Dylan J. Temples: Solution Set Two

Northwestern University, Quantum Mechanics I
Principles of Quantum Mechanics, Second Ed.- Shankar
October 9, 2015

Contents

1 Shankar 2.7.5. **2**

2 Shankar 2.8.3. **5**

3 Shankar 2.8.4. **7**
3.1 Verifying Infinitesimal Rotation is a Canonical Transformation 7
3.2 Determining Generator of Transformation 8

4 Shankar 2.8.5. **9**
4.1 Untransformed Frame 9
4.2 Transformed Frame 10
4.3 Comparison and Generalization 11

5 Problem #5: Hermitian Matrices **12**

1 Shankar 2.7.5.

Consider the polar coordinates, $\rho = \sqrt{x^2+y^2}$ and $\phi = \tan^{-1}(y/x)$, whose momenta are given by

$$p_\rho = \frac{xp_x + yp_y}{\sqrt{x^2+y^2}} \tag{1}$$

$$p_\phi = xp_y - yp_x \ , \tag{2}$$

where $x,\ y$ are the Cartesian coordinates and $p_x,\ p_y$ are their corresponding momenta. Let barred quantities be quantities in the transformed frame. In order to show these are canonical, the transformed coordinates must satisfy the conditions given in Shankar Equation 2.7.18,

$$\begin{cases} \{\bar{q}_j, \bar{q}_k\} = 0 = \{\rho, \phi\} = \{\phi, \rho\} \\ \{\bar{p}_j, \bar{p}_k\} = 0 = \{p_\rho, p_\phi\} = \{p_\phi, p_\rho\} \\ \{\bar{q}_j, \bar{p}_k\} = \delta_{jk} = \begin{cases} \{\rho, p_\phi\} = \{\phi, p_\rho\} = 0 \\ \{\rho, p_\rho\} = \{\phi, p_\phi\} = 1 \end{cases} \end{cases} , \tag{3}$$

where $\{\omega, \lambda\}$ is the Poisson Bracket, defined to be

$$\{\omega, \lambda\} = \sum_i \left(\frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i} \right) . \tag{4}$$

For the choice of initial coordinates the Poisson Bracket becomes,

$$\{\omega, \lambda\} = \left(\frac{\partial \omega}{\partial x}\frac{\partial \lambda}{\partial p_x} - \frac{\partial \omega}{\partial p_x}\frac{\partial \lambda}{\partial x} \right) + \left(\frac{\partial \omega}{\partial y}\frac{\partial \lambda}{\partial p_y} - \frac{\partial \omega}{\partial p_y}\frac{\partial \lambda}{\partial y} \right) . \tag{5}$$

To verify these coordinates are canonical, the first derivatives of the transformed positions and momenta with respect to the original coordinates and momenta, are calculated. The derivatives of ρ are:

$$\frac{\partial \rho}{\partial x} = (\rho)_x = \frac{x}{\sqrt{x^2+y^2}} \quad ; \quad \frac{\partial \rho}{\partial y} = (\rho)_y = \frac{y}{\sqrt{x^2+y^2}} \quad ; \quad \frac{\partial \rho}{\partial p_x} = (\rho)_{px} = 0 \quad ; \quad \frac{\partial \rho}{\partial p_y} = (\rho)_{py} = 0 \ .$$

The derivatives of ϕ are:

$$\frac{\partial \phi}{\partial x} = (\phi)_x = -\frac{y}{x^2+y^2} \quad ; \quad \frac{\partial \phi}{\partial y} = (\phi)_y = \frac{x}{x^2+y^2} \quad ; \quad \frac{\partial \phi}{\partial p_x} = (\phi)_{px} = 0 \quad ; \quad \frac{\partial \phi}{\partial p_y} = (\phi)_{py} = 0 \ .$$

