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April 20, 2016

Contents

1 Commutator of Electric and Magnetic Fields.

Calculate the commutator, $[B_i(\mathbf{r},t), E_j(\mathbf{r}',t)]$, of the electric and magnetic field operators. What can you say about the possibility of making simultaneous measurements of $\mathbf{B}$ and $\mathbf{E}$?

The quantum mechanical operators for the electric and magnetic fields of each independent mode, denoted $\mathbf{k},\lambda$, are given by

$$\hat{\mathbf{E}}_{\mathbf{k},\lambda}(\mathbf{r},t) = \left(\frac{i\omega_{\mathbf{k},\lambda}}{c}\right)\sqrt{\frac{2\pi\hbar c^2}{\omega_{\mathbf{k},\lambda}}}\left[\hat{a}_{\mathbf{k},\lambda}\mathbf{e}_{\mathbf{k},\lambda}\exp\left(i\theta_{\mathbf{k},\lambda}\right) - \hat{a}^\dagger_{\mathbf{k},\lambda}\mathbf{e}^*_{\mathbf{k},\lambda}\exp\left(-i\theta_{\mathbf{k},\lambda}\right)\right] \tag{1}$$

$$\hat{\mathbf{B}}_{\mathbf{k},\lambda}(\mathbf{r},t) = i\sqrt{\frac{2\pi\hbar c^2}{\omega_{\mathbf{k},\lambda}}}\left[\hat{a}_{\mathbf{k},\lambda}(\mathbf{k}\times\mathbf{e}_{\mathbf{k},\lambda})\exp\left(i\theta_{\mathbf{k},\lambda}\right) - \hat{a}^\dagger_{\mathbf{k},\lambda}(\mathbf{k}\times\mathbf{e}^*_{\mathbf{k},\lambda})\exp\left(-i\theta_{\mathbf{k},\lambda}\right)\right] , \tag{2}$$

with the wave carrier $\theta_{\mathbf{k},\lambda} = \mathbf{k}\cdot\mathbf{r} - \omega t$, where $\mathbf{k}$ is the wave vector and ω is the frequency of the EM wave. The total fields are the sum over all modes of these fields. The polarization state of the wave is represented by λ, and can take the values $1, 2$ representing the two polarization states, we will use the shorthand k to represent $\mathbf{k},\lambda$. The unit vector $\mathbf{e}_{\mathbf{k},\lambda}$ and its complex conjugate $\mathbf{e}^*_{\mathbf{k},\lambda}$ represent the basis vectors for the polarization.Additionally, $\hat{a}_{\mathbf{k},\lambda}$ and $\hat{a}^\dagger_{\mathbf{k},\lambda}$ are the photon creation and annihilation operators, respectively. Note the coefficient for the electric field can be written as $i\sqrt{2\pi\hbar\omega_k}$, and both these fields are measured in units of energy density, so there is an implicit factor of $1/V$ included in the square roots, which we will take to be unity.

The operators only act on a specific mode, and therefore the creation and annihilation operators of different phases commute, by this reasoning, we can find the commutator of a single mode and sum over all modes to find the commutator of the vector components of each total field:

$$\begin{cases} \hat{\mathbf{E}} &= \sum_k \hat{\mathbf{E}}_k(\mathbf{r},t) \\ \hat{\mathbf{B}} &= \sum_k \hat{\mathbf{B}}_k(\mathbf{r},t) \end{cases} \quad \Rightarrow \quad [\hat{\mathbf{E}}^{(i)}, \hat{\mathbf{B}}^{(j)}] = \sum_k [\hat{\mathbf{E}}^{(i)}_k, \hat{\mathbf{B}}^{(j)}_k] , \tag{3}$$

because $[\hat{a}_k, \hat{a}_{k'}] = 0 \ \forall\{k,k'\}$, and similarly for $\hat{a}^\dagger_k$. The commutators $[\hat{a}_k, \hat{a}^\dagger_{k'}] = [\hat{a}^\dagger_k, \hat{a}_{k'}] = 0$ only for $k \neq k'$. Note we used superscripts to denote which component of the vector we are referring to. In class, when we constructed these operators from the vector potential, we noted these fields depend on the coordinates of $\mathbf{r}$, *i.e.*, $\{x,y,z\}$, and not the quantum-mechanical position operators. Therefore we have that

$$[\hat{a}_k, x^a y^b z^c] = 0 = [\hat{a}^\dagger_k, x^a y^b z^c] , \tag{4}$$

so that we can expand the exponentials into infinite sums, where the operators $\hat{a}_k$ and $\hat{a}^\dagger_k$ commute with each term, so we are free to swap the positions of any factors we choose. The field operators for a single mode can be written

$$\hat{\mathbf{E}}_k(\mathbf{r},t) = i\sqrt{2\pi\hbar\omega_k}\left[\mathbf{e}_k\exp\left(i\mathbf{k}\cdot\mathbf{r}\right)\exp(-i\omega_k t)\hat{a}_k - \mathbf{e}^*_k\exp\left(-i\mathbf{k}\cdot\mathbf{r}\right)\exp(i\omega_k t)\hat{a}^\dagger_k\right] \tag{5}$$

$$\hat{\mathbf{B}}_k(\mathbf{r},t) = i\sqrt{\frac{2\pi\hbar c^2}{\omega_k}}\left[(\mathbf{k}\times\mathbf{e}_k)\exp\left(i\mathbf{k}\cdot\mathbf{r}\right)\exp(-i\omega_k t)\hat{a}_k - (\mathbf{k}\times\mathbf{e}^*_k)\exp\left(-i\mathbf{k}\cdot\mathbf{r}\right)\exp(i\omega_k t)\hat{a}^\dagger_k\right] . \tag{6}$$

Let us look at the ith component of the electric field and the jth component of the magnetic field, with $\{i,j\} \in \{1,2,3\}$, where the indices correspond to the x,y,z directions, respectively. This is accomplished by taking the dot product of the field operators with the unit vector in the specific direction

$$\hat{\mathbf{E}}^{(i)}(\mathbf{r},t) = \hat{\mathbf{i}} \cdot \hat{\mathbf{E}}_k(\mathbf{r},t) = i\sqrt{2\pi\hbar\omega_k}\left[(\hat{\mathbf{i}} \cdot \mathbf{e}_k)e^{i\theta_k(\mathbf{r})}\hat{a}_k - (\hat{\mathbf{i}} \cdot \mathbf{e}_k^*)e^{-i\theta_k(\mathbf{r})}\hat{a}_k^\dagger\right] \tag{7}$$

$$\hat{\mathbf{B}}^{(j)}(\mathbf{r}',t) = \hat{\mathbf{j}} \cdot \hat{\mathbf{B}}_k(\mathbf{r}',t) = i\sqrt{\frac{2\pi\hbar c^2}{\omega_k}}\left[\hat{\mathbf{j}} \cdot (\mathbf{k} \times \mathbf{e}_k)e^{i\theta_k(\mathbf{r}')}\hat{a}_k - \hat{\mathbf{j}} \cdot (\mathbf{k} \times \mathbf{e}_k^*)e^{-i\theta_k(\mathbf{r}')}\hat{a}_k^\dagger\right] \ , \tag{8}$$

