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Contents

1 Problem #1: Charged Particle Scattering from Neutral Hydrogen.

Consider the scattering of a particle (neither an electron nor a proton) of mass M, charge Ze and momentum $\hbar\mathbf{k}$ from neutral Hydrogen. Assume the electrostatic potential energy in which the incident particle travels is

$$V(r) = Ze\left(\frac{|e|}{r} - \int \mathrm{d}^3r' \frac{|e|n(r')}{|\mathbf{r}-\mathbf{r}'|}\right) \to Ze\left(\frac{|e|}{r} - \int \mathrm{d}^3s \frac{|e|n(s)}{|\mathbf{r}-\mathbf{s}|}\right) \tag{1}$$

where $n(r)$ is the electron probability density in the ground state of Hydrogen and $|e|(-|e|)$ is the charge of the proton (electron). Calculate the differential cross-section for elastic scattering of the charged particle from neutral Hydrogen in the Born approximation. Express your result in terms of the form factor of the electron probability distribution defined by $F(\mathbf{q}) = \frac{1}{Z}\int \mathrm{d}^3r e^{i\mathbf{q}\cdot\mathbf{r}} n(r)$. Give explicit forms for $\mathrm{d}\sigma/\mathrm{d}\Omega_0$ in the limits $qa \ll 1$ and $qa \gg 1$, where a is the Bohr radius and $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ is the momentum transfer. Explain how small-angle scattering can be used to determine the *rms* radius of the electron distribution. Express your results in terms of $Z, a, k, M, c, e, \hbar$ and the relevant scattering parameters.

Using the first Born approximation, we assume the outgoing wave is a plane wave denoted $|\mathbf{k}'\rangle$, which by energy conservation is $|\mathbf{k}'| = |\mathbf{k}|$. In the coordinate basis this wave function can be represented as

$$\langle \mathbf{r}'|\mathbf{k}'\rangle = \frac{e^{i\mathbf{k}'\cdot\mathbf{r}'}}{(2\pi)^{3/2}} \ , \tag{2}$$

and so the scattering amplitude is[1]

$$-f_{\mathbf{k}',\mathbf{k}} = \frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}^3r' \left(e^{i\mathbf{k}'\cdot\mathbf{r}'}\right)^* V(r') e^{i\mathbf{k}\cdot\mathbf{r}'} = \frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}^3r' e^{i\mathbf{q}\cdot\mathbf{r}'} V(r') = \frac{1}{4\pi}\frac{2m}{\hbar^2}\langle \mathbf{k}'|V(r')|\mathbf{k}\rangle \ , \tag{3}$$

which is simply the three-dimensional Fourier transform of the potential with respect to $\mathbf{q}$. If we insert the given potential, the scattering amplitude becomes the sum of two integrals:

$$\mathcal{I}_1 = \int \mathrm{d}^3r' e^{i\mathbf{q}\cdot\mathbf{r}'} \frac{1}{|\mathbf{r}'|}(Ze|e|) \tag{4}$$

$$\mathcal{I}_2 = \int \mathrm{d}^3r' e^{i\mathbf{q}\cdot\mathbf{r}'} \int \mathrm{d}^3s \frac{n(s)}{|\mathbf{r}'-\mathbf{s}|}(-Ze|e|) \ , \tag{5}$$

and let us define $\zeta \equiv Ze|e|$. The first integral is a well-known Fourier transform:

$$\mathcal{I}_1(\mathbf{q}) = \frac{4\pi\zeta}{q^2} \ , \tag{6}$$

while the integrals in the second can swap positions:

$$\mathcal{I}_2(\mathbf{q}) = -\zeta \int \mathrm{d}^3s \ n(s) \int \mathrm{d}^3r' \frac{e^{i\mathbf{q}\cdot\mathbf{r}'}}{|\mathbf{r}'-\mathbf{s}|} \ , \tag{7}$$

and we now identify the inner integral as the Fourier transform of the Green's function. Let us now only consider this Fourier transform:

$$\int \mathrm{d}^3r' \frac{e^{i\mathbf{q}\cdot\mathbf{r}'}}{|\mathbf{r}'-\mathbf{s}|} = e^{i\mathbf{q}\cdot\mathbf{s}} \int \mathrm{d}^3r' \frac{e^{i\mathbf{q}\cdot\mathbf{r}'}e^{-i\mathbf{q}\cdot\mathbf{s}}}{|\mathbf{r}'-\mathbf{s}|} = e^{i\mathbf{q}\cdot\mathbf{s}} \int \mathrm{d}^3r' \frac{e^{i\mathbf{q}\cdot(\mathbf{r}'-\mathbf{s})}}{|\mathbf{r}'-\mathbf{s}|} = e^{i\mathbf{q}\cdot\mathbf{s}}\frac{4\pi}{q^2} \ . \tag{8}$$

[1]Sakurai, Modern Quantum Mechanics, Rev ed. Sect 7.2, equation 2.

Note this is the same Fourier transform as the first integral, but instead of integrating over a sphere around the origin, we are integrating over a sphere centered at $\mathbf{s}$. Alternatively we can see this by defining a vector $\mathbf{R} = \mathbf{r}' - \mathbf{s}$, and so $\mathrm{d}^3 R = \mathrm{d}^3 r' - \mathrm{d}^3 s$, but since $\mathbf{s}$ is a constant integral (for the purpose of this integration) it's differential element is zero, and we see this as just the three-dimensional Fourier transform of $1/|\mathbf{R}|$. This result makes the second integral from the matrix element into

$$\mathcal{I}_2(\mathbf{q}) = -\zeta \int \mathrm{d}^3 s \; n(s) e^{i\mathbf{q}\cdot\mathbf{s}} \frac{4\pi}{q^2} = -\frac{4\pi\zeta Z}{q^2} F(\mathbf{q}) \; . \tag{9}$$

The form factor of the electron probability distribution can be found using the electron probability density in the ground state of Hydrogen:

$$n(r) = |\psi_{100}(r)|^2 = \frac{1}{\pi a^3} e^{-2r/a} \; , \tag{10}$$

and as such

$$F(\mathbf{q}) = \frac{1}{Z\pi a^3} \int \mathrm{d}^3 r e^{i\mathbf{q}\cdot\mathbf{r}} e^{-2r/a} = \frac{2\pi}{Z\pi a^3} \int_0^\infty \mathrm{d}r \; r^2 e^{-2r/a} \int_{-1}^{1} \mathrm{d}(\cos\theta) e^{iqr\cos\theta} \; , \tag{11}$$

where the azimuthal integration was carried out. The polar integration yields

$$F(\mathbf{q}) = \frac{2\pi}{Z\pi a^3} \int_0^\infty \mathrm{d}r \; r^2 e^{-2r/a} \frac{2\sin(qr)}{qr} = \frac{4}{Za^3 q} \int_0^\infty \mathrm{d}r \; r e^{-2r/a} \sin(qr) \; , \tag{12}$$

and letting Mathematica handle the radial integral, we see

$$F(\mathbf{q}) = \frac{4}{Za^3 q} \left(\frac{4a^3 q}{(a^2q^2+4)^2} \right) = \frac{16}{Z} \frac{1}{(a^2q^2+4)^2} \; , \tag{13}$$

with

$$q = |\mathbf{q}| = |\mathbf{k} - \mathbf{k}'| = \sqrt{|\mathbf{k}|^2 + |\mathbf{k}'|^2 - 2|\mathbf{k}||\mathbf{k}'|\cos\theta} = k\sqrt{2(1-\cos\theta)} = 2k\sqrt{\frac{1-\cos\theta}{2}} \; . \tag{14}$$

