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Lectures on Quantum Mechanics, J. A. Sauls
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Contents

1 Problem #1: Interference of Neutron Beams.

Consider a nearly monoenergetic beam of neutrons of mass M_N initially prepared as a wavepacket, $\psi(\mathbf{r})$, with a well defined momentum $\mathbf{p}_0 = p_0\hat{\mathbf{x}}$, i.e. a momentum spread $\Delta p_x \ll p_0$. The neutron beam is split into two coherent beams by Bragg reflection from a Silicon crystal at position A and reflected by Bragg mirrors at positions B and C,then recombined at position D in front of the detector as indicated in Figure 1. One beam travels along path ABD, while the other beam travels along path ABC. If the plane of the interferometer is perpendicular to the gravitational acceleration, $\mathbf{g} = -g\hat{\mathbf{z}}$, then neutrons propagating along either of the two paths pass through the same gravitational potential. However, if the plane of the interferometer is inclined by an angle δ about the axis AB then the gravitational potential is different for the neutrons propagating along path ABD and path ACD.

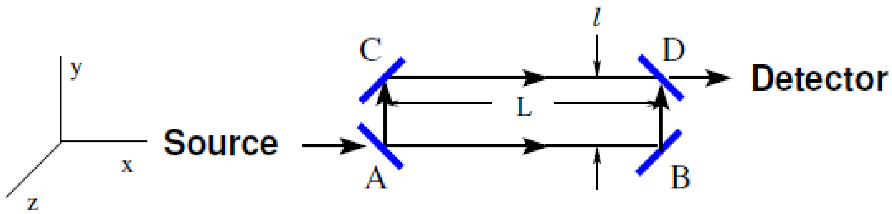

Figure 1: Diagram showing the coordinate system and Bragg mirror orientation for problem #1.

1.1 The Hamiltonian.

Write down the Hamiltonian that determines the time evolution of a neutron in a gravitational field of uniform acceleration, g. The Hamiltonian for a massive particle in a gravitational field is given by

$$\hat{H} = \frac{\hat{p}^2}{2M_n} - M_n g z \ , \tag{1}$$

and the time evolution of a state with this Hamiltonian is given by

$$\psi(t) = e^{\frac{i}{\hbar}t\hat{H}} |\psi\rangle \ , \tag{2}$$

where $|\psi\rangle$ is a properly normalized eigenstate of the Hamiltonian. If we tilt the system by an angle δ such that the path CD is at a height $\ell \sin\delta$, we can define the potential energy level to be at the path AB. Furthermore, the neutrons traveling along either path all must go against the same gradient to reach the potential, so the phase shift along the paths AC and BD are exactly the same, and only the phase shift along path CD relative to path AB contributes to the phase difference between path ACD and ABD. Let us consider the Hamiltonians of neutrons traveling along path CD and neutrons on path AB, which are respectively given by

$$\hat{H}_{CD} = \frac{\hat{p}^2}{2M_n} - M_n g\ell \sin\delta \tag{3}$$

$$\hat{H}_{AB} = \frac{\hat{p}^2}{2M_n} \ . \tag{4}$$

However, we must note some subtleties about the momenta along the different paths. As a neutron moves along path ACD, it begins with momentum p_0 and loses an amount Δp as it moves against the gradient of the gravitational field, which is converted to potential energy, reflected in the

Hamiltonian $\hat{H}_{CD}$. While it moves along the path CD, a neutron has momentum $p_0 - \Delta p$, while a neutron moving along path AB has momentum p_0. But when the neutron moves against the gravitational gradient on path BD, it also loses momentum Δp. From this it is clear to see the the neutrons are emitted with the same momenta (at point A) and detected with the same momentum (at point D). For this reason we can treat the momenta as equivalent through this problem.

1.2 Phase Difference.

Calculate the phase difference of the wavepackets which arrive at the detector from the two paths ABD and ACD when the interferometer is tilted by angle δ. Express the difference in phase in terms of the neutron mass, M_n, the acceleration due to gravity, g, the dimensions of the interferometer, L and ℓ, the de Broglie wavelength of the neutrons, λ, the tilt angle δ and relevant fundamental constants.

