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1 Shankar 12.2.4.

Consider a point in the $x-y$ plane that will be subjected to the following four transformations: an infinitesimal translation $\boldsymbol{\varepsilon} = \varepsilon_x \hat{\mathbf{i}} + \varepsilon_y \hat{\mathbf{j}}$, with $\varepsilon_y = 0$,

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{T(\vec{\varepsilon})} \begin{bmatrix} x+\varepsilon_x \\ y \end{bmatrix} \ ; \tag{1}$$

an infitesimal rotation by $\varepsilon_z \hat{\mathbf{k}}$, expanded to second order in ε_z,

$$\begin{bmatrix} x+\varepsilon_x \\ y \end{bmatrix} \xrightarrow{U[R(\varepsilon_z \hat{k})]} \begin{bmatrix} -\frac{1}{2}\left(\varepsilon_x + x\right)\left(\varepsilon_z^2 - 2\right) - y\varepsilon_z \\ \left(\varepsilon_x + x\right)\varepsilon_z - \frac{1}{2} y \varepsilon_z^2 + y \end{bmatrix} \ ; \tag{2}$$

by noting that to second order, for small θ

$$\cos\theta = 1 - \frac{\theta^2}{2} \qquad \sin\theta = \theta \qquad U[R(\varepsilon_z \hat{k})] = \begin{bmatrix} 1 - \frac{\varepsilon_z^2}{2} & -\varepsilon_z \\ \varepsilon_z & 1 - \frac{\varepsilon_z^2}{2} \end{bmatrix} \ ; \tag{3}$$

an infinitesimal translation $-\boldsymbol{\varepsilon} = -\varepsilon_x \hat{\mathbf{i}} - \varepsilon_y \hat{\mathbf{j}}$,

$$\begin{bmatrix} -\frac{1}{2}\left(\varepsilon_x + x\right)\left(\varepsilon_z^2 - 2\right) - y\varepsilon_z \\ \left(\varepsilon_x + x\right)\varepsilon_z - \frac{1}{2} y \varepsilon_z^2 + y \end{bmatrix} \xrightarrow{T(-\vec{\varepsilon})} \begin{bmatrix} -\frac{1}{2}\left(\varepsilon_x + x\right)\varepsilon_z^2 + x - y\varepsilon_z \\ \left(\varepsilon_x + x\right)\varepsilon_z - \frac{1}{2} y \varepsilon_z^2 + y \end{bmatrix} \ ; \tag{4}$$

and finally, an infinitesimal rotation by $-\varepsilon_z \hat{\mathbf{k}}$, expanded to second order in ε_z,

$$\begin{bmatrix} \left(\varepsilon_x + x\right)\left(1 - \frac{\varepsilon_z^2}{2}\right) - \varepsilon_x - y\varepsilon_z \\ \left(\varepsilon_x + x\right)\varepsilon_z + y\left(1 - \frac{\varepsilon_z^2}{2}\right) \end{bmatrix} \xrightarrow{U[R(-\varepsilon_z \hat{k})]} \begin{bmatrix} \frac{1}{4}\left(\varepsilon_x + x\right)\varepsilon_z^4 + \frac{1}{2}\varepsilon_x\varepsilon_z^2 + x \\ \varepsilon_x\varepsilon_z + \frac{y\varepsilon_z^4}{4} + y \end{bmatrix} \ . \tag{5}$$

Furthermore, dropping any terms of order ε_ℓ^4, the final state simplifies to

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{T(\vec{\varepsilon})\ U[R(\varepsilon_z \hat{k})]\ T(-\vec{\varepsilon})\ U[R(-\varepsilon_z \hat{k})]} \begin{bmatrix} x + \frac{1}{2}\varepsilon_x\varepsilon_z^2 \\ y + \varepsilon_x\varepsilon_z \end{bmatrix} \ . \tag{6}$$