Using the half-angle formula, we find

$$q = 2k \sin\left(\frac{\theta}{2}\right) \; , \tag{15}$$

so the form factor is then

$$F(\mathbf{q}) = \frac{16}{Z} \frac{1}{(4a^2k^2\sin^2(\theta/2)+4)^2} = \frac{1}{Z} \frac{1}{(a^2k^2\sin^2(\theta/2)+1)^2} = \frac{4Z^{-1}}{(2 + a^2k^2 - a^2k^2\cos(\theta))^2} \; , \tag{16}$$

where θ is the angle between the incoming and outgoing momenta. Combining the integrals $\mathcal{I}_1$ and $\mathcal{I}_2$, we have

$$-f_{\mathbf{k}',\mathbf{k}} = \frac{1}{4\pi} \frac{2m}{\hbar^2} (\mathcal{I}_1 + \mathcal{I}_2) = \frac{1}{4\pi} \frac{2m}{\hbar^2} \left(\frac{4\pi\zeta}{q^2} - \frac{4\pi\zeta Z}{q^2} F(\mathbf{q}) \right) = \frac{2m\zeta}{(\hbar q)^2} (1 - ZF(\mathbf{q})) \tag{17}$$

$$f_{\mathbf{k}',\mathbf{k}} = \frac{2m\zeta}{(\hbar q)^2} (ZF(\mathbf{q}) - 1) = Ze|e| \frac{2m}{(\hbar q)^2} \left(\frac{16}{(a^2q^2+4)^2} - 1 \right) \; . \tag{18}$$

Using this, we see the differential cross section is given by

$$\frac{\partial\sigma}{\partial\Omega} = |f_{\mathbf{k}',\mathbf{k}}|^2 = 4Z^2m^2\left(\frac{e}{\hbar q}\right)^4\left(\frac{1}{(1+\frac{a^2q^2}{4})^2}-1\right)^2 . \tag{19}$$

Let us Taylor expand $ZF(\mathbf{q})$ for small qa:

$$ZF(\mathbf{q}) = \left(1+\frac{a^2q^2}{4}\right)^{-2} \simeq 1-\frac{a^2q^2}{2} , \tag{20}$$

so in the limit $qa \ll 1$, we see

$$\frac{\partial\sigma}{\partial\Omega} = |f_{\mathbf{k}',\mathbf{k}}|^2 \simeq 4Z^2m^2\left(\frac{e}{\hbar q}\right)^4\left(-\frac{a^2q^2}{2}\right)^2 = Z^2m^2\left(\frac{ea}{\hbar}\right)^4 . \tag{21}$$

For large qa, the form factor takes the form

$$ZF(\mathbf{q}) = \left(1+\frac{a^2q^2}{4}\right)^{-2} \simeq \left(\frac{a^2q^2}{4}\right)^{-2} = \frac{16}{a^4q^4} , \tag{22}$$

so in the limit $qa \gg 1$:

$$\frac{\partial\sigma}{\partial\Omega} = |f_{\mathbf{k}',\mathbf{k}}|^2 \simeq 4Z^2m^2\left(\frac{e}{\hbar q}\right)^4\left(\frac{16}{a^4q^4}-1\right)^2 \simeq 4Z^2m^2\left(\frac{e}{\hbar q}\right)^4 , \tag{23}$$

in terms of the fine structure constant, this is

$$\frac{\partial\sigma}{\partial\Omega} = 4Z^2m^2\left(\frac{e^2}{\hbar c}\right)^2\left(\frac{c}{\sqrt{\hbar c}q}\right)^4 = (2Zm\alpha)^2\left(\frac{c}{\hbar}\right)^2\frac{1}{q^4} . \tag{24}$$

We are considering elastic scattering so $k = k'$, and thus small angle scattering refers to small momentum-transfer (q), see Equation 15. In this limit, the differential cross-section is a constant (Equation 21), so the distribution of scattered particles should be uniform in 4π, regardless of momentum transfer. The electron distribution of the ground state of neutral hydrogen has an *rms* radius related to the Bohr radius a. The remaining factors in Equation 21, are fundamental constants, so a measurement of the differential cross-section is essentially a measure of the Bohr radius, which is a metric for the *rms* radius of the electron distribution of the ground state of neutral hydrogen.

2 Problem #2: Quantum Hard Sphere Scattering.

Consider elastic scattering of a spinless particle of mass m from a "hard sphere" potential, $V(r) \to \infty$ for $r < a$ and otherwise $V(r) = 0$.

2.1 Energy Eigenstates.

Solve Schrödinger's equation for the energy eigenstates states. Show that for any energy eigenstate with orbital angular momentum quantum number ℓ the radial wave-function is "free-particle-like", but with a phase shift $\delta_\ell(k)$, *i.e.*, $R_{k\ell}(r) \sim \frac{\sin(kr-\ell\pi/2+\delta_\ell)}{kr}$ for $r > a$ and $kr \gg 1$, where $k \equiv \sqrt{2mE/\hbar^2}$.

In the region $r > a$, the Schrödinger equation is

$$\left(\nabla^2_{\{r,\theta,\varphi\}} + k^2\right)\psi(r,\theta,\varphi) = 0 \ , \tag{25}$$

with $k = \sqrt{2mE/\hbar^2}$. However, since the potential is spherically symmetric, neither the incoming or outgoing wave functions depend on the azimuthal angle φ about $\mathbf{k}$, so only the radial coordinate r and the polar angle θ (between the incoming momentum $\mathbf{k}$ and the scattered momentum $\mathbf{k'}$. Since these have no dependence on the azimuth, neither will the scattering amplitude, and thus it can be ignored for the remainder of this problem. Inserting the Laplacian in spherical coordinates yields

$$0 = \left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + k^2\right)\psi(r,\theta) \tag{26}$$

$$= \left(r^2\frac{\partial^2}{\partial r^2} + 2r\frac{\partial}{\partial r} + k^2r^2\right)\psi(r,\theta) + \left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right)\psi(r,\theta) \ . \tag{27}$$

If we assume a separable solution of the form $\psi(r,\theta) = R(r)Y(\theta)$, the Schrödinger equation becomes

$$0 = Y(\theta)\left(r^2\frac{\partial^2}{\partial r^2} + 2r\frac{\partial}{\partial r} + k^2r^2\right)R(r) + R(r)\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right)Y(\theta) \ .$$