The phase difference between the two states comes completely from the time evolution operator expressed in Equation 2. If we look at this operator, we see that the phase difference is completely determined by the extra term in the Hamiltonian on the path along CD. Let us look at the argument of the exponential for this path

$$\frac{i}{\hbar}t\left(\frac{p^2}{2M_n} - M_n g\ell\sin\delta\right) \ , \tag{5}$$

we can also note that the time the particle spends traveling along this path is $t = (L/p)M_n$, so the argument of the exponential becomes

$$i\left(\frac{Lp}{2\hbar} - \frac{M_n g\ell L\sin\delta}{p\hbar}\right) \ . \tag{6}$$

If we introduce the deBroglie wavelength $\lambda = (2\pi\hbar)/p$, the above expression simplifies to

$$i\left(\frac{L\pi}{\lambda} - \frac{\lambda M_n^2 g\ell L\sin\delta}{2\pi\hbar^2}\right) \ , \tag{7}$$

where the first term is the same as the argument of the exponential of the time evolution operator for the Hamiltonian along path AB. Therefore the relative phase shift for the neutron travelling along path ACD relative to the phase shift of the neutron traveling along path ABD is simply the second term above, however we must include the phase shift picked up by the reflection due to the Bragg mirrors. A neutron moving along ACD picks reflects once, while one moving along ABD is reflected twice, so there is an overall phase difference of π, so the total relative phase shift between the paths ACD and ABD is

$$\pi - \frac{\lambda M_n^2 g\ell L\sin\delta}{2\pi\hbar^2} \ , \tag{8}$$

which if we set $\delta = 0$ we get the expected relative phase shift of π.

1.3 Detection Rate.

In an actual version of this experiment (see Figure 2) the number of neutrons arriving at the detector is represented in the figure below. If the dimensions of the interferometer are $L = 10$cm and $\ell = 1$cm calculate the wavelength of the incident neutrons.

Given the overall phase shift

$$\epsilon = \pi - \frac{\lambda M_n^2 g \ell L \sin\delta}{2\pi\hbar^2} \ , \tag{9}$$

we get a minimum count rate (maximally out-of-phase) when the Bragg setup is not inclined, i.e. $\delta = 0$.

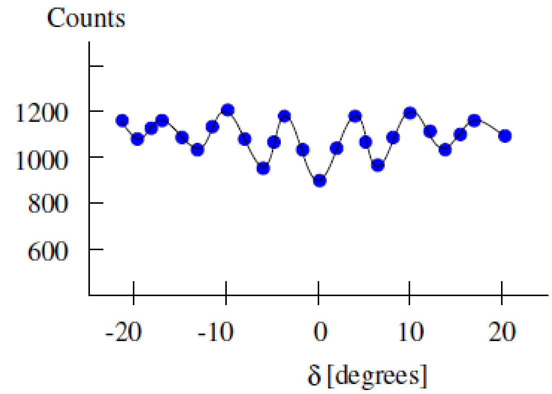

Figure 2: Neutron count rates at the detector as a function of the inclination of the Bragg mirror setup.

If we insert the dimensions of the Bragg interferometer and values of constants, we find

$$\epsilon = \pi - \lambda\alpha\sin\delta \tag{10}$$

where $\alpha = 3.934\,48\text{e}11\text{m}^{-1}$. We expect a maximum detection rate when the phase shift is at a minimum, or at

$$\pi = \lambda\alpha\sin\delta \ , \tag{11}$$

which if we estimate the maximum occurs at an inclination of $\delta = 2.5\,\text{deg}$, we see that

$$\lambda = 1.83\text{e}{-10}\ \text{m} = 1.83\text{Å} \ . \tag{12}$$

2 Problem #2: Electron in Uniform Magnetic Field.

Consider an otherwise free electron moving in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$.

2.1 Symmetric Gauge.

Show that the static vector potential, $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$, referred to as the "symmetric gauge", accounts for the magnetic field everywhere in space.

The magnetic field in all space is given by the curl of the vector potential,

$$\nabla \times \mathbf{A} = \frac{1}{2}\nabla \times (\mathbf{B} \times \mathbf{r}) = \frac{1}{2}\nabla \times \left[\hat{\mathbf{i}}(B_y z - B_z y) + \hat{\mathbf{j}}(B_z x - B_x z) + \hat{\mathbf{k}}(B_x y - B_y x)\right] \tag{13}$$

$$= \frac{1}{2}\left[\hat{\mathbf{i}}(B_x - (-B_x)) + \hat{\mathbf{j}}(B_y - (-B_y)) + \hat{\mathbf{k}}(B_z - (-B_z))\right] \tag{14}$$

$$= B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}} = \mathbf{B}\ . \tag{15}$$

Clearly, this is satisfied for any magnetic field $\mathbf{B}$, including our choice $\mathbf{B} = B\hat{\mathbf{Z}}$.