Therefore the transformation operator is equal to a translation:

$$U[R(-\varepsilon_z \hat{k})])\ T(-\vec{\varepsilon})\ U[R(\varepsilon_z \hat{k})])\ T(\vec{\varepsilon}) = T\left(\frac{1}{2}\varepsilon_x\varepsilon_z^2 \hat{\mathbf{i}} + \varepsilon_x\varepsilon_z \hat{\mathbf{j}}\right)\hat{P}_x \ , \tag{7}$$

note the swap in the listing of the operators from how they were performed. This is due to the rightmost operator acting on a state first. By changing each of these transofrmations into their generating functions (maintaining $\varepsilon_y = 0$) this becomes

$$\left(\mathcal{I} + \frac{i}{\hbar}\varepsilon_z \hat{L}_z - \frac{1}{2\hbar^2}\varepsilon_z^2 \hat{L}_z^2\right)\left[\mathcal{I} + \frac{i}{\hbar}(\varepsilon_x \hat{P}_x)\right]\left(\mathcal{I} - \frac{i}{\hbar}\varepsilon_z \hat{L}_z - \frac{1}{2\hbar^2}\varepsilon_z^2 \hat{L}_z^2\right)\left[\mathcal{I} - \frac{i}{\hbar}(\varepsilon_x \hat{P}_x)\right]$$
$$= \left(\mathcal{I} - \frac{i}{\hbar}\frac{1}{2}\varepsilon_x\varepsilon_z^2 \hat{P}_x - \frac{i}{\hbar}\varepsilon_x\varepsilon_z P_y\right) \ , \tag{8}$$

where $\mathcal{I}$ is the identity operator. Note the generator for rotations (and its second order Taylor expansion):

$$U[R(\theta)] = \exp[-\frac{i}{\hbar}\theta \hat{L}_z] = \mathcal{I} - \frac{i}{\hbar}(\theta)\hat{L}_z - \frac{1}{2\hbar^2}(\theta)^2 \hat{L}_z^2 \ . \tag{9}$$

By expanding out the transformation operators and matching coefficients (in terms of $\varepsilon_x^a \varepsilon_z^b$), the $\varepsilon_x \varepsilon_z^2$ term can be investigated. Let each term on the left hand side of Equation 8 be denoted by a letter A to J, starting from the left. Doing the multiplication out and only looking at powers of ε_x and ε_y gives

$$\begin{aligned}(A+B+C)\,[D+E]\,(F+G+H)\,[I+J] &\sim \left(1+\varepsilon_z+\varepsilon_z^2\right)[1+\varepsilon_x]\left(1+\varepsilon_z+\varepsilon_z^2\right)[1+\varepsilon_x] \\ &= \Big\{1+(\varepsilon_z)+(\varepsilon_z^2)+(\varepsilon_x)+(\varepsilon_z\varepsilon_x)+(\varepsilon_z^2\varepsilon_x)\Big\}\left(1+\varepsilon_z+\varepsilon_z^2\right)[1+\varepsilon_x] \\ &= \Big\{AD+BD+CD+AE+BE+CE\Big\}(F+G+H)\,[I+J]\end{aligned}$$

$$\begin{aligned}&= \Big\{1+(\varepsilon_z)+(\varepsilon_z^2)+(\varepsilon_x)+(\varepsilon_z\varepsilon_x)+(\varepsilon_z^2\varepsilon_x)+(\varepsilon_z)+(\varepsilon_z^2)+(\varepsilon_z^3)+(\varepsilon_x\varepsilon_z)+(\varepsilon_z^2\varepsilon_x)+(\varepsilon_z^3\varepsilon_x)+(\varepsilon_z^2) \\ &\quad +(\varepsilon_z^3)+(\varepsilon_z^4)+(\varepsilon_x\varepsilon_z^2)+(\varepsilon_z^3\varepsilon_x)+(\varepsilon_z^4\varepsilon_x)\Big\}[1+\varepsilon_x] \\ &= \Big\{ADF+BDF+CDF+AEF+BEF+CEF+ADG+BDG+CDG+AEG+BEG \\ &\quad +CEG+ADH+BDH+CDH+AEH+BEH+CEH\Big\}[I+J]\ .\end{aligned}$$