Let us introduce index notation, such that $\hat{\mathbf{i}} \cdot \mathbf{e}_k \to (\mathbf{e}_k)_i$ and $\hat{\mathbf{i}} \cdot \mathbf{e}_k^* \to (\mathbf{e}_k^*)_i$, the cross products then become

$$\hat{\mathbf{j}} \cdot (\mathbf{k} \times \mathbf{e}_k) = (\mathbf{k} \times \mathbf{e}_k)_j = \epsilon_{jlm}(\mathbf{k})_l(\mathbf{e}_k)_m \tag{9}$$

$$\hat{\mathbf{j}} \cdot (\mathbf{k} \times \mathbf{e}_k^*) = (\mathbf{k} \times \mathbf{e}_k^*)_j = \epsilon_{jlm}(\mathbf{k})_l(\mathbf{e}_k^*)_m \ , \tag{10}$$

where ϵ_{jlm} is the Levi-Civita tensor. Using this notation, the components of the field operators are

$$\hat{\mathbf{E}}^{(i)}(\mathbf{r},t) = i\sqrt{2\pi\hbar\omega_k}\left[(\mathbf{e}_k)_i e^{i\theta_k(\mathbf{r})}\hat{a}_k - (\mathbf{e}_k^*)_i e^{-i\theta_k(\mathbf{r})}\hat{a}_k^\dagger\right] \tag{11}$$

$$\hat{\mathbf{B}}^{(j)}(\mathbf{r}',t) = i\sqrt{\frac{2\pi\hbar c^2}{\omega_k}}\epsilon_{jlm}(\mathbf{k})_l\left[(\mathbf{e}_k)_m e^{i\theta_k(\mathbf{r}')}\hat{a}_k - (\mathbf{e}_k^*)_m e^{-i\theta_k(\mathbf{r}')}\hat{a}_k^\dagger\right] \ , \tag{12}$$

and their commutator is

$$[\hat{\mathbf{E}}_k^{(i)}(\mathbf{r}), \hat{\mathbf{B}}_k^{(j)}(\mathbf{r}')] = \\ -(2\pi\hbar c)\epsilon_{jlm}(\mathbf{k})_l\left[(\mathbf{e}_k)_i e^{i\theta_k(\mathbf{r})}\hat{a}_k - (\mathbf{e}_k^*)_i e^{-i\theta_k(\mathbf{r})}\hat{a}_k^\dagger \ , \ (\mathbf{e}_k)_m e^{i\theta_k(\mathbf{r}')}\hat{a}_k - (\mathbf{e}_k^*)_m e^{-i\theta_k(\mathbf{r}')}\hat{a}_k^\dagger\right] \ , \tag{13}$$

and since both operators are in the same mode, we have that $[\hat{a}_k, \hat{a}_k] = [\hat{a}_k^\dagger, \hat{a}_k^\dagger] = 0$, so only two terms of the commutator survive. In these terms, the time-varying exponential factors have opposite signs and multiply to unity, while the position exponentials sum.

$$[\hat{\mathbf{E}}_k^{(i)}(\mathbf{r}), \hat{\mathbf{B}}_k^{(j)}(\mathbf{r}')] = \\ (2\pi\hbar c)\epsilon_{jlm}(\mathbf{k})_l\left\{(\mathbf{e}_k)_i(\mathbf{e}_k^*)_m e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}[\hat{a}_k, \hat{a}_k^\dagger] + (\mathbf{e}_k^*)_i(\mathbf{e}_k)_m e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}[\hat{a}_k^\dagger, \hat{a}_k]\right\} \ , \tag{14}$$

with the commutation relations $[\hat{a}_k, \hat{a}_k^\dagger] = 1$ and $[\hat{a}_k^\dagger, \hat{a}_k] = -1$. Note we have four indices $\{i,j,l,m\} \in \{1,2,3\}$ which are being summed over. The terms where $m = j$ or $m = l$ are zero due to the Levi-Civita symbol (repeated indices: $\epsilon_{aba} = \epsilon_{abb} = 0$), so we the only surviving terms are with $m = i$. Using the commutator relations, and the fact that $(\mathbf{e}_k^*)_i(\mathbf{e}_k)_i = (\mathbf{e}_k)_i(\mathbf{e}_k^*)_i = 1$, we have

$$[\hat{\mathbf{E}}_k^{(i)}(\mathbf{r}), \hat{\mathbf{B}}_k^{(j)}(\mathbf{r}')] = (2\pi\hbar c)\epsilon_{jli}(\mathbf{k})_l\left\{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} - e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\right\} \ , \tag{15}$$

so the commutator over all modes is

$$[\hat{\mathbf{E}}^{(i)}(\mathbf{r}), \hat{\mathbf{B}}^{(j)}(\mathbf{r}')] = (2\pi\hbar c)\epsilon_{jli}\sum_k(\mathbf{k})_l\left\{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} - e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\right\} \tag{16}$$

which is a sum over all wave vectors and polarization states. Consider the infinite sum:

$$\sum_k\left\{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} + e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\right\} \ , \tag{17}$$

which after differentiating with respect to $\mathbf{r}_l$ is

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{r}_l}\sum_k \left\{ e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} + e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \right\} = \sum_k \left\{ (i\mathbf{k}_l) e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} + (-i\mathbf{k}_l) e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \right\} \tag{18}$$

$$= i\sum_k (\mathbf{k}_l) \left\{ e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} - e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \right\} \ . \tag{19}$$