2.2 The Hamiltonian.

Write down the Hamiltonian for the electron in this gauge. Identify the conserved variables and mutually commuting set of operator observables that include the Hamiltonian.

The Hamiltonian for a particle in a vector potential is

$$\hat{H} = \frac{1}{2\mu}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2\ , \tag{16}$$

where c is the speed of light and e is the magnitude of the electron charge and μ is the electron mass. If we expand the square, we find that

$$\hat{H} = \frac{1}{2\mu}\left(|\mathbf{p}|^2 + \frac{e^2}{c^2}|\mathbf{A}|^2 - \frac{e}{c}\left[\mathbf{p}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p}\right]\right)\ . \tag{17}$$

If we note the form of the magnetic field, we see that the vector potential is $\mathbf{A} = \frac{1}{2}(-By\hat{\mathbf{i}} + Bx\hat{\mathbf{j}})$. Then, the dot product with momentum is

$$\mathbf{p}\cdot\mathbf{A} = \frac{B}{2}\left(-p_x y + p_y x\right) \tag{18}$$

$$\mathbf{A}\cdot\mathbf{p} = \frac{B}{2}\left(-y p_x + x p_y\right)\ , \tag{19}$$

which if we note that $[p_x, y] = [p_y, x] = 0$, we can flip the order of the operators in Equation 18 and note both expressions are equivalent, and in fact their sum is

$$\mathbf{p}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p} = B\hat{L}_z\ , \tag{20}$$

so we can express the Hamiltonian as

$$\hat{H} = \frac{1}{2\mu}\left(|\mathbf{p}|^2 + \frac{e^2}{c^2}|\mathbf{A}|^2 - \frac{e}{c}B\hat{L}_z\right) = \frac{1}{2\mu}\left(p_x^2 + p_y^2 + p_z^2 + \frac{e^2}{4c^2}B^2(y^2 + x^2) - \frac{e}{c}B\hat{L}_z\right)\ . \tag{21}$$

Let us define a natural frequency $\omega_0 = eB/\mu c$, which allows us to write the Hamiltonian as

$$\hat{H} = \frac{1}{2\mu}(p_x^2 + p_y^2) + \frac{1}{2}\mu\left(\frac{\omega_0}{2}\right)^2 (x^2 + y^2) - \left(\frac{\omega_0}{2}\right)\hat{L}_z + \frac{p_z^2}{2\mu} \ . \tag{22}$$

This Hamiltonian has the form of a two dimensional isotropic harmonic oscillator of frequency $\omega_0/2$ plus an angular term, and a term representing a free particle in the z direction. The angular momentum projection operator commutes with the Hamiltonian for a two dimensional oscillator, and the z momentum operator commutes with both,

$$[\hat{H}, \hat{H}] = [\hat{H}, \hat{L}_z] = [\hat{H}, p_z^2] = 0 \ . \tag{23}$$

From this we know the conserved quantities are the total energy, the z projection of angular momentum, and the linear momentum in the z direction. A basis that diagonalizes the two dimensional isotropic oscillator will also diagonalize this Hamiltonian.

2.3 Two Dimensional Isotropic Harmonic Oscillator.

In the coordinate representation use section 2.2 to show that the energy eigenstates are determined by the equation for a two-dimensional, isotropic harmonic oscillator. Specify the set of quantum numbers that label each energy eigenstate.

In the cyindrical coordinate representation, we can denote the wave function as $\psi(\rho, \varphi, z) = R(\rho)\Phi(\varphi)Z(z)$, which due to the form of the Hamiltonian, we may assume solutions of the form

$$\Phi(\varphi) \sim e^{im\varphi} \quad \text{and} \quad Z(z) \sim e^{ik_z z} \ , \tag{24}$$

where m is an integer and k_z is a wavenumber. We may write the Hamiltonian as

$$\hat{H} = -\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right) + \frac{1}{2}\mu\left(\frac{\omega_0}{2}\right)^2\rho^2 + \left(\frac{\omega_0}{2}\right)\left(i\hbar\frac{\partial}{\partial\varphi}\right) - \frac{\hbar^2 k_z^2}{2\mu} \ . \tag{25}$$