Dividing both sides by the solution $\psi(r,\theta)$ allows us to write

$$\lambda = \frac{1}{R}\left(r^2\frac{\partial^2}{\partial r^2} + 2r\frac{\partial}{\partial r} + k^2r^2\right)R = -\frac{1}{Y}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right)Y \ , \tag{28}$$

where λ is a constant. This yields the two separated equations:

$$0 = \left(r^2\frac{\partial^2}{\partial r^2} + 2r\frac{\partial}{\partial r} + k^2r^2 - \lambda\right)R(r) \tag{29}$$

$$-\lambda Y(\theta,\varphi) = \left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right)Y(\theta,\varphi) \tag{30}$$

which are the radial and angular equations, respectively. The angular solutions are Legendre polynomials, and we have the constraint $\lambda = \ell(\ell+1)$, with $\ell \in \mathbb{Z}$. Therefore, the solution to the Schrödinger equation is

$$\psi(r,\theta) = \sum_{l=0}^{\infty} R_{\mathbf{k}\ell}(r)P_\ell(\cos\theta) \ , \tag{31}$$

where $R_{\mathbf{k}\ell}(r)$ satisfies the radial equation, which after dividing by r^2, yields

$$0 = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} + k^2 \right) R_{\mathbf{k}\ell}(r) \; . \tag{32}$$

This wave function is solved by the spherical Bessel functions of the first and second kind, denoted by $j_\ell(kr)$ and $y_\ell(kr)$, respectively. Therefore, the radial solution can be expressed as

$$R_{\mathbf{k}\ell} = A_\ell j_\ell(kr) + B_\ell y_\ell(kr) \; , \tag{33}$$

since the solution we found is only defined for $r > a$, both kinds of spherical Bessel functions are allowed. In the far-field, the asymptotic behavior of these functions are

$$j_\ell(kr) \xrightarrow[r\to\infty]{} \frac{\sin(kr - \ell\pi/2)}{kr} \tag{34}$$

$$y_\ell(kr) \xrightarrow[r\to\infty]{} -\frac{\cos(kr - \ell\pi/2)}{kr} \; . \tag{35}$$

We can write the far-field solution as a linear combination of these as

$$R_{\mathbf{k}\ell} \xrightarrow[r\to\infty]{} \frac{\sin(kr - \ell\pi/2 + \delta_\ell)}{kr} \; , \tag{36}$$

where δ_ℓ is a constant denoting the phase shift of the ℓth mode of the wave. Therefore, in the far-field, the total wave function can be expressed as

$$\psi(r,\theta) \xrightarrow[r\to\infty]{} \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) C_\ell \frac{\sin(kr - \ell\pi/2 + \delta_\ell)}{kr} P_\ell(\cos\theta) \; , \tag{37}$$

where the factor of $i^\ell(2\ell+1)$ is included to make normalization simpler, see below, and C_ℓ is a normalization constant.

Consider the wave function of the incoming plane wave (normalized to unit volume):

$$\psi_0(r,\theta) = e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos\theta} \; , \tag{38}$$

which can be written as an expansion in Legendre polynomials:

$$e^{ikr\cos\theta} = \sum_\ell a_\ell j_\ell(kr) P_\ell(\cos\theta) \; , \tag{39}$$

where a_ℓ are undetermined constants, we are not including the spherical Bessel function of the second kind, due to its divergence at the origin. Exploiting the orthogonality of Legendre polynomials, we can write

$$\int_{-1}^{1} \mathrm{d}(\cos\theta) P_m(\cos\theta) e^{ikr\cos\theta} = \sum_\ell a_\ell j_\ell(kr) \int_{-1}^{1} \mathrm{d}(\cos\theta) P_m(\cos\theta) P_\ell(\cos\theta) \tag{40}$$

$$= \sum_\ell a_\ell j_\ell(kr) \delta_{\ell m} \frac{2}{(2\ell+1)} = a_m j_m(kr) \frac{2}{2m+1} \; , \tag{41}$$

similarly the spherical Bessel function of the first kind can be expanded in terms of exponentials[2]:

$$j_\ell(kr) = \frac{(-i)^\ell}{2} \int_{-1}^{1} e^{ikr\cos\theta} P_\ell(\cos\theta) \mathrm{d}(\cos\theta) \; . \tag{42}$$

[2]Sakurai, Modern Quantum Mechanics, Rev ed. Sect 7.5, equation 19.

Isolating the integral in the above expression, and inserting this into the left-hand side of Equation 40, yields the relation

$$\frac{2}{(-i)^\ell} j_\ell(kr) = a_\ell j_\ell(kr) \frac{2}{2\ell+1} \ , \tag{43}$$

and thus $a_\ell = i^\ell(2\ell+1)$. Therefore the incident particle has the the wave function

$$\psi_0(r,\theta) = e^{ikr\cos\theta} = \sum_\ell i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos\theta) \ , \tag{44}$$

and the scattered wave function is

$$\psi(r,\theta) = \psi_0(r,\theta) + f(\theta)\frac{e^{ikr}}{r} \ , \tag{45}$$

which is also a solution to the free-particle Schrödinger equation for $r > a$.

2.2 Phase Shifts.

Calculate the phase shifts, $\delta_\ell(k)$ from the exact solution to the radial wave equation.

The exact solution to the radial wave equation is

$$R_{\mathbf{k}\ell}(r) = [A_\ell j_\ell(kr) + B_\ell y_\ell(kr)] \ , \tag{46}$$

so the total wave function, outside the scattering region ($r > a$) is

$$\psi(r,\theta) = \sum_{\ell=0}^{\infty} [A_\ell j_\ell(kr) + B_\ell y_\ell(kr)] \, P_\ell(\cos\theta) \ . \tag{47}$$

If we consider the far-field asymptotic behavior of this solution, we may write this as

$$\psi(r,\theta) \xrightarrow[r\to\infty]{} \sum_{\ell=0}^{\infty} C_\ell \frac{\sin(kr - \ell\pi/2 + \delta_\ell)}{kr} P_\ell(\cos\theta) \ , \tag{48}$$

with $A_\ell = C_\ell \cos\delta_\ell$ and $B_\ell = -C_\ell \sin\delta_\ell$. If we expand the sine, we may write this as

$$\psi(r,\theta) \xrightarrow[r\to\infty]{} \sum_{\ell=0}^{\infty} C_\ell \frac{e^{i(kr-\ell\pi/2+\delta_\ell)} - e^{-i(kr-\ell\pi/2+\delta_\ell)}}{2ikr} P_\ell(\cos\theta) \ , \tag{49}$$

which is the solution for the total wave function, and must include outgoing spherical waves and the incoming wave, ψ_0. In the far-field, we may write the incident wave function, from Equation 44 as

$$\psi_0(r,\theta) = e^{ikr\cos\theta} = \sum_\ell i^\ell(2\ell+1) \frac{e^{i(kr-\ell\pi/2)} - e^{-i(kr-\ell\pi/2)}}{2ikr} P_\ell(\cos\theta) \ . \tag{50}$$