The derivatives of p_ρ are:

$$\frac{\partial p_\rho}{\partial x} = (p_\rho)_x = \frac{p_x}{\sqrt{x^2+y^2}} - \frac{x(p_x x + p_y y)}{(x^2+y^2)^{3/2}} \quad ; \quad \frac{\partial p_\rho}{\partial y} = (p_\rho)_y = \frac{p_y}{\sqrt{x^2+y^2}} - \frac{y(p_x x + p_y y)}{(x^2+y^2)^{3/2}} \quad ;$$

$$\frac{\partial p_\rho}{\partial p_x} = (p_\rho)_{px} = \frac{x}{\sqrt{x^2+y^2}} \quad ; \quad \frac{\partial p_\rho}{\partial p_y} = (p_\rho)_{py} = \frac{y}{\sqrt{x^2+y^2}} \ .$$

The derivatives of p_ϕ are:

$$\frac{\partial p_\phi}{\partial x} = (p_\phi)_x = p_y \quad ; \quad \frac{\partial p_\phi}{\partial y} = (p_\phi)_y = -p_x \quad ; \quad \frac{\partial p_\phi}{\partial p_x} = (p_\phi)_{px} = -y \quad ; \quad \frac{\partial p_\phi}{\partial p_y} = (p_\phi)_{py} = x \ .$$

The first condition for the polar coordinates to be canonical is they satisfy $\{\rho, \phi\} = 0$. This is easy to see is true, because in each term of the sum there is a derivative of a coordinate with respect to a momentum. As is shown in the first line, neither coordinate depends on either momentum. Written out it is

$$\begin{aligned}
\{\rho, \phi\} &= [(\rho)_x(\phi)_{px} - (\rho)_{px}(\phi)_x] + [(\rho)_y(\phi)_{py} - (\rho)_{py}(\phi)_y] \\
&= [(\rho)_x(0) - (0)(\phi)_x] + [(\rho)_y(0) - (0)(\phi)_y] \\
&= 0 \ ,
\end{aligned}$$

which because it is zero also implies that $\{\phi, \rho\} = 0$. The next condition to be satisfied is $\{p_\rho, p_\phi\} = 0$,

$$\begin{aligned}
\{p_\rho, p_\phi\} &= [(p_\rho)_x(p_\phi)_{px} - (p_\rho)_{px}(p_\phi)_x] + [(p_\rho)_y(p_\phi)_{py} - (p_\rho)_{py}(p_\phi)_y] \\
&= \left[-y\left(\frac{p_x}{\sqrt{x^2+y^2}} - \frac{x(p_x x + p_y y)}{(x^2+y^2)^{3/2}}\right) - \frac{p_y x}{\sqrt{x^2+y^2}}\right] + \left[x\left(\frac{p_y}{\sqrt{x^2+y^2}} - \frac{y(p_x x + p_y y)}{(x^2+y^2)^{3/2}}\right) - \frac{-p_x y}{\sqrt{x^2+y^2}}\right] \\
&= -\frac{y p_x}{\sqrt{x^2+y^2}} + \frac{xy(p_x x + p_y y)}{(x^2+y^2)^{3/2}} + \frac{x p_y}{\sqrt{x^2+y^2}} - \frac{xy(p_x x + p_y y)}{(x^2+y^2)^{3/2}} - \frac{p_y x}{\sqrt{x^2+y^2}} + \frac{p_x y}{\sqrt{x^2+y^2}} \\
&= 0 \ ,
\end{aligned}$$

because each term exactly cancels with another. Again because the Poisson Bracket $\{p_\rho, p_\phi\} = 0$, so does $\{p_\phi, p_\rho\}$. Next, the conditions $\{\rho, p_\phi\} = 0$ and $\{\phi, p_\rho\} = 0$ must be satisfied,

$$\begin{aligned}
\{\rho, p_\phi\} &= [(\rho)_x(p_\phi)_{px} - (\rho)_{px}(p_\phi)_x] + [(\rho)_y(p_\phi)_{py} - (\rho)_{py}(p_\phi)_y] \\
&= [(\rho)_x(p_\phi)_{px} - (0)(\phi)_x] + [(\rho)_y(p_\phi)_{py} - (0)(\phi)_y] \\
&= \frac{x}{\sqrt{x^2+y^2}}(-y) + \frac{y}{\sqrt{x^2+y^2}}(x) \\
&= 0 \ ,
\end{aligned}$$