Before carrying on the multiplication, it is useful to note we are only interested in terms with $\varepsilon_x\varepsilon_z^2$. Therefore terms linear or lower order in ε_i can be dropped as well as any terms that are of third or higher order in ε_z. Furthermore, the ε_i that can be picked up in the final multiplication is ε_x so terms that are linear in both ε_x and ε_z can be dropped as well. The only surviving terms with $\varepsilon_x\varepsilon_z^2$ dependence are

$$-\frac{i}{2\hbar}\hat{P}_x = CEFI+BEGI+AEHI+CDFJ+BDGJ+ADHJ\ , \tag{10}$$

which when the multiplication of the coefficients is performed becomes

$$\begin{aligned}-\frac{i}{2\hbar}\hat{P}_x &= \left(-\frac{1}{2\hbar^2}\hat{L}_z^2\right)\left(\frac{i}{\hbar}\hat{P}_x\right)\mathcal{I}\mathcal{I}+\left(\frac{i}{\hbar}\hat{L}_z\right)\left(\frac{i}{\hbar}\hat{P}_x\right)\left(-\frac{i}{\hbar}\hat{L}_z\right)\mathcal{I}+\mathcal{I}\left(\frac{i}{\hbar}\hat{P}_x\right)\left(-\frac{1}{2\hbar^2}\hat{L}_z^2\right)\mathcal{I} \\ &\quad +\left(-\frac{1}{2\hbar^2}\hat{L}_z^2\right)\mathcal{I}\mathcal{I}\left(-\frac{i}{\hbar}\hat{P}_x\right)+\left(\frac{i}{\hbar}\hat{L}_z\right)\mathcal{I}\left(-\frac{i}{\hbar}\hat{L}_z\right)\left(-\frac{i}{\hbar}\hat{P}_x\right)+\mathcal{I}\mathcal{I}\left(-\frac{1}{2\hbar^2}\hat{L}_z^2\right)\left(-\frac{i}{\hbar}\hat{P}_x\right) \\ &= -\frac{i}{2\hbar^3}\hat{L}_z^2\hat{P}_x-\frac{i^3}{\hbar^3}\hat{L}_z\hat{P}_x\hat{L}_z-\frac{i}{2\hbar^3}\hat{P}_x\hat{L}_z^2+\frac{i}{2\hbar^3}\hat{L}_z^2\hat{P}_x+\frac{i^3}{\hbar^3}\hat{L}_z^2\hat{P}_x+\frac{i}{2\hbar^3}\hat{L}_z^2\hat{P}_x \\ h^2\hat{P}_x &= \hat{L}_z^2\hat{P}_x+2(i^2)\hat{L}_z\hat{P}_x\hat{L}_z+\hat{P}_x\hat{L}_z^2-\hat{L}_z^2\hat{P}_x-\hat{L}_z^2\hat{P}_x-2(i^2)\hat{L}_z^2\hat{P}_x \\ h^2\hat{P}_x &= \hat{L}_z^2\hat{P}_x-2\hat{L}_z\hat{P}_x\hat{L}_z+\hat{P}_x\hat{L}_z^2\ .\end{aligned}$$

This may seem to contradict the statement that "every consistency test will reduce to just another relation between the commutators of the generators", but using the commutator identity $-\Lambda\Omega\Lambda+\Omega\Lambda^2+\Lambda^2\Omega \equiv [\Lambda[\Lambda,\Omega]$ the above consistency constraint becomes

$$h^2\hat{P}_x = [\hat{L}_z,[\hat{L}_z,\hat{P}_x]] = \hat{L}_z[\hat{L}_z,\hat{P}_x]-[\hat{L}_z,\hat{P}_x]\hat{L}_z = [\hat{P}_x,\hat{L}_z]\hat{L}_z-\hat{L}_z[\hat{P}_x,\hat{L}_z]\ , \tag{11}$$

which is satisfied using the commutation relations given by Shankar Equation 12.2.17:

$$h^2\hat{P}_x = (-i\hbar\hat{P}_y)\hat{L}_z-L_z(-i\hbar\hat{P}_y) = (-i\hbar)(\hat{P}_y\hat{L}_z-\hat{L}_z\hat{P}_y) \quad\Rightarrow\quad i\hbar\hat{P}_x = [\hat{P}_y,\hat{L}_z]\ . \tag{12}$$