Using the definition of the scattered wave, we see

$$f(\theta)\frac{e^{ikr}}{r} = \psi - \psi_0 \ , \tag{51}$$

where there are only outgoing spherical waves on the left-hand side, and thus the coefficients of the incoming waves (e^{-ikr}) in ψ and ψ_0 must be the same. Factoring out the δ_ℓ from Equation 49, and using the fact stated previously, we see

$$i^\ell(2\ell+1) = C_\ell(e^{-i\delta_\ell}) \ , \tag{52}$$

and so $C_\ell = e^{i\delta_\ell} i^{-\ell}(2\ell+1) = (2\ell+1)e^{i(\ell\pi/2+\delta_\ell)}$. Using this result, we see the exact wave equation may be written

$$\psi(r,\theta) = \sum_{\ell=0}^{\infty} e^{i\delta_\ell} i^{-\ell}(2\ell+1)\left[\cos(\delta_\ell) j_\ell(kr) - \sin(\delta_\ell) y_\ell(kr)\right] P_\ell(\cos\theta) \ , \tag{53}$$

and we consider the exact solution to the radial wave function as

$$R_{\mathbf{k}\ell}(r) = e^{i\delta_\ell}\left[\cos(\delta_\ell) j_\ell(kr) - \sin(\delta_\ell) y_\ell(kr)\right] \ . \tag{54}$$

For the hard-sphere potential, the radial wave function must vanish at the surface of the infinite potential:

$$R_{\mathbf{k}\ell}(a) = 0 = e^{i\delta_\ell}\left[\cos(\delta_\ell) j_\ell(ka) - \sin(\delta_\ell) y_\ell(ka)\right] \ , \tag{55}$$

yielding the condition

$$\cos(\delta_\ell) j_\ell(ka) = \sin(\delta_\ell) y_\ell(ka) \quad \Rightarrow \quad \tan(\delta_\ell) = \frac{j_\ell(ka)}{y_\ell(ka)} \ , \tag{56}$$

which is an exact result[3].

2.3 Low-Energy Differential Cross-Section.

Calculate the differential cross-section to next-to-leading order in the $ka \ll 1$. Sketch the angular dependence of the probability distribution of the scattered particles.

The limit $ka \ll 1$ is low-energy scattering, in this limit, the asymptotic behavior of the spherical Bessel functions[4] is

$$j_\ell(ka) \xrightarrow[ka\ll 1]{} \frac{(ka)^\ell}{(2\ell+1)!!} \tag{57}$$

$$y_\ell(ka) \xrightarrow[ka\ll 1]{} -\frac{(2\ell-1)!!}{(ka)^{l+1}} \ , \tag{58}$$

and so, we have

$$\tan(\delta_\ell) = \frac{(ka)^\ell}{(2\ell+1)!!}\left(-\frac{(ka)^{l+1}}{(2\ell-1)!!}\right) = \frac{-(ka)^{2\ell+1}}{(2\ell+1)\left[(2\ell-1)!!\right]^2} \ , \tag{59}$$

which quickly vanishes for $\ell > 0$. For $\ell = 0$, we have

$$\tan(\delta_0) = \frac{j_0(ka)}{y_0(ka)} = \frac{\sin(ka)}{ka}\left(-\frac{ka}{\cos(ka)}\right) = -\tan(ka) \ , \tag{60}$$

so $\delta_0 = -ka$. For $\ell = 1$, in the low-energy limit, we have

$$\tan(\delta_1) = \frac{j_1(ka)}{y_1(ka)} \simeq \frac{ka}{3!!}\left(-\frac{(ka)^2}{(1)!!}\right) = -\frac{(ka)^3}{3} \ , \tag{61}$$

[3] There were no approximations made when solving for the phase shift. However, it may seem that an approximation was made when we solved for the relations of A_ℓ, B_ℓ, and C_ℓ. We investigated the far-field behavior of the radial wave function to determine these relations and then applied them to the exact result.

[4] Sakurai, Modern Quantum Mechanics, Rev ed. Sect 7.6, equation 46.

so $\delta_1 = \arctan(-(ka)^3/3)$. If we Taylor expand the arctangent for small values of ka, we see

$$\delta_1 = -(ka)^3/3! + \frac{(ka)^9/(3)^3}{3} \ , \tag{62}$$

or approximately, $\delta_1 = -(ka)^3/3$.

The differential cross-section, to next-to-leading order, is given[5] by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta)|^2 = \left| \frac{1}{k} \sum_{\ell=0}^{1} (2\ell+1) e^{i\delta_\ell} \sin\delta_\ell P_\ell(\cos\theta) \right|^2 = \frac{1}{k^2} \left| e^{i\delta_0}\sin\delta_0 + 3e^{i\delta_1}\sin(\delta_1)\cos\theta \right|^2 \tag{63}$$

$$= \frac{1}{k^2} \left(e^{i\delta_0}\sin\delta_0 + 3e^{i\delta_1}\sin(\delta_1)\cos\theta \right) \left(e^{-i\delta_0}\sin\delta_0 + 3e^{-i\delta_1}\sin(\delta_1)\cos\theta \right) \tag{64}$$

$$= \frac{1}{k^2} \left(\sin^2\delta_0 + 9\sin^2\delta_1\cos^2\theta + 3\sin\delta_0\sin\delta_1\cos\theta \left[e^{i(\delta_0-\delta_1)} + e^{i(\delta_1-\delta_0)} \right] \right) \tag{65}$$

$$= \frac{1}{k^2} \left(\sin^2\delta_0 + 9\sin^2\delta_1\cos^2\theta + 6\sin\delta_0\sin\delta_1\cos\theta\cos(\delta_0-\delta_1) \right) \ . \tag{66}$$

In the low-energy limit $\sin(\delta_\ell) \sim \delta_\ell$ (since $\delta_0 \propto k$ and $\delta_1 \propto k^3$) and $\cos\delta_\ell \sim 1$, as such, we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{k^2} \left\{ (\delta_0)^2 + 9(\delta_1)^2\cos^2\theta + 6\delta_0\delta_1\cos\theta \right\} \tag{67}$$

$$= \frac{1}{k^2} \left\{ (-ka)^2 + 9\left(-\frac{(ka)^3}{3} \right)^2 \cos^2\theta + 6(-ka)\left(-\frac{(ka)^3}{3} \right)\cos\theta \right\} \tag{68}$$

$$= a^2 \left\{ 1 + 2(ka)^2\cos\theta + (ka)^4\cos^2\theta \right\} \ . \tag{69}$$

If we retain only the leading and next-to-leading terms, this is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = a^2 \left\{ 1 + 2(ka)^2\cos\theta \right\} \ , \tag{70}$$

which is plotted for $ka = 0.5$ in Figure 1.

2.4 Low-Energy Total Cross-Section.

Calculate the total cross-section in the limit $k \to 0$.

In the limit $k \to 0$, the $\ell = 1$ phase shift is increasingly negligible to the $\ell = 0$ phase shift, and therefore, we can approximate the total cross-section as

$$\sigma = \int \mathrm{d}\Omega |f(\theta)|^2 = \int \mathrm{d}\Omega \frac{1}{k^2} \left| e^{i\delta_0}\sin\delta_0 \right|^2 = \int \mathrm{d}\Omega \frac{\sin^2(-ka)}{k^2} \ . \tag{71}$$

Furthermore, in this limit $\sin(ka) \to ka$, so the total cross section is

$$\sigma = \int \mathrm{d}\Omega \frac{(ka)^2}{k^2} = 4\pi a^2 \ , \tag{72}$$

[5] Sakurai, Modern Quantum Mechanics, Rev ed. Sect 7.6, equation 17.

which is four times the area we would expect classically (geometric cross-section of a sphere is πa^2). Alternatively, we may start from Equation 70, taking $k \to 0$, so that

$$\sigma = a^2 \int \mathrm{d}\Omega = 4\pi a^2 \ , \tag{73}$$

and we obtain the same result.