Using separation of variables, we find the Z function drops out, and the p_z term becomes a constant. We can now act on the angular solution, to get its eigenvalues, and divide through by the angular solution Φ, which yields

$$\hat{H}R = -\frac{\hbar^2}{2\mu}\left(\frac{\partial^2 R}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial\rho} + \frac{-m^2}{\rho^2}R\right) + \frac{1}{2}\mu\left(\frac{\omega_0}{2}\right)^2\rho^2 R + \left(\frac{\omega_0}{2}\right)(-m\hbar)R - \frac{\hbar^2 k_z^2}{2\mu}R = E_{nmk_z}R \ . \tag{26}$$

We can gather the constant terms to write

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2 R}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial\rho} - \frac{m^2}{\rho^2}R\right) + \frac{1}{2}\mu\left(\frac{\omega_0}{2}\right)^2\rho^2 R = \left[E_{nmk_z} + \frac{1}{2}m\hbar\omega_0 + \frac{\hbar^2 k_z^2}{2\mu}\right]R \ , \tag{27}$$

and denote a shifted energy ε_{nmk_z} as the coefficient of the radial function on the right hand side of the above equation, which yields the differential equation

$$-\frac{\hbar^2}{2\mu}\left(R'' + \frac{1}{\rho}R' - \frac{m^2}{\rho^2}R\right) + \frac{1}{2}\mu\left(\frac{\omega_0}{2}\right)^2\rho^2 R = \varepsilon_{nmk_z}R \ , \tag{28}$$

which has the same form as Shankar Equation 12.3.13, the differential equation that leads to a two dimensional isotropic harmonic oscillator. The quantum numbers that label each state are n the quantum number that defines the energy state of the two dimensional oscillator, m the angular momentum z-projection quantum number, and k_z the wavenumber that defines the plane wave solutions for the motion in the z direction.

2.4 Energy Eigenstates.

Solve the Schrödinger equation for the spectrum of energy eigenvalues and eigenfunctions. Tip: Choose a coordinate origin, then analyze the solutions in the limits $r \to \infty$ and $r \to 0$. Construct exact wave functions by "stitching together" the asymptotic solutions. What is the condition leading to quantization of the energy levels?

We can find the functional form of the radial solution by examining liiting cases[1]. If we examine the behavior of the solution in the limit $\rho \to 0$, we find that the constant term is negligible, as well as the potential because it goes like ρ^2, this has the form of a power,

$$R(\rho) \xrightarrow{\rho\to 0} \rho^\gamma \ . \tag{29}$$

If we plug this solution in to the modified differential equation (no potential or constant terms) we find[2] that $\gamma = |m|$. Additionally, if we explore the behavior of R as $\rho \to \infty$, we see the negligible terms are negative powers of ρ, so we get

$$-\frac{\hbar^2}{2\mu}R'' + \frac{1}{2}\mu\left(\frac{\omega_0}{2}\right)^2\rho^2 R = 0 \ , \tag{30}$$

which is amenable to a solution of the form $\exp[-\beta\rho^\delta]$. If we plug this back into the modified differential equation, we find[3]

$$R(\rho) \xrightarrow{\rho\to\infty} \exp\left[-\frac{\mu(\omega_0/2)}{2\hbar}\rho^2\right] \ . \tag{31}$$

With the asymptotic behavior of $R(\rho)$ known, we can fill in the behavior by defining a function $U(\rho)$ such that

$$R(\rho) = U(\rho)\rho^{|m|}\exp\left[-\frac{\mu\omega_0}{4\hbar}\rho^2\right] \ . \tag{32}$$

If we consider the change of variables

$$\epsilon = \frac{2}{\hbar\omega_0}\varepsilon_{nmk_z} \qquad\qquad y = \sqrt{\frac{\mu\omega_0}{2\hbar}}\rho \ , \tag{33}$$

we find that Equation 28 becomes[4]

$$0 = [2\epsilon - y^2 - m^2 y^{-2}]R(y) + y^{-1}R'(y) + R''(y) \ , \tag{34}$$

which when we plug in the solutions for R in terms of our dimensionless variable y, and its derivatives, we get a differential equation for $U(y)$,

$$0 = U'' + U'\left[\frac{1+2|m|}{y} - 2y\right] + U[2\epsilon - 2|m| - 2] \ . \tag{35}$$

[1] I have already solved this (see Shankar Problem 12. 3.7), so this will be a more abridged version. For more detailed steps, see Temples, Shankar Solution Set 5, section 3 (http://dylanjtemples.com/solutions/ShankarSolution05.pdf). 2015.