Inserting this result into the commutator, we see

$$[\hat{\mathbf{E}}^{(i)}(\mathbf{r}), \hat{\mathbf{B}}^{(j)}(\mathbf{r}')] = (2\pi\hbar c)\epsilon_{jli} \left(\frac{1}{i}\right) \frac{\mathrm{d}}{\mathrm{d}\mathbf{r}_l} \sum_k \left\{ e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} + e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \right\} \tag{20}$$

$$= -i(2\pi\hbar c)\epsilon_{jli} \frac{\mathrm{d}}{\mathrm{d}\mathbf{r}_l} \left\{ \delta(\mathbf{r}-\mathbf{r}') + \delta(\mathbf{r}'-\mathbf{r}) \right\} \tag{21}$$

where we have used the relation[1]

$$V\delta(\mathbf{r}'-\mathbf{r}) = \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \ , \tag{22}$$

and canceled the V with the two implicit factors of $1/\sqrt{V}$ mentioned earlier. Since the delta function is symmetric under swapping $\mathbf{r}$ and $\mathbf{r}'$, we can write the commutator as

$$[\hat{\mathbf{E}}^{(i)}(\mathbf{r}), \hat{\mathbf{B}}^{(j)}(\mathbf{r}')] = -i(4\pi\hbar c)\epsilon_{ijl} \frac{\mathrm{d}}{\mathrm{d}\mathbf{r}_l} \delta(\mathbf{r}-\mathbf{r}') \tag{23}$$

note we have permuted the Levi-Civita symbol. By induction, we have

$$[\hat{\mathbf{B}}^{(i)}(\mathbf{r}), \hat{\mathbf{E}}^{(j)}(\mathbf{r}')] = i(4\pi\hbar c)\epsilon_{ijl} \frac{\mathrm{d}}{\mathrm{d}\mathbf{r}_l} \delta(\mathbf{r}-\mathbf{r}') \ , \tag{24}$$

so we see that we may always simultaneously measure the electric and magnetic field amplitudes in any direction, except at the same point in space. The commutator is non zero when $\mathbf{r} = \mathbf{r}'$, however in the case $i = j$ the commutator is zero even at $\mathbf{r} = \mathbf{r}'$ due to the Levi-Civita symbol. This implies that we may simultaneously measure the electric and magnetic field amplitudes along the same direction even at the same point in space. We assumed a linear polarization basis in the beginning, but the result is the same as in a circular polarization basis, so the derivation is basis-independent.

[1] Wysin, "Quantization of the Free Electromagnetic Field: Photons and Operators", equation 58.

2 Free Radiation Field.

Consider the free radiation field in the gauge, $\Phi = 0$ and $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$.

2.1 Electric Field Operator.

Write down (or derive if necessary) the operator for the electric field in terms of creation and annihilation operators for photons with precise momentum and polarization.

Given a photon with precise momentum $\mathbf{k}$ and polarization λ (we denote $\{\mathbf{k}, \lambda\}$ as k), the electric field operator is given by

$$\hat{\mathbf{E}}_k(\mathbf{r}, t) = i\sqrt{2\pi\hbar\omega_k}\left[\mathbf{e}_k \exp\left(i\mathbf{k}\cdot\mathbf{r}\right)\exp(-i\omega t)\hat{a}_k - \mathbf{e}_k^* \exp\left(-i\mathbf{k}\cdot\mathbf{r}\right)\exp(i\omega t)\hat{a}_k^\dagger\right] \ , \tag{25}$$

see section 1 for an explanation.

2.2 Expectation Values of Electric Field of a Fock State.

Calculate the mean and rms values of the electric field for a single mode of the radiation field with exactly N photons in the mode $(k\hat{\mathbf{z}}, \hat{\mathbf{x}})$ where $\mathbf{k} = k\hat{\mathbf{z}}$ is the wavevector and $\hat{\mathbf{x}}$ is the polarization of the electric field.

A state with exactly N photons is a Fock state defined by $|N\rangle$; since we are considering a single state, we can drop the index k. The mean electric field in this state is given by $\langle N|\hat{\mathbf{E}}|N\rangle$. If we are linearly polarized along $\hat{\mathbf{x}}$, we can align the coordinate system such that we can express the electric field operator for this mode as[2]

$$\hat{\mathbf{E}}(\mathbf{r}, t) = i\sqrt{2\pi\hbar\omega}\hat{\mathbf{x}}\left[e^{i(kz-\omega_k t)}\hat{a} - e^{-i(kz-\omega_k t)}\hat{a}^\dagger\right] \ , \tag{26}$$

and its matrix element with a Fock state is then

$$\langle N|\hat{\mathbf{E}}|N\rangle = i\sqrt{2\pi\hbar\omega}\hat{\mathbf{x}}\left[e^{i(kz-\omega t)}\langle N|\hat{a}|N\rangle - e^{-i(kz-\omega t)}\langle N|\hat{a}^\dagger|N\rangle\right] \tag{27}$$

$$= i\sqrt{2\pi\hbar\omega}\hat{\mathbf{x}}\left[\sqrt{N}e^{i(kz-\omega t)}\langle N|N-1\rangle - \sqrt{N+1}e^{-i(kz-\omega t)}\langle N|N+1\rangle\right] = 0 \ , \tag{28}$$

because Fock states of different numbers (and modes) are orthogonal. Similarly, the root-mean-squared fluctuations in the electric field in this Fock state are given by

$$\Delta\hat{E}_{\text{rms}} = \sqrt{\langle N|\hat{E}^2|N\rangle} \ , \tag{29}$$

with the square of the electric field given by

$$\hat{E}^2 = \hat{\mathbf{E}}\cdot\hat{\mathbf{E}} = -(2\pi\hbar\omega)\left[e^{2i(kz-\omega t)}(\hat{a})^2 + e^{-2i(kz-\omega t)}(\hat{a}^\dagger)^2 - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\right] \tag{30}$$

$$= (2\pi\hbar\omega)(\hat{N}^\dagger + \hat{N}) \ , \tag{31}$$

where $\hat{N}$ is the photon number operator. We note that the action of raising or lowering the occupancy of a Fock state twice will result in taking the inner product of orthogonal states, which is zero[3]. We now take the matrix element with the Fock state of N photons

$$\langle N|\hat{E}^2|N\rangle = 2\pi\hbar\omega\left[\langle N|\hat{N}^\dagger|N\rangle + \langle N|\hat{N}|N\rangle\right] = 2\pi\hbar\omega\left[N + 1 + N\right] \ , \tag{32}$$

[2] As was done in class on April 15.

[3] See the reasoning presented in section 4.

so the rms fluctuation of the electric field of a Fock state is

$$\Delta \hat{E}_{\rm rms} = \sqrt{\langle N|\hat{E}^2|N\rangle} = \sqrt{2\pi\hbar\omega(1+2N)} \ . \tag{33}$$

2.3 Expectation Values of Electric Field of a Glauber State.

Repeat part 2.2 for a Glauber state of the radiation field with the same polarization, same mode number and mean number of photons N.