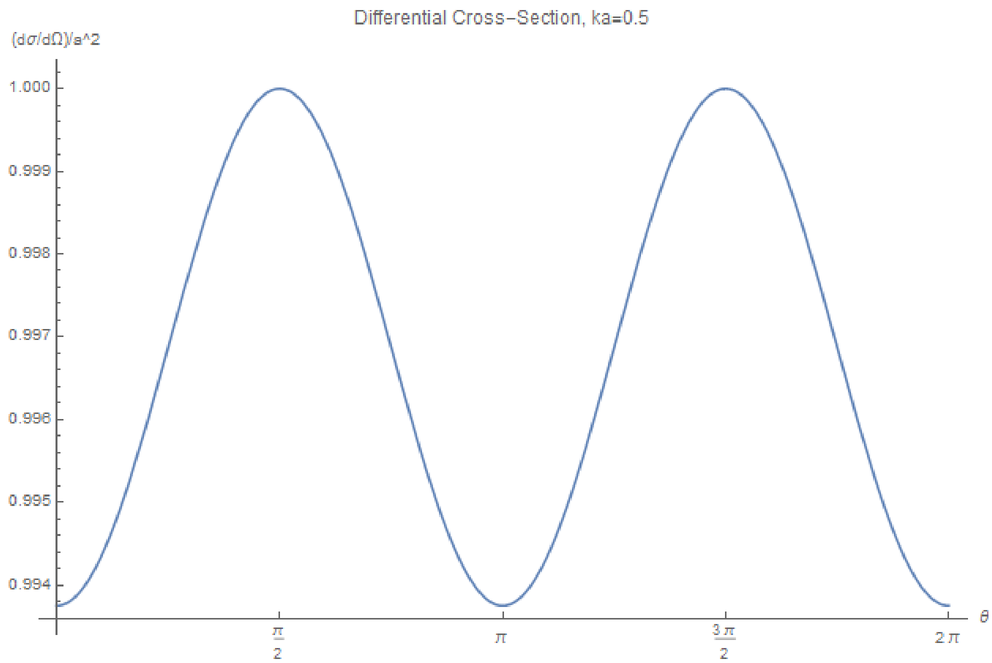

Figure 1: Differential cross-section, measured in units of the potential range squared (a^2), as a function of scattering angle.

3 Problem #3: The Deuteron.

The deuteron is a bound state of a neutron and proton. The nuclear potential is modeled as a short-ranged spherically symmetric potential well of radius $a = 2$fm and depth $V_0 = -36$MeV. The potential is sufficiently weak that there is only one 'shallow' bound state with binding energy $E_0 \simeq -2.2$MeV, *i.e.*, $|E_0| \ll |V_0|$. A monochromatic beam of linearly polarized gamma ray photons, with energy $\hbar\omega_\gamma$ near the ionization threshold of the deuteron, is incident on the deuteron (at rest).

3.1 Ground State Wave Function.

Calculate the ground state wave function of the deuteron. What is the range of the ground state wave function?

The potential is spherically symmetric, and thus all that matters is the separation of the neutron and proton r. Therefore, we work in the center-of-mass frame and we may consider only the radial wave-function:

$$\left\{ -\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\mathrm{d}r\right) + V(r) \right\} \psi(r) = -E_0\psi(r) \ , \tag{74}$$

where μ is the reduced mass of the proton-neutron system, which is, to a good approximation, $M/2$, where M is the mass of the nucleon. If we define a function $u(r)$ such that

$$\psi(r) \equiv \frac{u(r)}{r} \ , \tag{75}$$

the Schrödinger equation in the region $r \geq a$ becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} - \kappa^2 u = 0 \ , \tag{76}$$

where $\kappa^2 = 2\mu E_0/\hbar^2$. This equation has solutions of the form

$$u(r) = Ae^{-\kappa r} \ , \tag{77}$$

where A is a constant to be determined, note we have discarded the exponentially growing solution because it is unphysical (wave function must be normalizable). In the region $r \leq a$, the Schrödinger equation is

$$\frac{\partial^2 u}{\partial r^2} + q^2 u = 0 \ , \tag{78}$$

with $q^2 = 2\mu(V_0 - E_0)/\hbar^2$. This has solutions

$$u(r) = B\sin(qr) + C\cos(qr) \ , \tag{79}$$

with the constants B and C determined by normalization. The wave function can not diverge at the origin for it to be normalizable, so it must be that $u(0) = 0$, so the wave function is well-behaved at the origin. Consider expanding the wave function for small argument:

$$\psi_{r\leq a}(r) \simeq Bq + \frac{C}{r} \ , \tag{80}$$

which is only well-behaved at the origin if we must set $C = 0$, and thus we have

$$\psi_{r\leq a}(r) = B\frac{\sin(qr)}{r} \ . \tag{81}$$

Continuity of the wave-function and its first derivative dictates[6]

$$\begin{cases} \psi_{r\geq a}\big|_{r=a} & = \psi_{r\leq a}\big|_{r=a} \\ \dfrac{\mathrm{d}\psi_{r\geq a}}{\mathrm{d}r}\bigg|_{r=a} & = \dfrac{\mathrm{d}\psi_{r\leq a}}{\mathrm{d}r}\bigg|_{r=a} \end{cases} \quad \Rightarrow \quad \begin{cases} u_{r\geq a}\big|_{r=a} & = u_{r\leq a}\big|_{r=a} \\ \dfrac{\mathrm{d}u_{r\geq a}}{\mathrm{d}r}\bigg|_{r=a} & = \dfrac{\mathrm{d}u_{r\leq a}}{\mathrm{d}r}\bigg|_{r=a} \end{cases} \tag{82}$$

This gives the conditions

$$Ae^{-\kappa a} = B\sin(qa) \tag{83}$$

$$-\kappa Ae^{-\kappa a} = qB\cos(qa) \ , \tag{84}$$

which if we divide the first by the second yields

$$\tan(qa) = -\frac{q}{\kappa} = -\frac{qa}{\kappa a} \ . \tag{85}$$

The wave function for the Deuteron is

$$\psi(r) = \begin{cases} A\frac{e^{-\kappa r}}{r} & r \geq a \\ B\frac{\sin qr}{r} & r \leq a \ , \end{cases} \tag{86}$$

where κ and q are related by Equation 85, and A or B are normalization constants. The majority of the wave function is outside the interaction range a (the exponential tail), so if we normalize this function over all space, it is a good approximation to normalizing the entire wave function

$$1 = |A|^2 \int \frac{e^{-2\kappa r}}{r^2}\mathrm{d}^3r = |A|^2 \int_0^\infty r^2\mathrm{d}r\frac{e^{-2\kappa r}}{r^2}\int \mathrm{d}\Omega = \frac{4\pi|A|^2}{2\kappa} \ , \tag{87}$$

and so $A = \sqrt{\kappa/2\pi}$. Therefore, in the region $r \geq a$, the wave function is

$$\psi_{r\geq a}(r) = \sqrt{\frac{\kappa}{2\pi}}e^{-\kappa r} \ , \tag{88}$$

which has a range of $r_0 = 1/\kappa$.