2 Shankar 12.3.6.

Consider a particle of mass μ constrained to move on a circle of radius a. The two dimensional, zero potential, problem can be solved in the polar coordinates ρ and ϕ, due to it being rotationally invariant. The Hamiltonian operator for this particle is

$$\hat{H} = -\frac{\hbar^2}{2\mu}\nabla^2 = -\frac{\hbar^2}{2\mu}\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right] , \tag{13}$$

using the Laplacian in polar coordinates given by Shankar Equation 12.3.11. However, the radial component of the particle's motion is fixed to a value a, so all radial derivatives vanish:

$$\hat{H} = -\frac{\hbar^2}{2\mu}\left[\frac{1}{a^2}\frac{\partial^2}{\partial\phi^2}\right] = \frac{1}{2\mu a^2}\left[-\hbar^2\frac{\partial^2}{\partial^2\phi}\right] = \frac{1}{2\mu a^2}\left[i\hbar\frac{\partial}{\partial\phi}\right]^2 = \frac{1}{2\mu a^2}\hat{L}_z^2 . \tag{14}$$

The eigenvalue equation for this operator then becomes

$$\hat{H}\Phi_m = E\Phi_m \quad \Rightarrow \quad \frac{1}{2\mu a^2}\hat{L}_z^2\Phi_m = E\Phi_m , \tag{15}$$

where $\Phi_m(\phi)$ are the nondegenerate eigenfunctions of $\hat{L}_z$ (because the ρ coordinate is fixed, see Shankar Equation 12.3.9). The eigenvalues of these eigenfunctions are given by Shankar Equation 12.3.8, so that Equation 15 becomes

$$\hat{H}\Phi_m = \frac{1}{2\mu a^2}(\hbar m)^2\Phi_m = E\Phi_m , \tag{16}$$

so the eigenvalues of the Hamiltonian operator are $E_m = (\hbar^2/2\mu a^2)m^2$. Note that $E_m = E_{-m}$, so the states are two-fold degenerate. This corresponds to the particle moving clockwise around the circle or counterclockwise around the circle.

3 Shankar 12.3.7.

Consider the isotropic oscillator Hamiltonian

$$H = \frac{P_x^2 + P_y^2}{2\mu} + \frac{1}{2}\mu\omega^2(X^2 + Y^2) \ . \tag{17}$$

3.1 The Eigenvalue Problem.

The commutator of this Hamiltonian with the angular momentum projection operator is, up to factors of constants

$$\begin{aligned}[H, L_z] &\simeq [P_x^2 + P_y^2 + X^2 + Y^2, XP_y - YP_x] \\ &= [P_x^2, XP_y] + [P_x^2, -YP_x] + [P_y^2, XP_y] + [P_y^2, -YP_x] \\ &\quad + [X^2, XP_y] + [X^2, -YP_x] + [Y^2, XP_y] + [Y^2, -YP_x] \\ &= [P_x^2, XP_y] + 0 + 0 + [P_y^2, -YP_x] + 0 + [X^2, -YP_x] + [Y^2, XP_y] + 0 \ ,\end{aligned}$$

because every operator commutes with itself, and position operators and momentum operators commute if they are not along the same axis, i.e. $[X, P_y] = 0$. Continuing,

$$\begin{aligned}[H, L_z] &\simeq [P_x^2, XP_y] + [P_y^2, -YP_x] + [X^2, -YP_x] + [Y^2, XP_y] \\ &= [P_x^2, X]P_y + X[P_x^2, P_y] + [P_y^2, -Y]P_x - Y[P_y^2, P_x] \\ &\quad + [X^2, -Y]P_x - Y[X^2, P_x] + [Y^2, X]P_y + X[Y^2, P_y] \ ,\end{aligned}$$

using the identity $[A, BC] = [A, B]C + B[A, C]$. Noting that momentum operators along different axes commute (as well as position) this becomes