A Glauber state $|\alpha\rangle$ is an eigenstate of the annihilation operator $\hat{a}$ (with eigenvalue α), so the mean electric field can be found by evaluating the matrix element

$$\langle\alpha|\hat{\mathbf{E}}|\alpha\rangle = i\sqrt{2\pi\hbar\omega}\hat{\mathbf{x}}\left[e^{i(kz-\omega t)}\langle\alpha|\hat{a}|\alpha\rangle - e^{-i(kz-\omega t)}\langle\alpha|\hat{a}^\dagger|\alpha\rangle\right] \tag{34}$$

$$= i\sqrt{2\pi\hbar\omega}\hat{\mathbf{x}}\left[e^{i(kz-\omega t)}\alpha - e^{-i(kz-\omega t)}\alpha^*\right] \ . \tag{35}$$

If we write the complex value α in exponential form, $\alpha = |\alpha|e^{i\phi}$, we see the expectation value above becomes

$$\langle\alpha|\hat{\mathbf{E}}|\alpha\rangle = i\sqrt{2\pi\hbar\omega}|\alpha|\hat{\mathbf{x}}\left[e^{i(kz-\omega t)+i\phi} - e^{-i(kz-\omega t)-i\phi}\right] = i\sqrt{2\pi\hbar\omega}|\alpha|\left[2i\sin(kz-\omega t+\phi)\right]\hat{\mathbf{x}} \tag{36}$$

$$= -2|\alpha|\sqrt{2\pi\hbar\omega}\sin(kz-\omega t+\phi)\hat{\mathbf{x}} \ . \tag{37}$$

Let us now take an aside to examine the expectation value of the number operator in a Glauber state $|\alpha\rangle$:

$$N = \langle\alpha|\hat{N}|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = \left[\hat{a}\,|\alpha\rangle\right]^\dagger\hat{a}\,|\alpha\rangle = \alpha^*\alpha = |\alpha|^2 \ . \tag{38}$$

Using this, the mean value of the electric field of a Glauber state is

$$\langle\alpha|\hat{\mathbf{E}}|\alpha\rangle = -2\sqrt{2\pi\hbar\omega N}\sin(kz-\omega t+\phi)\hat{\mathbf{x}} \ . \tag{39}$$

The operator the square of the electric field is

$$\hat{\mathbf{E}}\cdot\hat{\mathbf{E}} = -(2\pi\hbar\omega)\left(e^{i\theta_k}\hat{a} - e^{-i\theta_k}\hat{a}^\dagger\right)\left(e^{i\theta_k}\hat{a} - e^{-i\theta_k}\hat{a}^\dagger\right)(\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}) \ , \tag{40}$$

using the notation $\theta_k = kz - \omega t$. Executing the dot product yields

$$\hat{\mathbf{E}}\cdot\hat{\mathbf{E}} = -(2\pi\hbar\omega)\left(e^{2i\theta_k}\hat{a}\hat{a} + e^{-2i\theta_k}\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\right) \ . \tag{41}$$

Consider the mean values of the following operators in a Glauber state:

$$\langle\alpha|\hat{a}\hat{a}|\alpha\rangle = (\alpha)\langle\alpha|\hat{a}|\alpha\rangle = \alpha^2 = |\alpha|^2e^{2i\phi} = Ne^{2i\phi} \tag{42}$$

$$\langle\alpha|\hat{a}\hat{a}^\dagger|\alpha\rangle = \langle\alpha|[\hat{a},\hat{a}^\dagger] + \hat{a}^\dagger\hat{a}|\alpha\rangle = 1 + \langle\alpha|\hat{N}|\alpha\rangle = N+1 \tag{43}$$

$$\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = N \tag{44}$$

$$\langle\alpha|\hat{a}^\dagger\hat{a}^\dagger|\alpha\rangle = (\alpha^*)^2 = Ne^{-2i\phi} \ , \tag{45}$$

note we have identified the operator $\hat{N}^\dagger$ which has eigenvalues $N+1$, with N as the occupancy number of a Fock state, or the mean occupancy number of a Glauber state. Using these mean values, we get the mean value of the squared electric field:

$$\langle\alpha|\hat{E}^2|\alpha\rangle = -(2\pi\hbar\omega)\left(e^{2i\theta_k}Ne^{2i\phi} + e^{-2i\theta_k}Ne^{-2i\phi} - (2N+1)\right) \tag{46}$$

$$= -2\pi\hbar\omega N\left(e^{2i(\theta_k+\phi)} + e^{-2i(\theta_k+\phi)} - 2 - \frac{1}{N}\right) \tag{47}$$

$$= -4\pi\hbar\omega N\left(\cos\{2(\theta_k+\phi)\} - 1 - \frac{1}{2N}\right) \tag{48}$$

$$= -4\pi\hbar\omega N\left(1 - 2\sin^2(\theta_k+\phi) - 1 - \frac{1}{2N}\right) \tag{49}$$

$$= 4\pi\hbar\omega N\left(2\sin^2(\theta_k+\phi) + \frac{1}{2N}\right) = 8\pi\hbar\omega N\left(\sin^2(\theta_k+\phi) + \frac{1}{4N}\right) . \tag{50}$$

The rms value of the electric field is then

$$\delta\hat{E}_{\text{rms}} = \sqrt{\langle\alpha|\hat{\mathbf{E}}\cdot\hat{\mathbf{E}}|\alpha\rangle - \langle\alpha|\hat{\mathbf{E}}|\alpha\rangle\cdot\langle\alpha|\hat{\mathbf{E}}|\alpha\rangle} \tag{51}$$

$$= \sqrt{4\pi\hbar\omega N\left(2\sin^2(\theta_k+\phi) + \frac{1}{2N}\right) - 4(2\pi\hbar\omega N)\sin^2(\theta_k+\phi)} \tag{52}$$

$$= \sqrt{2\pi\hbar\omega} . \tag{53}$$

The fractional error is then

$$\frac{\delta\hat{E}_{\text{rms}}}{\sqrt{|\mathbf{E}|^2}} = \frac{\sqrt{2\pi\hbar\omega}}{2\sqrt{2\pi\hbar\omega N}} = \frac{1}{2\sqrt{N}} , \tag{54}$$

if we ignore the carrier wave.

2.4 Probability Distribution of Mode Occupancy.

If the Glauber state in part 2.3 is described by the complex amplitude $\alpha = |\alpha|e^{i\phi}$, calculate the probability distribution for measuring n photons. Express your answer in terms of n, ϕ and the mean number of photons, N, for the Glauber state.

