1-x)\not{p}_1 + (1-y)\not{p}_2 - z\not{p}_3 + m \equiv \tau_3 \tag{99}$$

$$\not{k} + \not{p}_1 + \not{p}_2 + \not{p}_3 + m \to \not{k} + (1-x)\not{p}_1 + (1-y)\not{p}_2 + (1-z)\not{p}_3 + m \equiv \tau_4 \ , \tag{100}$$

which we insert into N, and evaluate using FeynCalc:

$$N^{\mu\nu\sigma\rho} = \mathrm{Tr}\left\{\tau_1\gamma^\mu\tau_2\gamma^\nu\tau_3\gamma^\sigma\tau_4\gamma^\rho\right\} \ , \tag{101}$$

This results in a whopping 6184 terms ... We can now compare this to what we would obtain from the effective field theory.

2.2 Effective Field Theory.

We can define an effective Lagrangian for the four-photon interaction. To do this we consider a theory with only photons and expand it to fourth order in the fields:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A\left[c_1(F^{\mu\nu}F_{\mu\nu})^2 + c_2(F^{\mu\nu}\tilde{F}_{\mu\nu})^2\right] + \dots \ , \tag{102}$$

where

$$\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\sigma\rho}F^{\sigma\rho} \ , \tag{103}$$

is F-dual. The overall constant A can be determined through dimensional analysis - since the operators $(FF)^2$ are mass-dimension 8, A must have mass dimension -4. Additionally, from the QED diagram, figure 3, we note there are four verteces (and four external photons), each contributing a factor of e, thus

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha^2}{m_e^4}\left[c_1(F^{\mu\nu}F_{\mu\nu})^2 + c_2(F^{\mu\nu}\tilde{F}_{\mu\nu})^2\right] \ , \tag{104}$$

neglecting the higher-order operators. This is the low-energy limit of the Euler-Heisenberg Lagrangian. We can determine the unknown (dimensionless) coefficients by comparison with QED theroy. The goal here is to match terms with a specific structure to those in the expansion of the trace from QED, in order to determine the coefficients c_1 and c_2. We can ignore the leading term in the effective Lagrangian becuase the terms from the loop integral are fourth order in the fields. If we expand the field tensors in terms of the photon fields A, we see that

$$\begin{aligned}\mathcal{L}_{\text{eff}} \supset c_1 &\left\{(p_1^\mu\epsilon_1^\nu - p_1^\nu\epsilon_1^\mu)(p_2^\mu\epsilon_2^\nu - p_2^\nu\epsilon_2^\mu)(p_3^\alpha\epsilon_3^{*\beta} - p_3^\beta\epsilon_3^{*\alpha})(p_4^\alpha\epsilon_4^{*\beta} - p_4^\beta\epsilon_4^{*\alpha})\right\} \\ &+ c_2\left\{\varepsilon_{\mu\nu\alpha\beta}(p_1^\mu\epsilon_1^\nu - p_1^\nu\epsilon_1^\mu)(p_2^\alpha\epsilon_2^\beta - p_2^\beta\epsilon_2^\alpha)\right\} \times \left\{\varepsilon_{\mu\nu\alpha\beta}(p_3^\mu\epsilon_3^{*\nu} - p_3^\nu\epsilon_3^{*\mu})(p_4^\alpha\epsilon_4^{*\beta} - p_4^\beta\epsilon_4^{*\alpha})\right\} \ ,\end{aligned} \tag{105}$$

and other terms of similar form where the momenta are permuted, corresponding to the six possible diagrams[1] one can draw (see Schwartz eq. 33.83). Now consider the first quantity in curly braces, each pair of indeces (μ,ν) and (α,β) can be contracted in four different ways, leading to 16 terms.

2.2.1 Terms from $(F^{\mu\nu}F_{\mu\nu})^2$.

The first term multiplied out yields terms of the form

$$\mathcal{L}_{\text{eff}} \supset c_1\left\{(p_1^\mu\epsilon_1^\nu)(p_2^\mu\epsilon_2^\nu)(p_3^\alpha\epsilon_3^\beta)(p_4^\alpha\epsilon_4^\beta)\right\} = c_1\left\{(p_1\cdot p_2)(\epsilon_1\cdot\epsilon_2)(p_3\cdot p_4)(\epsilon_3\cdot\epsilon_4)\right\} \tag{106}$$

$$= c_1\left\{(p_1\cdot p_2)(p_3\cdot p_4)g^{\mu\nu}g^{\rho\sigma}\right\}(\epsilon_1)_\mu(\epsilon_2)_\nu(\epsilon_3)_\rho(\epsilon_4)_\sigma \ , \tag{107}$$

however there are four terms that are identical from this, *i.e.*, you could contract the first two terms in each or the second two terms in each or contract second and first in each, yielding four identical combinations proportional to $(p_1\cdot p_2)(p_3\cdot p_4)$. Additionally, terms with this structure could be constructed from different terms, specifically from the F-dual terms, so we investigate a specific limit to eliminate this possibility. Consider the forward scattering limit $p_3 \to p_1$. We can whittle