$$\begin{aligned}[H, L_z] &\simeq [P_x^2, X]P_y + 0 + [P_y^2, -Y]P_x - 0 + 0 + -Y[X^2, P_x] + 0 + X[Y^2, P_y] \\ &= X[Y^2, P_y] = X(Y[Y, P_y] + [Y, P_y]Y) \\ &= 0 \ ,\end{aligned}$$

by noticing the first two nonzero commutators are the same form, but differ by a negative sign, these exactly cancel. Same for the last two nonzero commutators. Therefore, this Hamiltonian commutes with the angular momentum projection operator L_z. This Hamiltonian can be transformed to polar coordinates by noting that

$$\rho^2 = X^2 + Y^2 \qquad P_\rho^2 = P_x^2 + P_y^2 \ , \tag{18}$$

which implies there is no angular dependence for this Hamiltonian, and it can be reduced to a purely radial differential equation. Using Shankar Equation 12.3.13, the Hamiltonian becomes

$$H = \frac{P_\rho^2}{2\mu} + \frac{1}{2}\mu\omega^2\rho^2 = -\frac{\hbar^2}{2\mu}\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2}\right] + \frac{1}{2}\mu\omega^2\rho^2 \ , \tag{19}$$

which gives the differential equation for the radial equation $R(\rho)$:

$$HR_{Em}(\rho) = -\frac{\hbar^2}{2\mu}\left[\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \frac{m^2}{\rho^2}R\right] + \frac{1}{2}\mu\omega^2\rho^2 R = ER \ . \tag{20}$$

3.2 Small Radius Limit, $\rho \to 0$.

In the small ρ limit, the terms with the highest negative power of ρ dominate, but all derivatives must remain in the expansion. This makes Equation 20 into

$$-\frac{\hbar^2}{2\mu}\left[\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}-\frac{m^2}{\rho^2}R\right]+0=0 \tag{21}$$

Letting primes denote derivatives with respect to ρ, this is

$$R''+\frac{1}{\rho}R'-\frac{m^2}{\rho^2}R\ =0=\rho^2R''+\rho R'-m^2R\ , \tag{22}$$

which has a form that is amenable to a solution of the form $R=\rho^\alpha$, with $\alpha>0$. Plugging in this ansatz yields

$$0=\rho^2(\alpha)(\alpha-1)\rho^{\alpha-2}+\rho(\alpha)\rho^{(\alpha-1)}-m^2\rho^\alpha=\rho^\alpha[(\alpha)(\alpha-1)+\alpha-m^2]=\rho^\alpha[\alpha^2-m^2]\ , \tag{23}$$

so $\alpha^2=m^2$. This fact, along with the restrictions on α from the ansatz, yields that $\alpha=|m|$, so

$$R_{Em}(\rho)\xrightarrow{\rho\to 0}\rho^{|m|}\ . \tag{24}$$

3.3 Large Radius Limit, $\rho \to \infty$.

In the large ρ limit, the terms that dominate the expansion have the highest positive powers of ρ. Any terms with negative powers are negligible, so the differential equation becomes

$$-\frac{\hbar^2}{2\mu}\left[\frac{d^2R}{d\rho^2}\right]+\frac{1}{2}\mu\omega^2\rho^2R=0 \quad\Rightarrow\quad R''=\left(\frac{\mu\omega}{\hbar}\right)^2\rho^2R\ , \tag{25}$$

which has a form amenable to solutions of the form $\exp[\beta\rho^\gamma]+\exp[-\beta\rho^\gamma]$. However, due to physical constraints on this system, only the decaying exponential is allowed (otherwise the solution diverges at infinity). The derivatives of this solution are

$$R'=-\beta\gamma\rho^{\gamma-1}e^{-\beta\rho^\gamma} \qquad R''=\beta^2\gamma^2\rho^{2\gamma-2}e^{-\beta\rho^\gamma}-\beta(\gamma-1)\gamma\rho^{\gamma-2}e^{-\beta\rho^\gamma}\ , \tag{26}$$

but because we are in the large ρ limit, the term with higher powers of ρ dominates and the other is negligible, so

$$\beta^2\gamma^2\rho^{2\gamma-2}e^{-\beta\rho^\gamma}=\left(\frac{\mu\omega}{\hbar}\right)^2\rho^2e^{-\beta\rho^\gamma}\ , \tag{27}$$