A Glauber state can be represented as a sum over all Fock states (in this mode):

$$|\alpha\rangle = e^{-N/2}\sum_{m=0}^{\infty}\frac{N^{m/2}}{\sqrt{m!}}e^{im\phi}|m\rangle . \tag{55}$$

The probability of measuring a Glauber to have n photons is given by the square of the inner product of the Glauber state and the nth Fock state:

$$P(n) = |\langle n|\alpha\rangle|^2 = e^{-N}\sum_{m=0}^{\infty}\frac{N^m}{m!}e^{im\phi}e^{-im\phi}|\langle n|m\rangle|^2 = e^{-N}\frac{N^n}{n!} , \tag{56}$$

because the inner product results in a delta function δ_{mn} so the infinite sum is only nonzero for $m = n$. A plot of this distribution is shown in Figure 1 (on page 12) for $N = 10$ photons, and as expected is peaked at $n = 10$.

3 Momentum of Electromagnetic Waves.

The energy flux transported by the electromagnetic field is given by Poynting's vector, $\mathbf{S}$. The electromagnetic field also transports momentum, observable as radiation pressure, and is given by $\mathbf{g}(\mathbf{r},t) = \frac{1}{4\pi c}(\mathbf{E}\times\mathbf{B})$. Thus, the total momentum transported by the classical EM field is given by

$$\mathbf{P} = \frac{1}{4\pi c}\int \mathrm{d}^3 r (\mathbf{E}\times\mathbf{B}) \ . \tag{57}$$

Obtain the corresponding QM operator in terms of photon creation and annihilation operators, $\hat{a}^\dagger_{\mathbf{k}\lambda}$ and $\hat{a}_{\mathbf{k}\lambda}$. Consider radiation in a "box" of volume $V = L^3$ with periodic boundary conditions, in the limit $L \to \infty$. Note: Be careful to correctly identify the quantum mechanical operator corresponding to the classical momentum, $\mathbf{P}$.

The quantum mechanical operator for the total momentum must be a symmeterized version of Equation 57:

$$\hat{\mathbf{P}} = \frac{1}{4\pi c}\int \mathrm{d}^3 r \ \frac{1}{2}\left(\hat{\mathbf{E}}\times\hat{\mathbf{B}} - \hat{\mathbf{B}}\times\hat{\mathbf{E}}\right) \ , \tag{58}$$

which reduces to the classical expression if the field operators commute. Let us write the field operators using a linear polarization:

$$\hat{\mathbf{E}} = \frac{1}{\sqrt{L^3}}\sum_k i\sqrt{2\pi\hbar\omega_k}\mathbf{e}_k\left(e^{i\theta_k}\hat{a}_k - e^{-i\theta_k}\hat{a}^\dagger_k\right) \tag{59}$$

$$\hat{\mathbf{B}} = \frac{1}{\sqrt{L^3}}\sum_k i\sqrt{2\pi\hbar\omega_k}(\hat{\mathbf{k}}\times\mathbf{e}_k)\left(e^{i\theta_k}\hat{a}_k - e^{-i\theta_k}\hat{a}^\dagger_k\right) \ , \tag{60}$$

where $\theta_k = \mathbf{k}\cdot\mathbf{r} - \omega t$, $\mathbf{e}_k$ is a real unit vector, and $\hat{\mathbf{k}}$ is a unit vector in the direction of $\mathbf{k}$. We are still using the notation that k denotes a specific mode with wave vector $\mathbf{k}$ and polarization state λ. Note we have explicitly written the volume term above, whereas previously it was implied. Additionally, the box quantization leads to the canonical commutation relation.

Now consider the cross-products:

$$\hat{\mathbf{E}}\times\hat{\mathbf{B}} = -\frac{1}{L^3}\sum_k\sum_{k'}(2\pi\hbar\sqrt{\omega_k\omega_{k'}})\left[\mathbf{e}_k\times(\hat{\mathbf{k}'}\times\mathbf{e}_{k'})\right]\left(e^{i\theta_k}\hat{a}_k - e^{-i\theta_k}\hat{a}^\dagger_k\right)\left(e^{i\theta_{k'}}\hat{a}_{k'} - e^{-i\theta_{k'}}\hat{a}^\dagger_{k'}\right)$$

$$\hat{\mathbf{B}}\times\hat{\mathbf{E}} = -\frac{1}{L^3}\sum_k\sum_{k'}(2\pi\hbar\sqrt{\omega_k\omega_{k'}})\left[(\hat{\mathbf{k}'}\times\mathbf{e}_{k'})\times\mathbf{e}_k\right]\left(e^{i\theta_{k'}}\hat{a}_{k'} - e^{-i\theta_{k'}}\hat{a}^\dagger_{k'}\right)\left(e^{i\theta_k}\hat{a}_k - e^{-i\theta_k}\hat{a}^\dagger_k\right) \ ,$$

note the vector products in each are identical, up to a negative sign (obtained by swapping the order of the cross product outside the parenthesis), therefore the quantity $\frac{1}{2}\left(\hat{\mathbf{E}}\times\hat{\mathbf{B}} - \hat{\mathbf{B}}\times\hat{\mathbf{E}}\right)$ reduces to the classical value[4] $\mathbf{E}\times\mathbf{B}$.

Now let's investigate the product:

$$\int \mathrm{d}^3 x \ \left(e^{i\theta_k}\hat{a}_k - e^{-i\theta_k}\hat{a}^\dagger_k\right)\left(e^{i\theta_{k'}}\hat{a}_{k'} - e^{-i\theta_{k'}}\hat{a}^\dagger_{k'}\right) \ , \tag{61}$$

[4]This can be seen using the commutator of the field operators:

$$\frac{1}{2}\left(\hat{\mathbf{E}}\times\hat{\mathbf{B}} - \hat{\mathbf{B}}\times\hat{\mathbf{E}}\right)_i = \frac{1}{2}\epsilon_{ijk}(E_jB_k - B_jE_k) = \frac{1}{2}\epsilon_{ijk}(E_jB_k - [B_j,E_k] - E_kB_j) = \frac{1}{2}\epsilon_{ijk}(E_jB_k - B_jE_k)$$

which results in terms that, up to some operators, look like:

$$\int \mathrm{d}^3x\ e^{i\theta_k} e^{i\theta_{k'}} = e^{-2i\omega t} \int \mathrm{d}^3x\ e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} = e^{-2i\omega t}\delta(\mathbf{k}+\mathbf{k}')L^3 \tag{62}$$

$$\int \mathrm{d}^3x\ e^{i\theta_k} e^{-i\theta_{k'}} = \int \mathrm{d}^3x\ e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} = \delta(\mathbf{k}-\mathbf{k}')L^3 \tag{63}$$

$$\int \mathrm{d}^3x\ e^{-i\theta_k} e^{i\theta_{k'}} = \int \mathrm{d}^3x\ e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} = \delta(\mathbf{k}-\mathbf{k}')L^3 \tag{64}$$