[2] Temples, Shankar Solution Set 5. Equations 22-24. 2015.

[3] Temples, Shankar Solution Set 5. Equation 29. 2015.

[4] Temples, Shankar Solution Set 5. Section 3.4. 2015.

We will assume a power series solution $U(y) = \sum_{r=0}^{\infty} C_r y^r$, which when we insert in the above differential equation leads to the recursion relation[5]

$$C_{r+2}\left[2|m| + 2r - 2\epsilon + 2\right] = C_r\left[(r+2)(2|m| + r + 2)\right] \ . \tag{36}$$

In order to maintain the correct asymptotic behavior as $y \to \infty$, the power series must maximize at some power, so that it will cancel the power of y in the numerator, and keep the function finite. We can impose that the series terminates for some maximum value of r, which therefore means terms of higher power also vanish (see recursion relation). It was found that all $C_r = 0$ for odd r[6] , so all r values are even (including the maximum). The maximum value of r is given by setting C_r to zero and solving the polynomial on the left hand side of Equation 36 for r,

$$r = \epsilon - |m| - 1 \ , \tag{37}$$

so we can see the truncation point of the power series depends on the dimensionless energy ϵ. If we define a new variable k such that $r = 2k$ (so k can be any positive integer. If we insert this into the above expression, and solve for the dimensionless energy we find

$$\varepsilon_{nmk_z} = \frac{\hbar\omega_0}{2}(2k + |m| + 1) \tag{38}$$

$$E_{nmk_z} + \frac{1}{2}m\hbar\omega_0 + \frac{\hbar^2 k_z^2}{2\mu} = \frac{\hbar\omega_0}{2}(2k + |m| + 1) \tag{39}$$

$$E_{nmk_z} = \frac{\hbar\omega_0}{2}(2k + |m| - m + 1) - \frac{\hbar^2 k_z^2}{2\mu} \ , \tag{40}$$

after substituting in the definition of dimensionless energy and the shifted energy ε_{nmk_z}. If we define the quantum number for the energy state of the harmonic oscillator $n = k + \frac{1}{2}(|m| - m)$, we find the energy of the system to be

$$E_{nmk_z} = \hbar\omega_0(n + \tfrac{1}{2}) - \frac{\hbar^2 k_z^2}{2\mu} \ , \tag{41}$$

which are the Landau energy levels for a particle in this potential.

2.5 Degeneracy of Landau Levels.

Determine the degeneracy of each energy level ("Landau level"). From the exact energy level spectrum derive a formula to calculate the density of states as a function of energy.

We can note the degeneracy of this system dependent on the integers k and m, and the continuous wavenumber k_z. The quantum number k completely determines the value of n for $m > 0$. This is the case because for all positive values of m, $|m| - m = 0$, so $n = k$, and the energy level is therefore infinitely degenerate in m (for any given k). Additionally, the energy is two-fold degenerate with respect to the wave number because $k_z^2 = (-k_z)^2$. Therefore for $m < 0$, the states aree two-fold degenerate due to the wavenumber k_z.

[5] Temples, Shankar Solution Set 5. Equation 51. 2015.
[6] Temples, Shankar Solution Set 5. Page 9. 2015.

The field energy, If we define the wavevector of the nth Landau level as k_n, we see[7]

$$\frac{\hbar^2 k_n^2}{2\mu} = \hbar\omega_0(n + \tfrac{1}{2}) \ . \tag{42}$$

To find the density of states in k-space, we need to find the number of states that exists in the volume of a spherical shell in k-space between k_n and k_{n+1}, and divide by this k-space volume (area if the separation between wavenumbers is low). From the indicated reference, we see the degeneracy of a two dimensional phase space is

$$N = 2\frac{1}{\sqrt{2\pi}} A_r A_k \ , \tag{43}$$

where A_r is the area in r-space, A_k is the area in k-space and the factor 2 is due to the two possibilities of the electron's spin projection. From this we find

$$\frac{N}{A} = 2\frac{\pi k_{n+1}^2 - \pi k_n^2}{(2\pi)^2} \ , \tag{44}$$

which if we substitute in our definition of the wave number k_n (Equation 42), we find the density of states is

$$\frac{N}{A} = \frac{eB}{\pi\hbar} \ . \tag{45}$$

[7] Here we are following the treatment discussed in Magnetoresistance in two-dimensional systems and the quantum Hall effect. Page 112.