$$\begin{aligned}
\{\phi, p_\rho\} &= [(\phi)_x(p_\rho)_{px} - (\phi)_{px}(p_\rho)_x] + [(\phi)_y(p_\rho)_{py} - (\phi)_{py}(p_\rho)_y] \\
&= [(\phi)_x(p_\rho)_{px} - (0)(\rho)_x] + [(\phi)_y(p_\rho)_{py} - (0)(\rho)_y] \\
&= -\frac{y}{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}} + \frac{x}{x^2+y^2}\frac{y}{\sqrt{x^2+y^2}} \\
&= 0 \ .
\end{aligned}$$

The two final conditions are $\{\rho, p_\rho\} = 1$ and $\{\phi, p_\phi\} = 1$,

$$\begin{aligned}
\{\rho, p_\rho\} &= [(\rho)_x(p_\rho)_{px} - (\rho)_{px}(p_\rho)_x] + [(\rho)_y(p_\rho)_{py} - (\rho)_{py}(p_\rho)_y] \\
&= [(\rho)_x(p_\rho)_{px} - (0)(\rho)_x] + [(\rho)_y(p_\rho)_{py} - (0)(\rho)_y] \\
&= \frac{x}{\sqrt{x^2+y^2}}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{y}{\sqrt{x^2+y^2}}\left(\frac{y}{\sqrt{x^2+y^2}}\right) \\
&= \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} \\
&= 1
\end{aligned}$$

$$\begin{aligned}\{\phi, p_\phi\} &= [(\phi)_x(p_\phi)_{px} - (\phi)_{px}(p_\phi)_x] + [(\phi)_y(p_\phi)_{py} - (\phi)_{py}(p_\phi)_y] \\ &= [(\phi)_x(p_\phi)_{px} - (0)(\phi)_x] + [(\phi)_y(p_\phi)_{py} - (0)(\phi)_y] \\ &= -\frac{y}{x^2+y^2}(-y) + \frac{x}{x^2+y^2}(x) \\ &= 1 \ .\end{aligned}$$

Therefore the polar coordinates satisfy all the conditions set in Equation 3, which proves they are canonical.

2 Shankar 2.8.3.

Consider a Hamiltonian given by,

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2} m\omega^2 (x^2 + y^2) \ , \tag{6}$$

which is invariant under a rotation of both the coordinates and momenta, as well as invariant under a rotation of just the coordinates. However, this is a noncanonical transformation, which can be shown by checking that one of the conditions for a canonical transformation (Shankar Equation 2.7.18) fails. Let barred quantities be quantities in the transformed frame. Under an infinitesimal rotation of the coordinates, the new coordinates become (Shankar 2.8.2),

$$x \to \bar{x} = x\cos\epsilon - y\sin\epsilon = x - y\epsilon \tag{7}$$

$$y \to \bar{y} = x\sin\epsilon + y\cos\epsilon = x\epsilon + y \ , \tag{8}$$

and the momenta remain unchanged: $\bar{p}_x = p_x$ and $\bar{p}_y = p_y$. The derivatives of the coordinates and momenta are:

$$\begin{cases} \frac{\partial \bar{x}}{\partial x} = 1 \\ \frac{\partial \bar{x}}{\partial y} = -\epsilon \\ \frac{\partial \bar{x}}{\partial p_x} = 0 \\ \frac{\partial \bar{x}}{\partial p_y} = 0 \end{cases} ; \quad \begin{cases} \frac{\partial \bar{y}}{\partial x} = \epsilon \\ \frac{\partial \bar{y}}{\partial y} = 1 \\ \frac{\partial \bar{y}}{\partial p_x} = 0 \\ \frac{\partial \bar{y}}{\partial p_y} = 0 \end{cases} ; \quad \begin{cases} \frac{\partial \bar{p}_x}{\partial x} = 0 \\ \frac{\partial \bar{p}_x}{\partial y} = 0 \\ \frac{\partial \bar{p}_x}{\partial p_x} = 1 \\ \frac{\partial \bar{p}_x}{\partial p_y} = 0 \end{cases} ; \quad \begin{cases} \frac{\partial \bar{p}_y}{\partial x} = 0 \\ \frac{\partial \bar{p}_y}{\partial y} = 0 \\ \frac{\partial \bar{p}_y}{\partial p_x} = 0 \\ \frac{\partial \bar{p}_y}{\partial p_y} = 1 \end{cases} . \tag{9}$$