3.2 Constraints on Gamma Ray Energy.

Assume the photon wavelength is long compared to the typical diameter of the deuteron. What constraint does this place on the gamma ray energy?

Consider a photon with wave number k, which has wavelength $2\pi/k$. In the long-wavelength limit, the range of the wave function is negligible compared to the wavelength of the photon so (excluding factors of π), we have

$$r_0 \ll k^{-1} \quad \Rightarrow \quad kr_0 \ll 1 \ . \tag{89}$$

Using the fact that $r_0 = 1/\kappa$, we have the condition $k \ll \kappa$. The energy of a photon with wavenumber k is

$$E_k = \hbar ck \ , \tag{90}$$

[6]The condition on the derivative of the wave function is not obvious:

$$\partial_r\psi_1 = \partial_r\psi_2 \to \partial_r(u_1/r) = \partial_r(u_2/r) \to (\partial_r u_1)/r - u_1/r^2 = (\partial_r u_2)/r - u_2/r^2 \ ,$$

and since $u_1 = u_2$, and the r factors cancel, and we obtain $\partial_r u_1 = \partial_r u_2$.

so $k = E_k/\hbar c$, and so

$$\frac{E_k}{\hbar c} \ll \kappa \ . \tag{91}$$

Therefore the energy constraint on the photon is such that

$$E_k \ll \kappa\hbar c = \hbar c\sqrt{\frac{2\mu E_0}{\hbar^2}} = \sqrt{2\mu c^2 E_0} \simeq \sqrt{Mc^2 E_0} \simeq \sqrt{938(2.2)} \text{ MeV} = 45.43 \text{ MeV} \ , \tag{92}$$

so the energy of the incoming photon must be small compared to the square root of the product of the nucleon rest mass and the energy of the bound state, for the long wavelength approximation to be valid.

3.3 Photo-Ionization Cross-Section.

Calculate the photo-ionization cross-section of the deuteron as a function of photon energy, $\hbar\omega_\gamma$. Work out the general formula for the differential cross-section in terms of a matrix element of the initial and final state radial wave functions, then evaluate the formula to reasonable accuracy for the weakly bound ground state. Express your result in sensible units: $e^2/\hbar c$, relevant length squared, $\hbar\omega_\gamma/|E_0|$. Hint: Express the final state wave function in terms of scattering phase shifts and use your knowledge of the scaling of the phase shifts, $\delta_\ell(k_f a_s)$, with $k_f a_s$ where $\hbar k_f$ is the relative momentum of the final state neutron and proton, and a_s is the scattering length for neutron-proton scattering.

The initial state has the Deuteron in its ground state and N_k photons in mode k. after photo-ionization, the final state is $N_k - 1$ photons in mode k and two particles moving with relative momentum $\hbar\mathbf{k}_f$. This is a single photon process, so we may neglect the $|\hat{\mathbf{A}}|^2$ term in the interaction Hamiltonian, so we consider only

$$\hat{\mathcal{H}}_{\text{int}} = -\frac{e^*}{c}\left(\frac{\hat{\mathbf{p}}}{\mu}\cdot\hat{\mathbf{A}}\right) \ , \tag{93}$$

where e^* is the effective charge of the Deuteron, and $\hat{\mathbf{A}}$ is the free-radiation field operator,

$$\hat{\mathbf{A}} = \sqrt{\frac{2\pi\hbar c^2}{V}}\sum_{\mathbf{k},\lambda}\frac{1}{\sqrt{\omega_k}}\left[\hat{a}_{\mathbf{k},\lambda}\mathbf{e}_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + \hat{a}^\dagger_{\mathbf{k},\lambda}\mathbf{e}^*_{\mathbf{k},\lambda}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right] \ , \tag{94}$$

with $\hat{a}_{\mathbf{k},\lambda}$ ($\hat{a}^\dagger_{\mathbf{k},\lambda}$) representing the photon annihilation (creation) operator for the mode $\mathbf{k},\lambda$, with λ representing the polarization state of radiation. The incoming radiation is linearly polarized, so we have $\mathbf{e}_{\mathbf{k},\lambda} = \mathbf{e}^*_{\mathbf{k},\lambda}$, and we may factor it out. Using this, we see the matrix element relevant to Fermi's Golden rule is

$$M = \langle\mathbf{k}_f; N_k - 1| - \frac{e^*}{c}\left(\frac{\hat{\mathbf{p}}}{\mu}\cdot\hat{\mathbf{A}}\right)|0; N_k\rangle \tag{95}$$

$$= -\frac{e^*}{\mu c}\sqrt{\frac{2\pi\hbar c^2}{V\omega_k}}\langle\mathbf{k}_f; N_k - 1|\left(\hat{\mathbf{p}}\cdot\mathbf{e}_{\mathbf{k},\lambda}\left[\hat{a}_{\mathbf{k},\lambda}e^{i\mathbf{k}\cdot\mathbf{r}} + \hat{a}^\dagger_{\mathbf{k},\lambda}e^{-i\mathbf{k}\cdot\mathbf{r}}\right]\right)|0; N_k\rangle \ , \tag{96}$$

if we note that the final state has one *less* photon than the initial state, the term with the creation operator vanishes, and we are left with the matrix element

$$M = -\frac{e^*}{\mu}\sqrt{\frac{2\pi\hbar}{V\omega_k}}\langle\mathbf{k}_f|\hat{\mathbf{p}}\cdot\mathbf{e}_{\mathbf{k},\lambda}e^{i\mathbf{k}\cdot\mathbf{r}}|0\rangle\cdot\langle N_k - 1|\hat{a}_{\mathbf{k},\lambda}|N_k\rangle \ , \tag{97}$$

so the radiation factor contributes $\sqrt{N_k}$. Using this result, Fermi's Golden rule gives

$$\Gamma_{\mathbf{k},\lambda} = \frac{2\pi}{\hbar}|M|^2 = \frac{2\pi}{\hbar}\left(\frac{e^*}{\mu}\right)^2 \frac{2\pi\hbar}{V\omega_k} N_k \left|\langle \mathbf{k}_f|\hat{\mathbf{p}}\cdot\mathbf{e}_{\mathbf{k},\lambda}e^{i\mathbf{k}\cdot\mathbf{r}}|0\rangle\right|^2 \delta(E_f + E_0 - \hbar\omega_k) \ . \tag{98}$$

We are now interested in the matrix element

$$\mathcal{M} = \langle \mathbf{k}_f|\hat{\mathbf{p}}\cdot\mathbf{e}_{\mathbf{k},\lambda}e^{i\mathbf{k}\cdot\mathbf{r}}|0\rangle \ , \tag{99}$$

but we are in the long-wavelength limit, which leads to the electric dipole approximation. Therefore we approximate the exponential by retaining only the first term in the expansion for small arguments of the exponential, so $e^{i\mathbf{k}\cdot\mathbf{r}} \to 1$, so

$$\mathcal{M} = \langle \mathbf{k}_f|\hat{\mathbf{p}}\cdot\mathbf{e}_{\mathbf{k},\lambda}|0\rangle \ , \tag{100}$$

which can be expressed in the coordinate basis as

$$\mathcal{M} = -\frac{\hbar}{i}\int \mathrm{d}^3r \ \psi^*_{\mathbf{k}_f}(\mathbf{r})\left(\mathbf{e}_{\mathbf{k},\lambda}\cdot\boldsymbol{\nabla}\psi_0(\mathbf{r})\right) \ . \tag{101}$$