$$\int \mathrm{d}^3x\ e^{-i\theta_k} e^{-i\theta_{k'}} = e^{+2i\omega t} \int \mathrm{d}^3x\ e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} = e^{+2i\omega t}\delta(\mathbf{k}+\mathbf{k}')L^3\ , \tag{65}$$

by noting the orthogonality condition. We can represent the delta functions as Kronecker deltas: $\delta(\mathbf{k}+\mathbf{k}') \to \delta_{k,-k'}$ and $\delta(\mathbf{k}-\mathbf{k}') \to \delta_{k,k'}$. Using all of this, we can write the expression for the momentum operator as

$$\hat{\mathbf{P}} = -\frac{1}{L^3}\frac{\hbar}{2c}\sum_k\sum_{k'}\sqrt{\omega_k\omega_{k'}}\left[\mathbf{e}_k \times (\hat{\mathbf{k}'} \times \mathbf{e}_{k'})\right]\int \mathrm{d}^3r\left(e^{i\theta_k}\hat{a}_k - e^{-i\theta_k}\hat{a}_k^\dagger\right)\left(e^{i\theta_{k'}}\hat{a}_{k'} - e^{-i\theta_{k'}}\hat{a}_{k'}^\dagger\right) \tag{66}$$

$$= -\frac{\hbar}{2c}\sum_k\sum_{k'}\sqrt{\omega_k\omega_{k'}}\hat{\mathbf{p}}\left(\hat{a}_k\hat{a}_{k'}e^{-2i\omega t}\delta_{k,-k'} + \hat{a}_k^\dagger\hat{a}_{k'}^\dagger e^{+2i\omega t}\delta_{k,-k'} - \hat{a}_k\hat{a}_{k'}^\dagger\delta_{k,k'} - \hat{a}_k^\dagger\hat{a}_{k'}\delta_{k,k'}\right)\ , \tag{67}$$

where $\hat{\mathbf{p}}$ is a unit vector parallel to the momentum vector, given by

$$\hat{\mathbf{p}} = \mathbf{e}_k \times (\hat{\mathbf{k}'} \times \mathbf{e}_{k'})\ . \tag{68}$$

Consider the case when $\mathbf{k} = \mathbf{k}'$, in this case the inner cross product results in a unit vector mutually orthogonal to both $\mathbf{k}$ and $\mathbf{e}_k$, because they themselves are orthogonal. Then the remaining cross product is that of two orthogonal unit vectors, which must be the third mutually orthogonal unit vector[5], in this case $\mathbf{k}$. Now consider the case $-\mathbf{k} = \mathbf{k}'$, writing out k in full detail $(\mathbf{k}, \lambda)$ where λ is the polarization and can only take two values ± 1. The unit vector in the direction of momentum is then

$$\hat{\mathbf{p}} = \mathbf{e}_{\mathbf{k},\pm1} \times (-\hat{\mathbf{k}} \times \mathbf{e}_{-\mathbf{k},\pm1}) = \mathbf{e}_{\mathbf{k},\pm1} \times (-\hat{\mathbf{k}} \times \mathbf{e}_{\mathbf{k},\mp1}) = -\mathbf{e}_{\mathbf{k},\pm1} \times \mathbf{e}_{\mathbf{k},\pm1} = 0\ , \tag{69}$$

so the only nonzero components in the double sum are for $k = k'$ (wave vector and polarization). Let us now carry out the sum over k':

$$\hat{\mathbf{P}} = -\frac{\hbar}{2c}\sum_k \hat{\mathbf{k}}\left\{-\omega_k\left(\hat{a}_k\hat{a}_k^\dagger + \hat{a}_k^\dagger\hat{a}_k\right)\right\} = \frac{1}{2}\hbar\sum_k\frac{\omega_k}{c}\hat{\mathbf{k}}\left(\hat{a}_k\hat{a}_k^\dagger + \hat{a}_k^\dagger\hat{a}_k\right)\ . \tag{71}$$

Using the dispersion relation $\omega_k = c|\mathbf{k}|$ and identifying the operators $\hat{N}$ and $\hat{N}^\dagger$ (see Equation 43) using the box quantization (leading to the canonical commutation relation) allows us to write

$$\hat{\mathbf{P}} = \frac{1}{2}\sum_k \hbar\mathbf{k}\left(2\hat{N}+1\right) = \sum_{\mathbf{k}=-\infty}^{\infty}\hbar\mathbf{k}N_{\mathbf{k}}\ , \tag{72}$$

by noting that the constant term contributed by $\mathbf{k}$ is canceled by the term contributed by $-\mathbf{k}$.

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which simplifies to $\frac{1}{2}\epsilon_{ijk}(2E_jB_k - [B_j, E_k])$. If we integrate this expression over all space, we see

$$\epsilon_{ijk}\left\{\int \mathrm{d}^3r\ E_jB_k - \frac{1}{2}\int \mathrm{d}^3r\ [B_j, E_k]\right\} = \epsilon_{ijk}\left\{\int \mathrm{d}^3r\ E_jB_k - \frac{1}{2}\int \mathrm{d}^3r\ i(4\pi\hbar c)\epsilon_{ijl}\delta'(\mathbf{r}-\mathbf{r}')\right\} = \epsilon_{ijk}\int \mathrm{d}^3r\ E_jB_k$$

where we have used the identity $\int \mathrm{d}^3x\delta'(\mathbf{x}-\mathbf{x}_0)f(\mathbf{x}) = f'(\mathbf{x}_0)$.

[5] Consider aligning the wave vector with the z axis and $\mathbf{e}_k$ with the x, then:

$$\mathbf{k}' = \mathbf{k}:\ \hat{\mathbf{x}} \times (\hat{\mathbf{z}} \times \hat{\mathbf{x}}) = \hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}\ . \tag{70}$$

4 Electric Field Correlation Function.

Calculate the expectation value, $\langle 0|\hat{\mathbf{E}}(\mathbf{r})\cdot\hat{\mathbf{E}}(\mathbf{r}')|0\rangle$, for the "electric field-electric field correlation function" at different space points in vacuum.