Using definition of the Poisson Bracket (Equation 4) with the choice of untransformed coordinates, the Poisson Bracket $\{\bar{q}_j, \bar{p}_k\}$ (which is zero for a canonical transformation when $j \neq k$) is given by

$$\{\bar{q}_j, \bar{p}_k\} = 0 = \left(\frac{\partial \bar{q}_j}{\partial x}\frac{\partial \bar{p}_k}{\partial p_x} - \frac{\partial \bar{q}_j}{\partial p_x}\frac{\partial \bar{p}_k}{\partial x} \right) + \left(\frac{\partial \bar{q}_j}{\partial y}\frac{\partial \bar{p}_k}{\partial p_y} - \frac{\partial \bar{q}_j}{\partial p_y}\frac{\partial \bar{p}_k}{\partial y} \right) . \tag{10}$$

By picking a specific k for the transformed coordinates, say y, and noting $\frac{\partial \bar{p}_j}{\partial q_k} = 0$ for any $j,\ k$, this expression becomes

$$\begin{aligned} \{\bar{q}_j, \bar{p}_y\} &= \left(\frac{\partial \bar{q}_j}{\partial x}\frac{\partial \bar{p}_y}{\partial p_x} - \frac{\partial \bar{q}_j}{\partial p_x}\frac{\partial \bar{p}_y}{\partial x} \right) + \left(\frac{\partial \bar{q}_j}{\partial y}\frac{\partial \bar{p}_y}{\partial p_y} - \frac{\partial \bar{q}_j}{\partial p_y}\frac{\partial \bar{p}_y}{\partial y} \right) && (11) \\ &= \left(\frac{\partial \bar{q}_j}{\partial x}(0) - \frac{\partial \bar{q}_j}{\partial p_x}(0) \right) + \left(\frac{\partial \bar{q}_j}{\partial y}(1) - \frac{\partial \bar{q}_j}{\partial p_y}(0) \right) && (12) \\ &= \frac{\partial \bar{q}_j}{\partial y} \ . && (13) \end{aligned}$$

For this to be a canonical transformation both of the following conditions must be met,

$$\{\bar{x}, \bar{p}_y\} = 0 \tag{14}$$

$$\{\bar{y}, \bar{p}_y\} = 1 \ , \tag{15}$$

so if one of these is found to be false, it is a noncanonical transformation. The transformed coordinates' derivatives with respect to y make these two conditions

$$\{\bar{x}, \bar{p}_y\} = -\epsilon = 0 \tag{16}$$

$$\{\bar{y}, \bar{p}_y\} = 1 = 1 \ , \tag{17}$$

so clearly it is a noncanonical transformation. In such a transformation, it is not possible to write the change in $\mathcal{H}$ as $\epsilon\{\mathcal{H}, g\}$ for any $g(q_i, p_i)$. To verify this, compare the transformation in Equations 7 and 8, to the dynamical consequences of a canonical transformation on Shankar page 99. Define the transformation as

$$x \to \bar{x} = x - y\epsilon = x + \epsilon \frac{\partial g}{\partial p_x} \tag{18}$$

$$y \to \bar{y} = x\epsilon + y = y + \epsilon \frac{\partial g}{\partial p_y} \tag{19}$$

$$p_x \to \bar{p}_x = p_x = p_x - \epsilon \frac{\partial g}{\partial x} \tag{20}$$

$$p_y \to \bar{p}_y = p_y = p_y - \epsilon \frac{\partial g}{\partial y} \ , \tag{21}$$

which implies

$$\begin{cases} \frac{\partial g}{\partial p_x} = -y \\ \frac{\partial g}{\partial p_y} = x \end{cases} \quad ; \quad \begin{cases} \frac{\partial g}{\partial x} = 0 \\ \frac{\partial g}{\partial y} = 0 \end{cases} \quad . \tag{22}$$

However, there can not be a function $g(x, y, p_x, p_y)$ for which this is true. For the derivative of a p_i to be a q_j, the position would have to be a coefficient on the momentum, *e.g.* $g = -yp_x + xp_y$. This function satisfies the conditions on the left in Equation 22, but does not satisfy the ones on the right. There is no solution that exists and satisfies all 4 conditions simultaneously. Therefore an equation of the form of Shankar Equation 2.8.4 cannot be written.