[1]The diagrams not shown in figure 3 are: interchange p_3 and p_4, interchange p_3 and p_2, and the three remaining are identical but with the direction of U(1) charge flow reversed.

down the FeynCalc output to terms that are proportional to $(p_1 \cdot p_2)^2 g^{\mu\nu} g^{\rho\sigma}$, after noting from kinematics, $p_1 \cdot p_2 = p_3 \cdot p_4$. We then get:

$$N^{\mu\nu\rho\sigma} \supset 4\left(4(y-1)y(x+z-1)^2\right)(p_1 \cdot p_2)^2 g^{\mu\nu} g^{\rho\sigma} \ , \tag{108}$$

so we need to evaluate the integral(s)

$$3!(4\pi\alpha)^2 (p_1 \cdot p_2)^2 g^{\mu\nu} g^{\rho\sigma} \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \int_0^y \mathrm{d}z \int \frac{\mathrm{d}^d k}{(2\pi)^d} 4 \frac{4(y-1)y(x+z-1)^2}{(k^2-\Delta)^4} \ ,$$

in the low-energy limit, we have $p_i^2 \ll m^2$ so $\Delta \to m^2$. The momentum integral yields, from Peskin's appendix:

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2-\Delta)^4} = \frac{i}{(4\pi)^2} \frac{\Gamma(2)}{\Gamma(4)} \Delta^{-2} = \frac{1}{3!} \frac{i}{(4\pi)^2} \frac{1}{m^4} \ . \tag{109}$$

The integral over Feynman parameters is

$$4 \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \int_0^y \mathrm{d}z \left(4(y-1)y(x+z-1)^2\right) = -4\left(\frac{1}{90}\right) \ , \tag{110}$$

so

$$\mathcal{M}_1^{\mu\nu\rho\sigma} \supset i3!(4\pi\alpha)^2 4\left(\frac{-1}{90}\right) \frac{1}{3!} \frac{i}{(4\pi)^2} \frac{1}{m^4} (p_1 \cdot p_2)^2 g^{\mu\nu} g^{\rho\sigma} = \frac{\alpha^2}{m^4} \frac{4}{90} (p_1 \cdot p_2)^2 g^{\mu\nu} g^{\rho\sigma} \ . \tag{111}$$

We are finally making some progress, but we should stop to examine potential missed factors of 2. Initially, we see the 4 here cancels the 4 from the different combinations of terms which yield $(p_1 \cdot p_2)(p_3 \cdot p_4)$. Additionally, we should pick up a factor of two in the matrix element from the loop diagram, due to the two possible directions for the U(1) charge current flow. However, when we took the forward scattering limit ($p_3 \to p_1$), we set $(p_1 \cdot p_2)(p_3 \cdot p_4) = (p_1 \cdot p_2)(p_1 \cdot p_4)$, which in this limit is equivalent to the permutation obtained by swapping p_2 and p_4, so in effect we have double counted a permutation, so we introduce a factor of 2 in the effective Lagrangian to account for this. We see all possible errant factors of 2 cancel (both the calculated matrix element and effective Lagrangian pick up the same number of factors), and we are left with (by comparison of equations 111 and 107):

$$c_1 = \frac{1}{90} \ . \tag{112}$$

2.2.2 Terms from $(F^{\mu\nu}\tilde{F}_{\mu\nu})^2$.

The allowable, non-reduntant (*i.e.*, not included in the fact of two obtained from reversing the direction of U(1) current) permutations are interchanging p_3 and p_4 and interchanging p_2 and p_3. Thus the first term in the effective Lagrangian, $(F^{\mu\nu}F_{\mu\nu})^2$ cannot create terms that are proportional to $p_1 \cdot \epsilon_4^*$. So we are interested in the terms generated by $(F^{\mu\nu}\tilde{F}_{\mu\nu})^2$ which are proportional to $p_1 \cdot \epsilon_4^*$:

$$\mathcal{L}_{\text{eff}} \supset c_2 \left\{ \varepsilon_{\mu\nu\alpha\beta} (p_1^\mu \epsilon_1^\nu)(p_2^\alpha \epsilon_2^\beta) \right\} \times \left\{ \varepsilon_{\rho\sigma\gamma\delta} (-p_3^\sigma \epsilon_3^{*\rho})(-p_4^\delta \epsilon_4^{*\gamma}) \right\} \ . \tag{113}$$