The ground state of Deuteron is isotropic, which leads to

$$\mathcal{M} = -\frac{\hbar}{i}\int \mathrm{d}^3r \ \psi^*_{\mathbf{k}_f}(\mathbf{r})(\hat{\mathbf{r}}\cdot\mathbf{e}_{\mathbf{k},\lambda})\frac{\partial\psi_0(\mathbf{r})}{\partial r} \ , \tag{102}$$

where $\hat{\mathbf{r}}$ denotes a unit vector in the direction of $\mathbf{r}$. The scattered wave function can be written as an expansion over partial waves:

$$\psi_{\mathbf{k}_f}(\mathbf{r}) = \frac{1}{\sqrt{(2\pi)^3V}}\sum_{\ell\geq 0} i^\ell(2\ell+1)\mathcal{R}_\ell(k_f r)P_\ell(\hat{\mathbf{k}}_f\cdot\hat{\mathbf{r}}) \ , \tag{103}$$

where $P_\ell(x)$ is the ℓth Legendre polynomial, and $\hat{\mathbf{k}}_f$ is a unit vector in the direction of the relative momentum of the neutron and proton. We previously stated that the ground state wave function is isotropic ($\ell = 0$), and so by the selection rules of electric dipole transitions, the final state must have $\ell = 1$. We can therefore write the scattered wave function as an $\ell = 1$ wave:

$$\psi_{\mathbf{k}_f}(\mathbf{r}) = \frac{3}{\sqrt{(2\pi)^3V}} i\mathcal{R}_1(k_f r)(\hat{\mathbf{k}}_f\cdot\hat{\mathbf{r}}) \ , \tag{104}$$

and from the far-field scattering solution (for a constant radial potential) in Equation 105, we see the radial wave function for p-waves is

$$R_1(r) = e^{i\delta_1}\left[\cos(\delta_1)j_1(k_f r) - \sin(\delta_1)y_1(k_f r)\right] \ , \tag{105}$$

where the phase shift δ_1 is given by Equation 62, in the low-energy limit:

$$\delta_1 = -(ka)^3/3 \ , \tag{106}$$

but since $ka \ll 1$ in this limit, we may approximate the radial wave function for $\ell = 1$ as

$$R_1(r) \sim e^{-i(ka)^3/3}\left[j_1(k_f r) - \frac{(ka)^3}{3}y_1(k_f r)\right] \simeq j_1(k_f r) = \frac{k_f r}{3} \tag{107}$$

in the far-field, see Equation 57, so the wave function is

$$\psi_{\mathbf{k}_f}(\mathbf{r}) = \frac{3}{\sqrt{(2\pi)^3 V}} i \frac{k_f r}{3} (\hat{\mathbf{k}}_f \cdot \hat{\mathbf{r}}) \ , \tag{108}$$

Using this, the matrix element becomes

$$\mathcal{M} = \frac{-(3i)}{\sqrt{(2\pi)^3 V}} \frac{\hbar}{i} \int \mathrm{d}^3 r \ R_1^*(k_f r)(\hat{\mathbf{k}}_f \cdot \hat{\mathbf{r}})(\hat{\mathbf{r}} \cdot \mathbf{e}_{\mathbf{k},\lambda}) \frac{\partial \psi_0(\mathbf{r})}{\partial r} \ , \tag{109}$$

and the angular integrals can be evaluated immediately. Consider the integral of two unit vectors over all solid angle:

$$\int \frac{\mathrm{d}\Omega_{\hat{\mathbf{r}}}}{4\pi} \hat{\mathbf{r}}_i \hat{\mathbf{r}}_j \ , \tag{110}$$

the result of which is a rank two tensor, which must be invariant (rotations of the vectors will not change the outcome because they are isotropic and are integrated over the unit sphere surface). Therefore it is proportional to the Kronecker delta; if we contract over the indices $\hat{\mathbf{r}}_i \hat{\mathbf{r}}_j = 1$ so the integral is unity, while the contraction of the Kronecker delta (in 3 dimensional space) is 3. Therefore the proportionality constant must be 1/3. If we now perform the indicated contraction (two dot products), we see

$$(\hat{\mathbf{k}}_f \cdot \hat{\mathbf{r}})(\hat{\mathbf{r}} \cdot \mathbf{e}_{\mathbf{k},\lambda}) = \frac{4\pi}{3} (\hat{\mathbf{k}}_f \cdot \mathbf{e}_{\mathbf{k},\lambda}) \ , \tag{111}$$

which is independent of $\mathbf{r}$, and can then be factored out of the integral. The sum over polarizations is due to the radiation field operator having a sum over wave vector and polarization, so considering only one k does not do the sum over λ. The matrix element simplifies to

$$\sqrt{(2\pi)^3 V} \mathcal{M} = -4\pi\hbar \sum_\lambda (\hat{\mathbf{k}}_f \cdot \mathbf{e}_{\mathbf{k},\lambda}) \int_0^\infty (r^2 \mathrm{d}r) \ R_1^*(k_f r) \frac{\partial \psi_0(\mathbf{r})}{\partial r} \ , \tag{112}$$

where the integral is

$$\int_0^\infty (r^2 \mathrm{d}r) \ R_1^*(k_f r) \frac{\partial \psi_0(\mathbf{r})}{\partial r} = \int_0^\infty (r^2 \mathrm{d}r) \frac{k_f r}{3} \frac{\mathrm{d}}{\mathrm{d}r} \sqrt{\frac{\kappa}{2\pi}} \frac{e^{-\kappa r}}{r} \tag{113}$$

$$= \frac{k_f}{3} \sqrt{\frac{\kappa}{2\pi}} \int_0^\infty (r^3 \mathrm{d}r) \left(-\frac{e^{-\kappa r}}{r^2} - \frac{\kappa e^{-\kappa r}}{r} \right) \tag{114}$$

$$= -\frac{k_f}{3} \sqrt{\frac{\kappa}{2\pi}} \int_0^\infty \mathrm{d}r \left(r e^{-\kappa r} + r^2 \kappa e^{-\kappa r} \right) \tag{115}$$

$$= -\frac{k_f}{3} \sqrt{\frac{\kappa}{2\pi}} \left(\frac{3}{\kappa^2} \right) = -\frac{k_f}{\sqrt{2\pi\kappa^3}} \ . \tag{116}$$