3 Shankar 2.8.4.

Consider the one dimensional Hamiltonian given by

$$\mathcal{H} = \frac{1}{2}p^2 + \frac{1}{2}x^2 \ , \tag{23}$$

which is invariant under an infinitesimal rotation in phase space, the $x - p$ plane.

3.1 Verifying Infinitesimal Rotation is a Canonical Transformation

To verify this transformation is canonical, the transformed coordinate and momentum must be found. A rotation in the $x - p$ plane is given by the matrix equation

$$\begin{bmatrix} \bar{x} \\ \bar{p} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \ , \tag{24}$$

however, in an infinitesimal rotation, $\theta \to \epsilon$, where ϵ is an infinitesimal angle. Applying the first order approximations for $\sin\theta$ and $\cos\theta$ for small values of θ (as in Shankar Equations 2.8.2 and 2.8.3), which will eliminate any ϵ^2 or higher order terms, Equation 24 becomes

$$\begin{bmatrix} \bar{x} \\ \bar{p} \end{bmatrix} = \begin{bmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \ , \tag{25}$$

which says the transformation has the form

$$x \to \bar{x} = x - \epsilon p \tag{26}$$

$$p \to \bar{p} = \epsilon x + p \ . \tag{27}$$

To verify this transformation is canonical, this coordinate and momentum must satisfy the one dimensional equivalent of Shankar Equation 2.7.18, $\{\bar{x}, \bar{p}\} = 1$. The first step to calculating this is to determine the derivatives of the transformed position and momentum with respect to the untransformed quantities, which are

$$\begin{cases} \frac{\partial \bar{x}}{\partial x} = 1 \\ \frac{\partial \bar{x}}{\partial p} = -\epsilon \end{cases} \quad ; \quad \begin{cases} \frac{\partial \bar{p}}{\partial x} = \epsilon \\ \frac{\partial \bar{p}}{\partial p} = 1 \end{cases} \quad . \tag{28}$$

Now their Poisson Bracket is

$$\{\bar{x}, \bar{p}\} = \frac{\partial \bar{x}}{\partial x}\frac{\partial \bar{p}}{\partial p} - \frac{\partial \bar{x}}{\partial p}\frac{\partial \bar{p}}{\partial x} \tag{29}$$

$$= (1)(1) - (-\epsilon)(\epsilon) \tag{30}$$

$$= 1 + \epsilon^2 \ , \tag{31}$$

any ϵ^2 terms were discarded when finding the transformation equations, so it may be discarded here. Therefore the Poisson Bracket is one, as expected. This confirms an infinitesimal rotation is canonical.

3.2 Determining Generator of Transformation

The generator of the transformation can be found by applying Shankar Equation 2.8.3 to the transformed coordinates and momenta. For this problem the equations for the transformed position and momentum can be found in Equations 26 and 27, and the equations equivalent to Shankar Equation 2.8.3 are

$$x \rightarrow \bar{x} = x + \epsilon \frac{\partial g}{\partial p} \tag{32}$$

$$y \rightarrow \bar{y} = p - \epsilon \frac{\partial g}{\partial x} \ , \tag{33}$$

where $g(x,p)$ is the generator. Equating these with Equations 26 and 27 gives a set of coupled differential equations for g,

$$x + \epsilon \frac{\partial g}{\partial p} = x - \epsilon p \tag{34}$$

$$p - \epsilon \frac{\partial g}{\partial x} = \epsilon x + p \ , \tag{35}$$

which reduce to

$$\frac{\partial g}{\partial p} = -p \rightarrow g(p) = -\frac{1}{2}p^2 \tag{36}$$

$$\frac{\partial g}{\partial x} = -x \rightarrow g(x) = -\frac{1}{2}x^2 \ , \tag{37}$$

which gives the solution $g(x,p) = -\frac{1}{2}(x^2 + p^2)$.