3 Problem #3: Coherent States.

Consider a coherent state defined by $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$, where $\alpha = |\alpha|e^{i\theta}$ is a complex number.

3.1 Unitary Transformation Form.

Show that the normalized coherent state can be represented by a unitary transformation of the ground state of the form $|\alpha\rangle = \hat{\mathcal{D}}(\alpha)\,|0\rangle$. Determine $\hat{\mathcal{D}}(\alpha)$. Hint: Use the Campbell, Baker, Hausdorff identity.

A coherent state (Glauber state) is a linear combination of energy eigenstates of the harmonic oscillator (number states),

$$|\alpha\rangle = \sum_{n=0}^{\infty} = c_n\,|n\rangle \quad . \tag{46}$$

but we can also insert a complete set of states on the left of this ket to find

$$|\alpha\rangle = \sum_{n=0}^{\infty} |n\rangle\,\langle n|a\rangle \quad , \tag{47}$$

and by equating the two we find $c_n = \langle n|\alpha\rangle$. We can write a number eigenstate as a series of operations of the creation operator acting on the vacuum state

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\,|0\rangle \quad , \tag{48}$$

which allows us to calculate c_n

$$\langle n|\alpha\rangle = \frac{1}{\sqrt{n!}}\,\langle 0|(\hat{a})^n|\alpha\rangle \quad , \tag{49}$$

by taking the conjugate transpose of Equation 48. If we note the action of the annihilation operator on a coherent state, we can replace the operator with the complex number α,

$$\langle n|\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}\,\langle 0|\alpha\rangle \quad . \tag{50}$$

We can now rewrite coherent states (Equation 47) as

$$|\alpha\rangle = \langle 0|\alpha\rangle \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle \quad , \tag{51}$$

and impose the normalization condition

$$1 = \langle\alpha|\alpha\rangle = |\,\langle 0|\alpha\rangle\,|^2 \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!}\,\langle n|n\rangle = |\,\langle 0|\alpha\rangle\,|^2 \sum_{n=0}^{\infty} \frac{(\alpha^2)^n}{n!} = |\,\langle 0|\alpha\rangle\,|^2 e^{|\alpha|^2} \quad , \tag{52}$$

from which we get $\langle 0|\alpha\rangle = \exp[-|\alpha|^2/2]$. Furthermore, if we substitute this result and Equation 48 into the expression for a coherent state, we find

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\,|0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha\hat{a}^\dagger)^n}{n!}\,|0\rangle = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^\dagger}\,|0\rangle \quad . \tag{53}$$

Note that we specified that the operator that propagates $|0\rangle$ to $|\alpha\rangle$ must be unitary, and hence must obey $\mathcal{D}(\alpha)^\dagger = \mathcal{D}(-\alpha)$,

$$\mathcal{D}(\alpha)^\dagger = \exp\left(-|\alpha|^2/2\right)\exp\left(\alpha^*\hat{a}\right) \neq \exp\left(-|\alpha|^2/2\right)\exp\left(-\alpha\hat{a}^\dagger\right) = \mathcal{D}(-\alpha)\ , \tag{54}$$

so there must be another component to $\mathcal{D}$. We note that this operator is of unit norm, but does not satisfy unitarity, if we introduce another term in the operator that still preserves the norm but makes it unitary, we will have a definition of $\mathcal{D}(\alpha)$ which satisfies our constraints. Consider the operator $\exp[-\alpha^*\hat{a}]$, which we can expand to be

$$\sum_{n=0}^{\infty}\frac{(-\alpha^*\hat{a})^n}{n!} = 1-(\alpha^*\hat{a})+\frac{1}{2}(\alpha^*\hat{a})^2+\ldots \tag{55}$$

which when acting on the vacuum state, only the first term survives ($\hat{a}\,|0\rangle = 0$) so we get the eigenvalue equation

$$\exp[-\alpha^*\hat{a}]\,|0\rangle = 1\,|0\rangle \quad \Rightarrow \quad e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}\,|0\rangle = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}\,|0\rangle\ . \tag{56}$$