First we should note that

$$\varepsilon_{\mu\nu\alpha\beta}\varepsilon_{\rho\sigma\gamma\delta} = \det \begin{pmatrix} g_{\mu\rho} & g_{\mu\sigma} & g_{\gamma\mu} & g_{\delta\mu} \\ g_{\nu\rho} & g_{\nu\sigma} & g_{\gamma\nu} & g_{\delta\nu} \\ g_{\alpha\rho} & g_{\alpha\sigma} & g_{\alpha\gamma} & g_{\alpha\delta} \\ g_{\beta\rho} & g_{\beta\sigma} & g_{\beta\gamma} & g_{\beta\delta} \end{pmatrix} \ , \tag{114}$$

we're being a bit cavalier with the co-/contra- variant indeces, but in Minkowski space, it is largely irrelevant. Now we can beat FeynCalc into evaluating this determinant and contract it with the Lorentz vectors for the polarizations. Selecting terms which are proportional to $p_1 \cdot \epsilon_4^*$ and $p_4 \cdot \epsilon_1$, we have

$$-\left(p_1 \cdot \epsilon_4^*\right)\left(p_2 \cdot p_3\right)\left(p_4 \cdot \epsilon_1\right)\left(\epsilon_2 \cdot \epsilon_3^*\right) \ , \tag{115}$$

and thus

$$\mathcal{L}_{\text{eff}} \supset -4\frac{\alpha^2}{m^4} c_2 \left(p_1 \cdot \epsilon_4^*\right)\left(p_2 \cdot p_3\right)\left(p_4 \cdot \epsilon_1\right)\left(\epsilon_2 \cdot \epsilon_3^*\right) \ , \tag{116}$$

note that we again get a factor of 4 from the number of possible contractions leading to the same tensor structure.

Returning to the matrix element calculated from the Feynman diagram. Selecting the terms with the desired tensor structure from FeynCalc result of the trace evaluation:

$$-8(x-1)(x-z)(y+z)\left(p_1 \cdot \epsilon_4^*\right)\left(p_2 \cdot p_3\right)\left(p_4 \cdot \epsilon_1\right)\left(\epsilon_2 \cdot \epsilon_3^*\right) \ , \tag{117}$$

then integrating over the Feynman parameters yields 7/90, thus

$$\mathcal{M}_1 \supset i3!(4\pi)^2\alpha^2 \frac{7}{90}\left(p_1 \cdot \epsilon_4^*\right)\left(p_2 \cdot p_3\right)\left(p_4 \cdot \epsilon_1\right)\left(\epsilon_2 \cdot \epsilon_3^*\right) \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2-\Delta)^4} \ . \tag{118}$$

Noting the integral over k is identical to the previous one, we have

$$\mathcal{M}_1 \supset i3!(4\pi)^2\alpha^2 \frac{7}{90}\left(\frac{1}{3!}\frac{i}{(4\pi)^2}\frac{1}{m^4}\right)\left(p_1 \cdot \epsilon_4^*\right)\left(p_2 \cdot p_3\right)\left(p_4 \cdot \epsilon_1\right)\left(\epsilon_2 \cdot \epsilon_3^*\right) \tag{119}$$

$$\supset -\frac{7}{90}\frac{\alpha}{m^4}\left(p_1 \cdot \epsilon_4^*\right)\left(p_2 \cdot p_3\right)\left(p_4 \cdot \epsilon_1\right)\left(\epsilon_2 \cdot \epsilon_3^*\right) \ . \tag{120}$$

By comparison, we find

$$c_2 = \frac{7}{360} = \frac{1}{90}\frac{7}{4} \ . \tag{121}$$

2.3 Euler-Heisenberg Lagrangian

The terms in the effective Lagrangian that contribute to the scattering amplitude, to leading order, for the process $\gamma\gamma \to \gamma\gamma$ is

$$\mathcal{L}_{\text{eff}} = \frac{\alpha^2}{90m^4}\left[(F^{\mu\nu}F_{\mu\nu})^2 - \frac{7}{4}(F^{\mu\nu}\tilde{F}_{\mu\nu})^2\right] \ , \tag{122}$$

thus the scattering amplitude is

$$\mathcal{M} = \frac{\alpha^2}{90m^4}\left\{(p_1^\mu\epsilon_1^\nu - p_1^\nu\epsilon_1^\mu)(p_2^\mu\epsilon_2^\nu - p_2^\nu\epsilon_2^\mu)(p_3^\alpha\epsilon_3^{*\beta} - p_3^\beta\epsilon_3^{*\alpha})(p_4^\alpha\epsilon_4^{*\beta} - p_4^\beta\epsilon_4^{*\alpha})\right\}$$
$$+\frac{7\alpha^2}{360m^4}\left\{\varepsilon_{\mu\nu\alpha\beta}(p_1^\mu\epsilon_1^\nu - p_1^\nu\epsilon_1^\mu)(p_2^\alpha\epsilon_2^\beta - p_2^\beta\epsilon_2^\alpha)\right\}\times\left\{\varepsilon_{\mu\nu\alpha\beta}(p_3^\mu\epsilon_3^{*\nu} - p_3^\nu\epsilon_3^{*\mu})(p_4^\alpha\epsilon_4^{*\beta} - p_4^\beta\epsilon_4^{*\alpha})\right\} + \text{perms.} \ , \tag{123}$$

where perms. refers to permutations obtained by interchange of the momenta p_3 and p_4 or p_2 and p_3.