4 Shankar 2.8.5.

Noncanonical transformations that leave $\mathcal{H}$ invariant, do not map a solution into another solution. In another view, this is equivalent to saying an experiment and its transformed version do not give the same result even though the transformation leaves $\mathcal{H}$ invariant, but is noncanonical. This can be shown to be true for a specific case. Consider the two dimensional harmonic oscillator potential,

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2) \ , \tag{38}$$

with initial conditions $(x_0, y_0, p_{x0}, p_{y0}) = (a, 0, b, 0)$ in the untransformed frame. Considering the same transformation as Shankar 2.8.3, which rotates the coordinates but leaves the momenta unchanged, the initial conditions in the transformed frame are $(\bar{x}_0, \bar{y}_0, \bar{p}_{x0}, \bar{p}_{y0}) = (0, a, b, 0)$. To see how this noncanonical transformation effects the time evolution of this system, we solve for the equations of motion in both the original and transformed frames. Let barred quantities be quantities in the transformed frame.

4.1 Untransformed Frame

The Hamiltonian in the untransformed frame is given by,

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2) \ , \tag{39}$$

which yields Hamilton's canonical equations,

$$\begin{cases} \frac{\partial \mathcal{H}}{\partial x} = -\dot{p}_x = m\omega^2 x \\ \frac{\partial \mathcal{H}}{\partial y} = -\dot{p}_y = m\omega^2 y \end{cases} \quad ; \quad \begin{cases} \frac{\partial \mathcal{H}}{\partial p_x} = -\dot{x} = p_x/m \\ \frac{\partial \mathcal{H}}{\partial p_y} = -\dot{y} = p_y/m \end{cases} . \tag{40}$$

The equations resulting from derivatives with respect to p_i can be differentiated with respect to time to yield,

$$\ddot{x} = \dot{p}_x/m = -\omega^2 x \tag{41}$$

$$\ddot{y} = \dot{p}_y/m = -\omega^2 y \ , \tag{42}$$

by substituting in the equations resulting from derivatives with respect to q_i. This system of differential equations is trivial to solve, and gives solutions of the form

$$x(t) = A\sin(\omega t) + B\cos(\omega t) \tag{43}$$

$$y(t) = C\sin(\omega t) + D\cos(\omega t) \ , \tag{44}$$

where the constants A, B, C, and D can be found by imposing initial conditions. By imposing the initial conditions on the position,

$$x(0) = a = A\sin(0) + B\cos(0) \tag{45}$$

$$y(0) = 0 = C\sin(0) + D\cos(0) \ , \tag{46}$$

which implies $B = a$ and $D = 0$. Now impose initial conditions on the velocity by noting that in Equation 41, $p_i = m\dot{q}_i$. By differentiating Equations 43 and 44 with respect to time and multiplying through by mass m, equations for the momenta as a function of time can be found,

$$p_x(t) = m\omega A\cos(\omega t) - am\omega\sin(\omega t) \tag{47}$$

$$p_y(t) = m\omega C\cos(\omega t) \ , \tag{48}$$

note that the initial conditions of position were applied. Applying the momentum initial conditions yields,

$$p_x(0) = b = m\omega A\cos(0) - a\omega\sin(0) \tag{49}$$

$$p_y(0) = 0 = m\omega C\cos(0) \ , \tag{50}$$

which implies $C = 0$ and $A = b/(m\omega)$, yielding the true equations of motion of the untransformed frame,

$$x(t) = \frac{b}{m\omega}\sin(\omega t) + a\cos(\omega t) \tag{51}$$

$$y(t) = 0 \ . \tag{52}$$

These equations say that because there was no initial y displacement or momentum, nothing will ever happen in that direction, and the system is basically one-dimensional.

