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1 Shankar 1.8.8.

1.1 Part 1.

The Hermitian matrices M^1, M^2, M^3, M^4 obey

$$M^i M^j + M^j M^i = 2\delta^{ij} I \ , \tag{1}$$

for $i, j = 1, 2, 3, 4$. In the eigenbasis of M^i, M^i is diagonal, and there exist eigenvectors, $|m_i\rangle$, such that,

$$M^i |m_i\rangle = \lambda_i |m_i\rangle \ , \tag{2}$$

where λ_i is the eigenvalue. When $i = j$, Equation 1 simplifies to $2M^i M^i = 2I$, which when acting on a eigenvector gives,

$$2M^i M^i |m_i\rangle = 2 |m_i\rangle \Rightarrow 2M^i \lambda_i |m_i\rangle = 2 |m_i\rangle \Rightarrow \lambda_i^2 |m_i\rangle = |m_i\rangle \ . \tag{3}$$

Hermitian operators must have real eigenvalue because they are observables, so $\lambda_i = \pm 1$.

1.2 Part 2.

Using the relationship of the Hermitian operators M^i given in Equation 1, when $i \neq j$,

$$M^i M^j = -M^j M^i \ . \tag{4}$$

From the same identity when $i = j$, $M^i M^i = I$. Therefore acting on the left with another M^i gives,

$$M^i M^i M^j = -M^i M^j M^i \Rightarrow I M^j = -M^i M^j M^i \ , \tag{5}$$

and taking the trace of both sides yields,

$$\mathrm{Tr}[M^j] = -\,\mathrm{Tr}[M^i M^j M^i] \ , \tag{6}$$

which after noting $\mathrm{Tr}[ABC] = \mathrm{Tr}[CBA]$, becomes

$$\mathrm{Tr}[M^j] = -\,\mathrm{Tr}[M^i M^i M^j] \Rightarrow \mathrm{Tr}[M^j] = -\,\mathrm{Tr}[M^j] \ , \tag{7}$$

and the only way that is possible is if $\mathrm{Tr}[M^j] = 0$.

1.3 Part 3.

In any basis, the trace of M^i is the same, which can be shown by again noting the trace is cyclic. Using a unitary operator U to change the basis of M^i, and taking the trace

$$\mathrm{Tr}[U^\dagger M^i U] = -\,\mathrm{Tr}[U U^\dagger M^i] = \mathrm{Tr}[M^i] \ . \tag{8}$$

The trace of M^i in the eigenbasis is zero, and therefore is zero in any basis. In the eigenbasis, the trace is the sum of the eigenvalues, ± 1. In order for this to be zero, there must be an even number of eigenvalues, which in a diagonal matrix corresponds to the dimensionality of the matrix. Therefore M^i must be even-dimensional.

2 Shankar 1.8.9.

A collection of masses m_a, located at $\mathbf{r}_a$, is rotating around a common axis with angular velocity vector $\boldsymbol{\omega}$. Their total angular momentum is given by

$$\mathbf{l} = \sum_a m_a(\mathbf{r}_a \times \mathbf{v}_a) = \sum_a m_a \mathbf{r}_a \times (\boldsymbol{\omega} \times \mathbf{r}_a) = \sum_a m_a[\boldsymbol{\omega}(\mathbf{r}_a \cdot \mathbf{r}_a) - \mathbf{r}_a(\boldsymbol{\omega} \cdot \mathbf{r}_a)] = \sum_a m_a[(r_a)^2\boldsymbol{\omega} - \mathbf{r}_a(\boldsymbol{\omega} \cdot \mathbf{r}_a)] \ , \tag{9}$$

using the identity for the vector triple product. To find the i^{th} Cartesian coordinate of angular momentum the vectors $\boldsymbol{\omega}$ and $\mathbf{r_a}$ are reduced to their i^{th} components, ω_i and $(r_i)_a$. The i^{th} component of angular momentum becomes

$$\begin{aligned} l_i &= \sum_a m_a \left[(r_a)^2 \omega_i - (r_i)_a \sum_{j=1}^{3} \omega_j (r_j)_a \right] \\ &= \sum_a m_a \left[(r_a)^2 \sum_{j=1}^{3} [\omega_j \delta_{ij}] - (r_i)_a \sum_{j=1}^{3} [\omega_j (r_j)_a] \right] \ , \end{aligned} \tag{10}$$