So Fermi's Golden rule (Equation 98) becomes

$$\Gamma_{\mathbf{k},\lambda} = \left(\frac{e^*}{\mu} \right)^2 \frac{(2\pi)^2}{V\omega_k} N_k \left| \mathcal{M} \right|^2 \delta(E_f + E_0 - \hbar\omega_k) \ , \tag{117}$$

and the kinematics (delta function) give the condition $\hbar\omega_k - E_0 = \hbar k_f^2 / 2\mu$. Inserting the result for $\mathcal{M}$ gives

$$\Gamma_{\mathbf{k}_f} = \left(\frac{e^*}{\mu} \right)^2 \frac{(2\pi)^2}{V\omega_k} N_k (4\pi\hbar)^2 \frac{k_f^2}{2\pi\kappa^3} \left| \sum_\lambda (\hat{\mathbf{k}}_f \cdot \mathbf{e}_{\mathbf{k},\lambda}) \right|^2 \left(\frac{1}{\sqrt{(2\pi)^3 V}} \right)^2 \delta(E_f + E_0 - \hbar\omega_k) \tag{118}$$

$$= \Sigma \left(\frac{e^*}{\mu} \right)^2 \frac{4(2\pi)^3 \hbar^2 k_f^2}{V\omega_k \kappa^3} N_k \frac{1}{(2\pi)^3 V} \delta(E_f + E_0 - \hbar\omega_k) = \Sigma \left(\frac{e^*}{\mu} \right)^2 \frac{4\hbar^2 k_f^2}{V^2 \omega_k \kappa^3} N_k \delta(\Delta E) \ , \tag{119}$$

where $\Delta E = E_f + E_0 - \hbar\omega_k$ and $\Sigma = \left|\sum_\lambda (\hat{\mathbf{k}}_f \cdot \mathbf{e}_{\mathbf{k},\lambda})\right|^2$. To find the total rate, we sum this over all possible final states $\mathbf{k}_f$, which we can approximate as a three-dimensional integral:

$$\Gamma = \frac{V}{8\pi^3}\int_0^\infty k_f^2 \mathrm{d}k_f \int \mathrm{d}\Omega_{\hat{\mathbf{k}}_f} \Gamma_{\mathbf{k}_f} \tag{120}$$

$$= \frac{V}{8\pi^3}\left(\frac{e^*}{\mu}\right)^2 \frac{4\hbar^2}{V^2\omega_k\kappa^3} N_k \int_0^\infty (k_f^2 \mathrm{d}k_f) k_f^2 \delta(E_f - E_0 - \hbar\omega_k) \int \mathrm{d}\Omega_{\hat{\mathbf{k}}_f} \left|\sum_\lambda (\hat{\mathbf{k}}_f \cdot \mathbf{e}_{\mathbf{k},\lambda})\right|^2 . \tag{121}$$

Consider only the angular integral:

$$I_{\{\theta,\phi\}} = \int \mathrm{d}\Omega_{\hat{\mathbf{k}}_f} \left|\sum_\lambda (\hat{\mathbf{k}}_f \cdot \mathbf{e}_{\mathbf{k},\lambda})\right|^2 = \int \mathrm{d}\Omega_{\hat{\mathbf{k}}_f} \sum_\lambda \left[(\hat{\mathbf{k}}_f^i \mathbf{e}^i_{\mathbf{k},\lambda})(\hat{\mathbf{k}}_f^j \mathbf{e}^j_{\mathbf{k},\lambda})\right]$$
$$= \int \mathrm{d}\Omega_{\hat{\mathbf{k}}_f} \sum_\lambda \left[(\mathbf{e}^i_{\mathbf{k},\lambda})(\mathbf{e}^j_{\mathbf{k},\lambda})\right] \hat{\mathbf{k}}_f^i \hat{\mathbf{k}}_f^j = \int \mathrm{d}\Omega_{\hat{\mathbf{k}}_f} \left[\mathbf{e}^i_{\mathbf{k},+1}\mathbf{e}^j_{\mathbf{k},+1} + \mathbf{e}^i_{\mathbf{k},-1}\mathbf{e}^j_{\mathbf{k},-1}\right] \hat{\mathbf{k}}_f^i \hat{\mathbf{k}}_f^j$$
$$= \int \mathrm{d}\Omega_{\hat{\mathbf{k}}_f} \left[\delta^{ij} - \hat{\mathbf{k}}^i\hat{\mathbf{k}}^j\right] \hat{\mathbf{k}}_f^i \hat{\mathbf{k}}_f^j = \int \mathrm{d}\Omega_{\hat{\mathbf{k}}} \left[\hat{\mathbf{k}}_f^i \hat{\mathbf{k}}_f^j - \hat{\mathbf{k}}^i\hat{\mathbf{k}}^j \hat{\mathbf{k}}_f^i \hat{\mathbf{k}}_f^j\right] = \int \mathrm{d}\Omega_{\hat{\mathbf{k}}} \left[1 - (\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}_f)^2\right] .$$

We now define the scattering angle θ which is the angle between the incoming photon momentum $\mathbf{k}$ and the scattered relative momentum $\mathbf{k}_f$, so

$$I_{\{\theta,\phi\}} = \int \mathrm{d}\Omega_{\hat{\mathbf{k}}} \left[1 - \cos^2\theta\right] = 4\pi\sin^2\theta , \tag{122}$$

because $\hat{\mathbf{k}}$ has no dependence on the scattering angle. The radial integral is

$$I_r = \int_0^\infty (k_f^2 \mathrm{d}k_f) k_f^2 \delta\left(\frac{\hbar^2 k_f^2}{2\mu} - E_0 - \hbar\omega_k\right) , \tag{123}$$

the Dirac delta is satisfied if $k_f^2 = (2\mu/\hbar^2)(E_0 + \hbar\omega_k)$, where $\omega_k = ck$. The integral is simply

$$I_r = (k_f^2)^2\big|_{k_f^2 = (2\mu/\hbar^2)(E_0+\hbar\omega_k)} = \frac{4\mu^2}{\hbar^4}(E_0 + \hbar ck)^2 . \tag{124}$$

Collecting all the results, the rate is given by

$$\Gamma = \frac{V}{8\pi^3}\left(\frac{e^*}{\mu}\right)^2 \frac{4\hbar^2}{V^2\omega_k\kappa^3} N_k \frac{4\mu^2}{\hbar^4}(E_0 + \hbar ck)^2 4\pi\sin^2\theta = \frac{N_k}{V}\left(\frac{4^3(e^*)^2}{2^3\pi^2\hbar^2\omega_k\kappa^3}\right)(E_0 + \hbar ck)^2 \sin^2\theta . \tag{125}$$

The scattering rate is related to the differential cross section by the flux of incident particles $J = (N/V)c$, by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\Gamma}{J} = \left(\frac{8(e^*)^2}{\pi^2 c\hbar^2\omega_k\kappa^3}\right)(E_0 + \hbar ck)^2 \sin^2\theta = \left(\frac{8(e^*)^2}{\pi^2 c\hbar^2\omega_k\kappa^3}\right)(E_0 + \hbar ck)^2 \sin^2\theta . \tag{126}$$

3.4 Energy and Angular Dependence of Outgoing Protons.

Sketch both the energy and angular dependences of the outgoing protons for gamma rays near threshold.

The angular dependence of the differential cross section is proportional to $\sin^2\theta$ while the energy dependence is proportional to $(E_0 + \hbar ck)^2$, but since the photon is near threshold, we may approximate this as $4E_0^2$. The plots are shown on the back of this page.