4.2 Transformed Frame

The transformed frame has rotated coordinates, but unchanged momentum, and barred symbols represent these transformed quantities. The true equations of motion in the transformed frame can be found following the same methodology as in the untransformed frame. The Hamiltonian in the transformed frame is given by,

$$\bar{\mathcal{H}} = \frac{\bar{p}_x^2 + \bar{p}_y^2}{2m} + \frac{1}{2}m\omega^2(\bar{x}^2 + \bar{y}^2) \ , \tag{53}$$

which yields identical Hamilton's canonical equations and equations of motion but with barred positions and momenta. Following the same process as in Section 1.4.1, the equations of motion have the form

$$\bar{x}(t) = \bar{A}\sin(\omega t) + \bar{B}\cos(\omega t) \tag{54}$$

$$\bar{y}(t) = \bar{C}\sin(\omega t) + \bar{D}\cos(\omega t) \ , \tag{55}$$

but the initial conditions are now changed, so to find the barred constants, the initial conditions on the coordinates are applied,

$$\bar{x}(0) = 0 = \bar{A}\sin(0) + \bar{B}\cos(0) \tag{56}$$

$$\bar{y}(0) = a = \bar{C}\sin(0) + \bar{D}\cos(0) \ , \tag{57}$$

which implies $\bar{B} = 0$ and $\bar{D} = a$. Applying these changes, and imposing the initial conditions on the momenta yields

$$\bar{p}_x(0) = b = m\omega\bar{A}\cos(0) \tag{58}$$

$$\bar{p}_y(0) = 0 = m\omega\bar{C}\cos(0) - am\omega\sin(0) \ , \tag{59}$$

which implies $\bar{A} = b/(m\omega)$ and $\bar{C} = 0$, yielding the true equations of motion for the transformed frame,

$$\bar{x}(t) = \frac{b}{m\omega}\sin(\omega t) \tag{60}$$

$$\bar{y}(t) = a\cos(\omega t) \ . \tag{61}$$

These equations say that because in one direction there was no initial displacement, but there was some initial momentum, a particle will feel the potential in that direction. It also says in the other direction there was no initial momentum, but there is an initial displacement, a particle will also feel the potential in that dimension. Therefore it cannot be reduced to a one dimensional potential.

4.3 Comparison and Generalization

For the potential specified in the previous sections, the noncanonical transformation does not preserve the state of the system after any given time, *i.e.* the states in both the untransformed and transformed frames will not be related by the same transformation at time t. This can be verified by considering measuring the positions of the particle at a time t in the transformed frame by applying the transformation to the state. The measured state at time t in the untransformed frame is

$$x(t) = \frac{b}{m\omega}\sin(\omega t) + a\cos(\omega t) \tag{62}$$

$$y(t) = 0 \tag{63}$$

$$p_x(t) = b\cos(\omega t) - am\omega\sin(\omega t) \tag{64}$$

$$p_y(t) = 0 \ , \tag{65}$$

which by applying the transformation that rotates coordinates but leaves the momenta unchanged gives the state of the system at time t inn the transformed (barred) frame,

$$\bar{x}(t) = 0 \tag{66}$$

$$\bar{y}(t) = \frac{b}{m\omega}\sin(\omega t) + a\cos(\omega t) \tag{67}$$

$$\bar{p}_x(t) = \frac{b}{\omega}\cos(\omega t) - am\omega\sin(\omega t) \tag{68}$$

$$\bar{p}_y(t) = 0 \ . \tag{69}$$

Compare this to the state of the transformed frame at time t measured by using the barred time evolution, Equations 60 and 61,

$$\bar{x}(t) = \frac{b}{m\omega}\sin(\omega t) \tag{70}$$

$$\bar{y}(t) = a\cos(\omega t) \tag{71}$$

$$\bar{p}_x(t) = b\cos(\omega t) \tag{72}$$

$$\bar{p}_y(t) = -am\omega\sin(\omega t) \ , \tag{73}$$

which do not describe the same motion as Equations 66 through 69. So the states are no longer related through the same transformation that was initially imposed. Had this been a canonical transformation, Hamilton's canonical equations would give correct answers, so under a noncanonical transformation, Hamilton's canonical equations do not hold.