Immediately, we can argue that this quantity must be zero for $\mathbf{r}\neq\mathbf{r}'$. The electromagnetic field at two different points in space, in a vacuum state must be uncorrelated, otherwise you could determine the electric field at any point with just the electric field at another. Consider the operator

$$\hat{\mathbf{E}}(\mathbf{r})\cdot\hat{\mathbf{E}}(\mathbf{r}') = \left[\sum_k \hat{\mathbf{E}}_k(\mathbf{r})\right]\cdot\left[\sum_k \hat{\mathbf{E}}_k(\mathbf{r}')\right] \ , \tag{73}$$

which has terms with factors of $\hat{a}_k\hat{a}_{k'}$, $\hat{a}_k\hat{a}^\dagger_{k'}$, $\hat{a}^\dagger_k\hat{a}_{k'}$, and $\hat{a}^\dagger_k\hat{a}^\dagger_{k'}$ (allowing for the possibility $k=k'$). Consider the matrix element of these operators with the vacuum:

$$\langle 0|\hat{a}_k\hat{a}_{k'}|0\rangle = \left[\hat{a}^\dagger_k|0\rangle\right]^\dagger \hat{a}_{k'}|0\rangle = \langle 1_k|(0) = 0 \tag{74}$$

$$\langle 0|\hat{a}_k\hat{a}^\dagger_{k'}|0\rangle = \left[\hat{a}^\dagger_k|0\rangle\right]^\dagger \hat{a}^\dagger_{k'}|0\rangle = \langle 1_k|1_{k'}\rangle \tag{75}$$

$$\langle 0|\hat{a}^\dagger_k\hat{a}_{k'}|0\rangle = \left[\hat{a}_k|0\rangle\right]^\dagger \hat{a}_{k'}|0\rangle = (0)(0) = 0 \tag{76}$$

$$\langle 0|\hat{a}^\dagger_k\hat{a}^\dagger_{k'}|0\rangle = \left[\hat{a}_k|0\rangle\right]^\dagger \hat{a}^\dagger_{k'}|0\rangle = (0)|1_{k'}\rangle = 0 \ . \tag{77}$$

So we have the condition that

$$\langle 0|\hat{\mathbf{E}}(\mathbf{r})\cdot\hat{\mathbf{E}}(\mathbf{r}')|0\rangle = 0 \quad \text{if} \quad \langle 1_k|1_{k'}\rangle = 0 \ , \tag{78}$$

so we must prove that Fock states occupied by a single photon in different modes are orthogonal. The creation and annihilation operators of different modes obey the condition that

$$[\hat{a}_k,\hat{a}_{k'}] = [\hat{a}_k,\hat{a}^\dagger_{k'}] = [\hat{a}^\dagger_k,\hat{a}_{k'}] = [\hat{a}^\dagger_k,\hat{a}^\dagger_{k'}] = 0 \quad \text{for} \quad k\neq k' \ , \tag{79}$$

so we can express the nonzero matrix element from above as

$$\langle 0|\hat{a}_k\hat{a}^\dagger_{k'}|0\rangle = \langle 0|\hat{a}^\dagger_{k'}\hat{a}_k|0\rangle = \left[\hat{a}_{k'}|0\rangle\right]^\dagger \hat{a}_k|0\rangle = 0 \quad \text{so} \quad \langle 1_k|1_{k'}\rangle = 0 \quad \text{for} \quad k\neq k' \ , \tag{80}$$

note we could apply this logic to Equation 76 and make it equal to $\langle 1_{k'}|1_k\rangle$, but this again would be the inner product of orthogonal states, and be identically zero. This will not matter when we look at products of these operators of the same mode, because they do not commute. Therefore the only nonzero contributions for the "electric field-electric field correlation function" are from the terms

$$\langle 0|\hat{a}_k\hat{a}^\dagger_k|0\rangle = \left[\hat{a}^\dagger_k|0\rangle\right]^\dagger \hat{a}^\dagger_k|0\rangle = \langle 1_k|1_k\rangle = 1 \ , \tag{81}$$

which allows the electric field-electric field correlation operator to be written as

$$\hat{\mathbf{E}}(\mathbf{r})\cdot\hat{\mathbf{E}}(\mathbf{r}') = \sum_k \hat{\mathbf{E}}_k(\mathbf{r})\cdot\hat{\mathbf{E}}_k(\mathbf{r}') \ . \tag{82}$$

If we consider one mode (one element of the sum), from Equation 5

$$\hat{\mathbf{E}}_k(\mathbf{r})\cdot\hat{\mathbf{E}}_k(\mathbf{r}') = -(2\pi\hbar\omega_k)\left[e^{i\theta_k(\mathbf{r})}\hat{a}_k\mathbf{e}_k - e^{-i\theta_k(\mathbf{r})}\hat{a}^\dagger_k\mathbf{e}^*_k\right]\cdot\left[e^{i\theta_k(\mathbf{r}')}\hat{a}_k\mathbf{e}_k - e^{-i\theta_k(\mathbf{r}')}\hat{a}^\dagger_k\mathbf{e}^*_k\right] \ , \tag{83}$$

and it is important to note that in the complex plane, if $\mathbf{n}$ is a unit vector, it satisfies

$$\mathbf{n} \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathbf{n} \cdot \mathbf{n}^* = 1 \ , \tag{84}$$

which implies that only the cross terms survive the dot product. Using this, and the result of Equation 81, the only surviving term is

$$\hat{\mathbf{E}}_k(\mathbf{r}) \cdot \hat{\mathbf{E}}_k(\mathbf{r}') = 2\pi\hbar\omega_k(\hat{a}_k\hat{a}_k^\dagger)e^{i\theta_k(\mathbf{r})-i\theta_k(\mathbf{r}')} = 2\pi\hbar\omega_k\hat{N}_k^\dagger e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \ . \tag{85}$$

We may sum this over all k and obtain

$$\hat{\mathbf{E}}(\mathbf{r}) \cdot \hat{\mathbf{E}}(\mathbf{r}') = 4\pi\hbar\sum_{\mathbf{k}}\omega_k\hat{N}_k^\dagger e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \ , \tag{86}$$

where we have carried out the sum over the polarization to pick up the extra factor of two. The mean of this quantity in the vacuum is

$$\langle 0|\hat{\mathbf{E}}(\mathbf{r}) \cdot \hat{\mathbf{E}}(\mathbf{r}')|0\rangle = 4\pi\hbar\sum_{\mathbf{k}}\omega_k \langle 0|\hat{N}_k^\dagger|0\rangle\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} = 4\pi\hbar c\sum_{\mathbf{k}}|\mathbf{k}|e^{-i\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})} \ . \tag{87}$$