clearly $\gamma=2$ (by matching powers of ρ). The equation for β is then

$$\beta^2 2^2=\left(\frac{\mu^2\omega^2}{\hbar^2}\right)^2 \quad\Rightarrow\quad 2\beta=\frac{\mu\omega}{\hbar}\ , \tag{28}$$

so in the large ρ limit

$$R_{Em}(\rho)\xrightarrow{\rho\to\infty}\exp[-\frac{\mu\omega}{2\hbar}\rho^2]\ . \tag{29}$$

3.4 Change of Variables.

From the previous two sections, the form of $R_{Em}(\rho)$ can be assumed to be

$$R_{Em}(\rho) = \rho^{|m|} \exp[-\frac{\mu\omega}{2\hbar}\rho^2] U_{Em}(\rho) \ . \tag{30}$$

Now consider the dimensionless variables $\epsilon = E\hbar/\omega$ and $y = (\mu\omega/\hbar)^{1/2}\rho$. Changing to these variables, the radial equation becomes (up to a constant)

$$R(y) = y^{|m|} \exp[-y^2/2] U(y); \quad R' = \frac{dR}{d\rho} = \frac{dR}{dy}\frac{dy}{d\rho} = \sqrt{\frac{\mu\omega}{\hbar}} R'_y; \quad R'' = \frac{\mu\omega}{\hbar} R''_y \ . \tag{31}$$

The differential equation for R (Equation 20) becomes, after dividing through by $\hbar\omega$,

$$-\frac{\hbar}{2\mu\omega}\left[R'' + \frac{1}{\rho}R' - \frac{m^2}{\rho^2}R\right] + \frac{1}{2}\frac{\mu\omega}{\hbar}\rho^2 R = \epsilon R \tag{32}$$

$$-\frac{\hbar}{2\mu\omega}\left[\frac{\mu\omega}{\hbar}R''_y + \left(\sqrt{\frac{\mu\omega}{\hbar}}\frac{1}{y}\right)\sqrt{\frac{\mu\omega}{\hbar}}R'_y - m^2\frac{\mu\omega}{\hbar}\frac{1}{y^2}R_y\right] + \frac{1}{2}\frac{\mu\omega}{\hbar}\rho^2 R_y = \epsilon R_y \tag{33}$$

$$-\left[R''_y + \frac{1}{y}R'_y - \frac{m^2}{y^2}R\right] + y^2 R = 2\epsilon R \tag{34}$$

$$0 = [2\epsilon - y^2 - m^2 y^{-2}]R_y + y^{-1}R'_y + R''_y \ . \tag{35}$$

3.5 Differential Equation for $U_{Em}(y)$

To turn the differential equation for R_y into one for U, the derivatives of R_y must be found. Note the differentiation chain rule, expanded to three functions, states:

$$\frac{d}{dx}[a(x)\{b(x)c(x)\}] = b(x)c(x)[a'] + a(x)\frac{d}{dx}[b(x)c(x)] = a'bc + a(b'c + c'b) = a'bc + ab'c + abc' \ . \tag{36}$$

Using this, R_y and its derivatives are

$$R_y = y^{|m|}\exp[-y^2/2]U(y) \tag{37}$$

$$R'_y = e^{-\frac{y^2}{2}} y^{|m|-1}\left(U(y)\left(|m| - y^2\right) + yU'(y)\right) \tag{38}$$

$$R''_y = e^{-\frac{y^2}{2}} y^{|m|-2}\left[y\left(2\left(|m| - y^2\right)U'(y) + yU''(y)\right) + U(y)\left(|m|\left(|m| - 2y^2 - 1\right) + y^4 - y^2\right)\right] \ . \tag{39}$$

Substituting these into Equation 35, and dividing through by $y^{|m|}\exp[-y^2/2]$, yields

$$\begin{aligned} 0 = {} & [2\epsilon - y^2 - m^2 y^{-2}]U + y^{-1}\left(U\left(|m|\, y^{-1} - y\right) + U'\right) \\ & + \left[2\,|m|\, U' y^{-1} + |m|^2\, U y^{-2} - |m|\, U y^{-2} - 2\,|m|\, U + U'' - 2yU' + y^2 U - U\right] \ . \end{aligned} \tag{40}$$