where $(r_a)^2$ is the squared norm of the displacement vector, $\mathbf{r}_a \cdot \mathbf{r}_a$. The sum over j on the first line was introduced to replace the dot product. The Kronecker Delta was introduced on the bottom line in order to change the index of ω, allowing the first term to be written as a sum over the index j. This allows the sum over a of two sums of j to be written as a sum over j of a two-term sum over a. Now the i^{th} component of angular momentum is given by,

$$l_i = \sum_{j=1}^{3} \left[\omega_j \sum_a m_a [(r_a)^2 \delta_{ij} - (r_i)_a (r_j)_a] \right] = \sum_j M_{ij} \omega_j \ , \tag{11}$$

where $M_{ij} = \sum_a m_a[(r_a)^2\delta_{ij} - (r_i)_a(r_j)_a]$, which in Dirac notation is $|l\rangle = M\,|\omega\rangle$.

2.1 Part 1.

In order for the angular momentum and angular velocities to always be parallel the i^{th} component of angular momentum must be only a scaling factor of the i^{th} component of angular velocity, therefore all M matrices must be diagonal. Otherwise the i^{th} component of $\mathbf{l}$ will couple with other components of $\boldsymbol{\omega}$. By looking at Equation 11, and noting that for off-diagonal entries $i \neq j$, it can be shown the off-diagonal entries are of the form,

$$M_{ij} = \sum_a m_a[-(r_i)_a(r_j)_a] \ . \tag{12}$$

A mass distribution can be chosen such that this quantity will be nonzero, which implies the angular momentum and angular velocities will not be parallel.

2.2 Part 2.

If the matrix M is Hermitian, then $M^\dagger = M$. This requires $M_{ij} = M_{ji}^*$, but since M is real, $M_{ji}^* = M_{ji}$, so if $M_{ij} = M_{ji}$, then M is Hermitian. The matrix element M_{ji} is calculated,

$$\begin{aligned} M_{ji} &= \sum_a m_a[(r_a)^2\delta_{ji} - (r_j)_a(r_i)_a] \\ &= \sum_a m_a[(r_a)^2\delta_{ij} - (r_i)_a(r_j)_a] \\ &= M_{ij} \ . \end{aligned} \tag{13}$$

The facts that multiplication is commutative, and the Kronecker Delta is a function of dummy indeces, which can be swapped, makes the first two lines of Equation 30 equal. Therefore the moment of inertia matrix M is Hermititan.

2.3 Part 3.

By noting M is Hermitian, there exisits a basis $\{|e_i\rangle\}$ that will diagonalize this matrix. The angular momentum, $|\omega\rangle$, can be represented in this basis as $\sum_i \omega_i |e_i\rangle$. In this basis, because M is diagonal, the angular momentum can be expressed as

$$|l\rangle = M \sum_i \omega_i |e_i\rangle = \sum_i \omega_i M |e_i\rangle = \sum_i \omega_i m_i |e_i\rangle \ , \tag{14}$$

where m_i is the eigenvalue on the i^{th} basis vector. Since M is the moment of inertia matrix in three-dimensional Cartesian space, it is a 3×3 matrix, which has 3 bases and 3 eigenvalues. If the angular velocity points directly along one of these bases, such that $|\omega\rangle = \omega_i |e_i\rangle$, it will be parallel with angular momentum. This gives three directions for which $|l\rangle \,||\, |w\rangle$, these correspond to the eigenvectors of M.

2.4 Part 4.

Due to the complete symmetry of a sphere, the moment of inertia matrix, M, must be diagonal. Any direction is an eigendirection for rotation, which means the eigenvalues are triply degenerate. This says the form of M for a sphere is

$$M = mI_3 \ , \tag{15}$$

where m is the triply degenerate eigenvalue, and I_3 is the three dimensional identity matrix.

3 Shankar 1.8.10.

The two Hermitian matrices,

$$\Omega = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \ , \ \Lambda = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} \ , \tag{16}$$

can be diagonalized simultaneously if their commutator is zero. In this case the commutator,

$$[\Omega, \Lambda] = \Omega\Lambda - \Lambda\Omega = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \tag{17}$$

$$= \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix} = 0 \tag{18}$$

is zero, which confirms Ω and Λ can be diagonalized simultaneously. In order to diagonalize both matrices with a unitary transformation, a basis must be chosen. Ω is degenerate in its eigenvalues while Λ is not, so the eigenvectors of Λ will create a good basis. Solving the characteristic equation of Λ,