To evaluate the infinite sum, we approximate it as an integral

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}^3k}{(2\pi)^3}\ |\mathbf{k}|e^{-i\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})} = \frac{1}{(2\pi)^3}\int_0^{2\pi}\mathrm{d}\phi\int_{-1}^{1}\mathrm{d}(\cos)\theta\int_0^\infty(k^2\mathrm{d}k)ke^{-ik|\mathbf{r}-\mathbf{r}'|\cos\theta} \ , \tag{88}$$

after writing $\mathbf{k}$ in spherical coordinates. We can immediately carry out the azimuthal integral yielding

$$\frac{1}{(2\pi)^2}\int_0^\infty k^3\mathrm{d}k\int_{-1}^{1}\mathrm{d}(\cos\theta)e^{-ik|\mathbf{r}-\mathbf{r}'|\cos\theta} = \frac{1}{2\pi^2}\int_0^\infty\frac{k^3\mathrm{d}k}{k|\mathbf{r}-\mathbf{r}'|}\left(e^{ik|\mathbf{r}-\mathbf{r}'|}-e^{-ik|\mathbf{r}-\mathbf{r}'|}\right) \ , \tag{89}$$

simplifying to

$$\langle 0|\hat{\mathbf{E}}(\mathbf{r}) \cdot \hat{\mathbf{E}}(\mathbf{r}')|0\rangle = \frac{2}{\pi}\frac{\hbar}{|\mathbf{r}-\mathbf{r}'|}\int_0^\infty k^2\left(e^{ik|\mathbf{r}-\mathbf{r}'|}-e^{-ik|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}k \ . \tag{90}$$

Let us now consider the integrals:

$$\int_0^\infty k^2e^{ik|\mathbf{r}-\mathbf{r}'|}\mathrm{d}k - \int_0^\infty k^2e^{-ik|\mathbf{r}-\mathbf{r}'|}\mathrm{d}k = \int_0^{-\infty}\kappa^2e^{-i\kappa|\mathbf{r}-\mathbf{r}'|}(-\mathrm{d}\kappa) - \int_0^\infty k^2e^{-ik|\mathbf{r}-\mathbf{r}'|}\mathrm{d}k \ , \tag{91}$$

and the derivative

$$\frac{\partial^2}{\partial|\mathbf{r}-\mathbf{r}'|^2}e^{-ik|\mathbf{r}-\mathbf{r}'|} = (-ik)^2e^{-ik|\mathbf{r}-\mathbf{r}'|} = -k^2e^{-ik|\mathbf{r}-\mathbf{r}'|} \ , \tag{92}$$

so we may write the integral in Equation 90 as

$$\int_0^\infty k^2\left(e^{ik|\mathbf{r}-\mathbf{r}'|}-e^{-ik|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}k = \frac{\partial^2}{\partial|\mathbf{r}-\mathbf{r}'|^2}\left\{\int_0^{-\infty}e^{-ik|\mathbf{r}-\mathbf{r}'|}\mathrm{d}k + \int_0^\infty e^{-ik|\mathbf{r}-\mathbf{r}'|}\mathrm{d}k\right\} \ , \tag{93}$$

using Equation 91 after rewriting $\kappa \to k$. We may write these over the same interval by introducing the Heaviside function

$$\int_0^{-\infty}e^{-ik|\mathbf{r}-\mathbf{r}'|}\mathrm{d}k + \int_0^\infty e^{-ik|\mathbf{r}-\mathbf{r}'|}\mathrm{d}k = -\int_{-\infty}^\infty e^{-ik|\mathbf{r}-\mathbf{r}'|}\Theta(-k)\mathrm{d}k + \int_{-\infty}^\infty e^{-ik|\mathbf{r}-\mathbf{r}'|}\Theta(k)\mathrm{d}k \ , \tag{94}$$

which are now simply Fourier transforms of the Heaviside function of positive and negative arguments, given by

$$\mathcal{F}[\Theta(k)](x) = \sqrt{\frac{\pi}{2}}\delta(x) + \frac{i}{\sqrt{2\pi}x} \tag{95}$$

$$\mathcal{F}[\Theta(-k)](x) = \sqrt{\frac{\pi}{2}}\delta(x) - \frac{i}{\sqrt{2\pi}x} \ . \tag{96}$$

Inserting these results into Equation 93 yields

$$\int_0^\infty k^2 \left(e^{ik|\mathbf{r}-\mathbf{r}'|} - e^{-ik|\mathbf{r}-\mathbf{r}'|}\right) \mathrm{d}k = \frac{\partial^2}{\partial|\mathbf{r}-\mathbf{r}'|^2}\left\{-\left(\sqrt{\frac{\pi}{2}}\delta(|\mathbf{r}-\mathbf{r}'|) - \frac{i}{\sqrt{2\pi}|\mathbf{r}-\mathbf{r}'|}\right) + \sqrt{\frac{\pi}{2}}\delta(|\mathbf{r}-\mathbf{r}'|) + \frac{i}{\sqrt{2\pi}|\mathbf{r}-\mathbf{r}'|}\right\} \ , \tag{97}$$

simplifying to

$$\int_0^\infty k^2 \left(e^{ik|\mathbf{r}-\mathbf{r}'|} - e^{-ik|\mathbf{r}-\mathbf{r}'|}\right) \mathrm{d}k = \sqrt{\frac{2}{\pi}}i\frac{\partial^2}{\partial|\mathbf{r}-\mathbf{r}'|^2}\frac{1}{|\mathbf{r}-\mathbf{r}'|} = 2i\sqrt{\frac{2}{\pi}}\frac{1}{|\mathbf{r}-\mathbf{r}'|^3} \ . \tag{98}$$

Therefore the final result of the correlation function is

$$\langle 0|\hat{\mathbf{E}}(\mathbf{r})\cdot\hat{\mathbf{E}}(\mathbf{r}')|0\rangle = 4i\hbar\sqrt{\frac{2}{\pi^3}}\frac{1}{|\mathbf{r}-\mathbf{r}'|^4} \ . \tag{99}$$

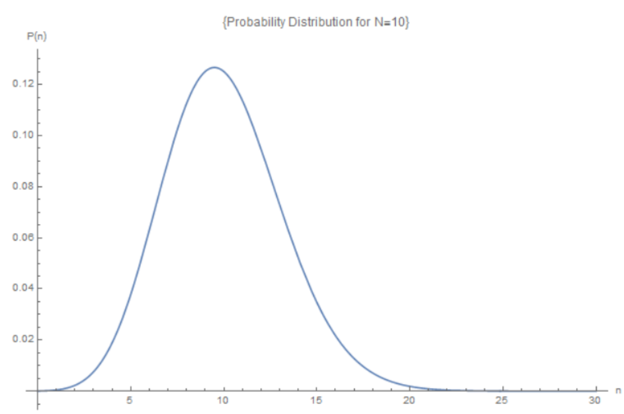

Figure 1: Probability distribution for measuring a Glauber state to contain n photons, with mean photon number $N = 10$.