Shankar gives the definition of the time evolution of a parameter as the Poisson Bracket of that parameter with the Hamiltonian (Shankar Equation 2.7.13). It says $\dot{\omega} = \{\omega, \mathcal{H}\}$ for any parameter ω. Following his method of finding the equations of motion for $\bar{q}_i$ and $\bar{p}_i$, the equations for the time evolution of q_i and p_i (Shankar Equations 2.7.16 and 2.7.17) only reduce to the canonical equations if the transformed coordinates and momenta are canonical. For a noncanonical transformation, the time evolution of the parameters will not be given by Hamilton's canonical equations (Shankar Equation 2.5.12). In other words, a particle with a given Hamiltonian will have a particular trajectory through phase space. Any canonical transformation to the coordinates and momenta will preserve this phase space trajectory. However, a noncanonical transformation will make the particle move subject to different equations of motion and will give it a different trajectory through phase space.

5 Problem #5: Hermitian Matrices

In order to find the commutator and anti-commutator for each pair of the matrices

$$H = \begin{bmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{bmatrix},\ Q = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix},\ Q^\dagger = \begin{bmatrix} 0 & A^\dagger \\ 0 & 0 \end{bmatrix},$$

the products of each pair were found, shown below. Notice H is Hermitian, $H = H^\dagger$, because $(A^\dagger A)^\dagger = A^\dagger (A^\dagger)^\dagger = A^\dagger A$, so H does not change under a conjugate transpose.

$$HQ = \begin{bmatrix} 0 & 0 \\ AA^\dagger A & 0 \end{bmatrix} \quad ; \quad QH = \begin{bmatrix} 0 & 0 \\ AA^\dagger A & 0 \end{bmatrix}$$

$$HQ^\dagger = \begin{bmatrix} 0 & A^\dagger AA^\dagger \\ 0 & 0 \end{bmatrix} \quad ; \quad Q^\dagger H = \begin{bmatrix} 0 & A^\dagger AA^\dagger \\ 0 & 0 \end{bmatrix}$$

$$QQ^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & AA^\dagger \end{bmatrix} \quad ; \quad Q^\dagger Q = \begin{bmatrix} A^\dagger A & 0 \\ 0 & 0 \end{bmatrix}$$

$$HH = \begin{bmatrix} A^\dagger AA^\dagger A & 0 \\ 0 & AA^\dagger AA^\dagger \end{bmatrix}$$

$$QQ = Q^\dagger Q^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Using these products, the commutator, $[A, B] = AB - BA = -[B, A]$, and the anti-commutator, $\{A, B\} = AB + BA = \{B, A\}$ can be found directly. It is unnecessary to calculate the commutator of a matrix and itself because it is zero by definition, while this is untrue for the anti-commutator.

$$[H, Q] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = -[Q, H] = 0 \tag{74}$$

$$[H, Q^\dagger] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = -[Q^\dagger, H] = 0 \tag{75}$$

$$[Q, Q^\dagger] = -[Q^\dagger, Q] = \begin{bmatrix} -A^\dagger A & 0 \\ 0 & AA^\dagger \end{bmatrix} \tag{76}$$

$$\{Q, Q^\dagger\} = \{Q^\dagger, Q\} = \begin{bmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{bmatrix} \tag{77}$$

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{78}$$

$$\{H, Q\} = \{Q, H\} = 2\begin{bmatrix} 0 & 0 \\ AA^\dagger A & 0 \end{bmatrix} = 2QQ^\dagger Q = 2HQ = 2QH \tag{79}$$

$$\{H, Q^\dagger\} = \{Q^\dagger, H\} = 2\begin{bmatrix} 0 & A^\dagger AA^\dagger \\ 0 & 0 \end{bmatrix} = 2Q^\dagger QQ^\dagger = 2HQ^\dagger = 2Q^\dagger H \tag{80}$$

$$\{H, H\} = 2\begin{bmatrix} A^\dagger AA^\dagger A & 0 \\ 0 & AA^\dagger AA^\dagger \end{bmatrix} \tag{81}$$