$$0 = \begin{vmatrix} 2-\omega & 1 & 1 \\ 1 & -\omega & -1 \\ 1 & -1 & 2-\omega \end{vmatrix} \tag{19}$$

for ω will return the three non-degenerate eigenvalues of Λ.Using Mathematica to calculate the determinant and solve the characteristic equation yields $\omega = 3, 2, -1$. Using these values of ω, the eigenvectors can be found by solving the equation,

$$\Lambda |v_i\rangle = \omega_i |v_i\rangle \ \ , \tag{20}$$

where $|v_i\rangle$ is one eigenvector. Starting with an arbitrary eigenvector, $|v\rangle = (x, y, z)$, Eqaution 20 results in a system of three equations which can be solved for x, y, z to obtain the three eigenvectors. This system of equations is

$$2x + y + z = \omega x \tag{21}$$

$$x - z = \omega y \tag{22}$$

$$x - y + 2z = \omega z \tag{23}$$

The solutions to these equations for each choice of ω are,

$$\omega = 3 : \begin{cases} x = z \\ y = 0 \end{cases} \qquad \omega = 2 : \begin{cases} x = y \\ z = -y \end{cases} \qquad \omega = -1 : \begin{cases} y = -2x \\ z = -x \end{cases} \quad . \tag{24}$$

From these solutions eigenvectors can be found. These eigenvectors create the basis that diagonalizes both matrices. The unitary matrix required to transform these matrices into this basis has these normalized eigenvectors as its columns. The normalized eigenvectors are,

$$\omega_1 = 3 \ , \ |v_1\rangle = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \omega_2 = 2 \ , \ |v_2\rangle = \sqrt{\frac{1}{3}} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \qquad \omega_3 = -1 \ , \ |v_3\rangle = \sqrt{\frac{1}{6}} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \ , \tag{25}$$

from which the unitary operator required for the transformation to the eigenbasis can be determined to be

$$U = \begin{bmatrix} \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{6}} \\ 0 & -\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{6}} \\ \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{6}} \end{bmatrix} . \tag{26}$$

To verify this diagonalizes both Λ and Ω, the quantities $U^\dagger \Lambda U$ and $U^\dagger \Omega U$ are calculated,

$$D_\Lambda = U^\dagger \Lambda U = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad D_\Omega = U^\dagger \Omega U = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \tag{27}$$

which are diagonal matrices with the eigenvalues of Λ and Ω down the diagonals, respectively.

4 Shankar 1.10.4.

A string fixed at endpoints $x = 0$ and $x = L$ has initial the initial condition,

$$\Psi(x, t = 0) = \begin{cases} \frac{2h}{L}x, & 0 \leq x \leq \frac{L}{2} \\ \frac{2h}{L}(L - x), & \frac{L}{2} \leq x \leq L \end{cases} . \tag{28}$$

Before taking the time evolution into account, the string is subject to the same boundary conditions as the string in Shankar Example 1.10.1, which gives the time evolution of a string with fixed endpoints (Shankar Equation 1.10.59) to be

$$\Psi(x, t) = \sum_{m=1}^{\infty} \sqrt{\frac{2}{L}} \sin\left(\frac{m\pi x}{L}\right) \cos(\omega_m t) \langle m|\psi(0)\rangle \ , \tag{29}$$

where m is an integer and the matrix inner product is given by the integral

$$\langle m|\psi(0)\rangle = \sqrt{\frac{2}{L}} \int_0^L \sin\left(\frac{m\pi x}{L}\right) \Psi(x, 0) dx \ . \tag{30}$$

For the given $\Psi(x, 0)$, Equation 30 becomes

$$\begin{aligned} \langle m|\psi(0)\rangle &= \sqrt{\frac{2}{L}} \Bigg[\int_0^{L/2} \left(\frac{2h}{L}\right) x \sin\left(\frac{m\pi x}{L}\right) dx + \int_{L/2}^{L} \left(\frac{2h}{L}\right) (L - x) \sin\left(\frac{m\pi x}{L}\right) dx \Bigg] \\ &= \sqrt{\frac{2}{L}} \Bigg[\int_0^{L/2} \left(\frac{2h}{L}\right) x \sin\left(\frac{m\pi x}{L}\right) dx + \int_{L/2}^{L} \left(\frac{2h}{L}\right) L \sin\left(\frac{m\pi x}{L}\right) dx - \int_{L/2}^{L} \left(\frac{2h}{L}\right) x \sin\left(\frac{m\pi x}{L}\right) dx \Bigg] \\ &= \sqrt{\frac{2}{L}} \left[\frac{4hL}{m^2\pi^2} \sin\left(\frac{m\pi}{2}\right) - \frac{2hL}{m^2\pi^2} \cos\left(\frac{m\pi}{2}\right) \right] \\ &= \sqrt{\frac{2}{L}} \frac{4hL}{m^2\pi^2} \sin\left(\frac{m\pi}{2}\right) . \end{aligned} \tag{31}$$