We can now define the displacement operator

$$\mathcal{D}(\alpha)\,|0\rangle = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}\,|0\rangle = |\alpha\rangle\ . \tag{57}$$

If we examine the operators in the exponentials of the displacement operator, $\hat{A} = \alpha\hat{a}^\dagger$, and $\hat{B} = -\alpha^*\hat{a}$, we find

$$[A,B] = [\alpha\hat{a}^\dagger, -\alpha^*\hat{a}] = -|\alpha|^2[\hat{a}^\dagger,\hat{a}] = -|\alpha|^2(-1) = |\alpha|^2 \tag{58}$$

$$[[A,B],A] = [|\alpha|^2, A] = 0 \tag{59}$$

$$[[A,B],B] = [|\alpha|^2, B] = 0\ . \tag{60}$$

Additionally, we can move the first exponential term in the displacement operator around as we choose because it is just a real number, this allows us to an identity[8] and we can write

$$\mathcal{D}(\alpha) = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}} = e^{\alpha\hat{a}^\dagger-\alpha^*\hat{a}}\ . \tag{61}$$

Let us show quickly this is a unitary operator,

$$\mathcal{D}(\alpha)^\dagger = e^{\alpha^*\hat{a}-\alpha\hat{a}^\dagger} = e^{(-\alpha)\hat{a}^\dagger-(-\alpha)^*\hat{a}} = \mathcal{D}(-\alpha), \tag{62}$$

so indeed all of our constraints have been met.

3.2 Displacement Operator.

Show that $\hat{\mathcal{D}}(\alpha)$ "displaces" the annihilation operator by α, i.e.

$$\hat{\mathcal{D}}^\dagger(\alpha)\hat{a}\hat{\mathcal{D}}(\alpha) = \hat{a}+\alpha\ . \tag{63}$$

Give a physical interpretation of the generator of the unitary transformation.

[8] Reinhold Bertlmann, Time-Dependent Schrödinger Equation. Equation 2.72

Consider the operation

$$\hat{\mathcal{D}}^\dagger(\alpha)\hat{a}\hat{\mathcal{D}}(\alpha) = e^{\alpha^*\hat{a}-\alpha\hat{a}^\dagger}\hat{a}e^{\alpha\hat{a}^\dagger-\alpha^*\hat{a}} = e^{\alpha^*\hat{a}-\alpha\hat{a}^\dagger}\hat{a}e^{-(\alpha^*\hat{a}-\alpha\hat{a}^\dagger)} \ , \tag{64}$$

and the commutation relations

$$[\alpha^*\hat{a}-\alpha\hat{a}^\dagger,\ \hat{a}] = [\alpha^*\hat{a},\ \hat{a}] - [\alpha\hat{a}^\dagger,\ \hat{a}] = \alpha^*[\hat{a},\hat{a}] - \alpha[\hat{a}^\dagger,\hat{a}] = \alpha \tag{65}$$

$$[\alpha^*\hat{a}-\alpha\hat{a}^\dagger,\ [\alpha^*\hat{a}-\alpha\hat{a}^\dagger,\ \hat{a}]] = [\alpha^*\hat{a}-\alpha\hat{a}^\dagger,\ \alpha] = 0 \ , \tag{66}$$

because the commutator with an imaginary number is zero. Then using the Baker-Campbell-Hausdorff formula, we can write

$$\hat{\mathcal{D}}^\dagger(\alpha)\hat{a}\hat{\mathcal{D}}(\alpha) = \hat{a} + \alpha \ , \tag{67}$$

which proves Equation 63.

Coherent states (Glauber states) are linear combinations of stationary states of the harmonic oscillator that saturate the uncertainty principle. Since they are eigenstates of the annihilation operator, a single state can be measured (annihilated) without affecting the overall coherent state. For example, a coherent state (laser) would remain unchanged if one photon annihilated (detected).[9] The generator G of a unitary transformation is given by $U = e^{iG}$, so for this unitary transformation the generator is given by $G(\alpha) = -i(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, which is a linear combination of the annihilation and creation operators. If we compare this generator to the generator of translations (momentum), we can call the generator of a displacement of a coherent state, as some kind of phase-momentum operator acting on a coherent state.

[9] Reinhold Bertlmann, Harmonic Oscillator and Coherent States. Page 111.