The above integrals were evaluated using MATHEMATICA. Since m is an integer, the cosine term is identically zero in the second to last step above. Substituting this value into Equation 32, yields the final result,

$$\begin{aligned} \Psi(x, t) &= \sum_{m=1}^{\infty} \sqrt{\frac{2}{L}} \sin\left(\frac{m\pi x}{L}\right) \cos(\omega_m t) \sqrt{\frac{2}{L}} \frac{4hL}{m^2\pi^2} \sin\left(\frac{m\pi}{2}\right) \\ &= \sum_{m=1}^{\infty} \left(\frac{8h}{m^2\pi^2}\right) \sin\left(\frac{m\pi x}{L}\right) \cos(\omega_m t) \sin\left(\frac{m\pi}{2}\right) . \end{aligned} \tag{32}$$

5 Problem #5: Hermitian Matrices.

In order to find the commutator and anti-commutator for each pair of the matrices

$$H = \begin{bmatrix} A^\dagger & 0 \\ 0 & A \end{bmatrix},\ Q = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix},\ Q^\dagger = \begin{bmatrix} 0 & A^\dagger \\ 0 & 0 \end{bmatrix},$$

the products of each pair were found, shown below.

$$HQ = \begin{bmatrix} 0 & 0 \\ A^2 & 0 \end{bmatrix} \tag{33}$$

$$QH = \begin{bmatrix} 0 & 0 \\ AA^\dagger & 0 \end{bmatrix} \tag{34}$$

$$HQ^\dagger = \begin{bmatrix} 0 & (A^\dagger)^2 \\ 0 & 0 \end{bmatrix} \tag{35}$$

$$Q^\dagger H = \begin{bmatrix} 0 & A^\dagger A \\ 0 & 0 \end{bmatrix} \tag{36}$$

$$QQ^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & AA^\dagger \end{bmatrix} \tag{37}$$

$$Q^\dagger Q = \begin{bmatrix} A^\dagger A & 0 \\ 0 & 0 \end{bmatrix} \tag{38}$$

$$HH = \begin{bmatrix} (A^\dagger)^2 & 0 \\ 0 & A^2 \end{bmatrix} \tag{39}$$

$$QQ = Q^\dagger Q^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{40}$$

Using these products, the commutator, $[A, B] = AB - BA$, and the anti-commutator, $\{A, B\} = AB + BA$ can be found directly. It is unnecessary to calculate the commutator of a matrix and itself because it is zero by definition, while this is untrue for the anti-commutator.

$$[H, Q] = \begin{bmatrix} 0 & 0 \\ A^2 - AA^\dagger & 0 \end{bmatrix} = -[Q, H] \tag{41}$$

$$[H, Q^\dagger] = \begin{bmatrix} 0 & (A^\dagger)^2 - A^\dagger A \\ 0 & 0 \end{bmatrix} = -[Q^\dagger, H] \tag{42}$$

$$[Q, Q^\dagger] = -[Q^\dagger, Q] = \begin{bmatrix} -A^\dagger A & 0 \\ 0 & AA^\dagger \end{bmatrix} \tag{43}$$

$$\{Q, Q^\dagger\} = \{Q^\dagger, Q\} = \begin{bmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{bmatrix} \tag{44}$$

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{45}$$

$$\{H, Q\} = \{Q, H\} = \begin{bmatrix} 0 & 0 \\ A^2 + AA^\dagger & 0 \end{bmatrix} \tag{46}$$

$$\{H, Q^\dagger\} = \{Q^\dagger, H\} = \begin{bmatrix} 0 & (A^\dagger)^2 + A^\dagger A \\ 0 & 0 \end{bmatrix} \tag{47}$$

$$\{H, H\} = \{H, H\} = \begin{bmatrix} 2(A^\dagger)^2 & 0 \\ 0 & 2A^2 \end{bmatrix} \tag{48}$$