Combining terms of each order derivative yields

$$0 = U'' + U'[y^{-1} + 2|m|y^{-1} - 2y] + U[2\epsilon - y^2 - m^2y^{-2} + |m|y^{-2} - 1 + |m|^2y^{-2} - |m|y^{-2} - 2|m| + y^2 - 1] \tag{41}$$

$$0 = U'' + U'\left[\frac{1+2|m|}{y} - 2y\right] + U[2\epsilon - 2|m| - 2 - m^2y^{-2} + |m|^2y^{-2}] \tag{42}$$

$$0 = U'' + U'\left[\frac{1+2|m|}{y} - 2y\right] + U[2\epsilon - 2|m| - 2] \ , \tag{43}$$

by arguing that if m is real $m^2 = |m|^2$.

3.6 Power Series Solution

. Assume the solution to the final differential equation for U is a power series,

$$U(y) = \sum_{r=0}^{\infty} C_r y^r \ . \tag{44}$$

The first and second derivatives of which have powers y^{r-1} and y^{r-2}, respectively. Noting that the second derivative and the original function are both only multiplied by terms that have no y dependence, these terms will contribute two different powers of y: y^r and y^{r-2}. The first derivative is multiplied by a term linear in y and a term inversely proportional to y, so it will also contribute two powers of y: y^r and y^{r-2}. Therefore there are only two different power series, with the same coefficients, just paired with different powers of y. Performing a change of variables on the sum index such that both power series have y^i dependence with i being the sum index will result in terms with C_i and C_{i+2}. This results in a two term recursion relation, with separate ladders corresponding to C_0 and C_1.

3.7 Recursion Relation.

Substituting the ansatz from Equation 44 into Equation 43 yields

$$0 = \sum_{r=0}^{\infty} C_r r(r-1)y^{r-2} + \left[\frac{1+2|m|}{y} - 2y\right] \sum_{r=0}^{\infty} C_r r y^{r-1} + [2\epsilon - 2|m| - 2] \sum_{r=0}^{\infty} C_r y^r \tag{45}$$

$$0 = \sum_{r=0}^{\infty} C_r r(r-1)y^{r-2} + \sum_{r=0}^{\infty} C_r r(1+2|m|)y^r - 2\sum_{r=0}^{\infty} C_r r(1+2|m|)y^{r-2} + [2\epsilon - 2|m| - 2] \sum_{r=0}^{\infty} C_r y^r \tag{46}$$

$$0 = \sum_{r=0}^{\infty} C_r r(r-1)y^{r-2} + \sum_{r=0}^{\infty} C_r r(1+2|m|)y^{r-2} - 2\sum_{r=0}^{\infty} C_r r y^r + [2\epsilon - 2|m| - 2] \sum_{r=0}^{\infty} C_r y^r \tag{47}$$

$$0 = \sum_{r=0}^{\infty} C_r [r(r-1) + r(1+2|m|)]y^{r-2} + \sum_{r=0}^{\infty} C_r [2\epsilon - 2|m| - 2 - 2r]y^r \ . \tag{48}$$

Performing a change of variables on the first sum such that the transformation is $n = r-2$, makes Equation 48

$$0 = \sum_{n=-2}^{\infty} C_{n+2}[(n+2)((n+2)-1) + (n+2)(1+2|m|)]y^n + \sum_{r=0}^{\infty} C_r [2\epsilon - 2|m| - 2 - 2r]y^r \tag{49}$$

$$0 = \sum_{n=-2}^{\infty} C_{n+2}[(n+2)(2|m|+n+2)]y^n + \sum_{r=0}^{\infty} C_r [2\epsilon - 2|m| - 2 - 2r]y^r \ . \tag{50}$$

We are now free to change the dummy variable n back to r, and pull out the first two terms, yielding

$$0 = C_0(0) + C_1[(1)(2|m|+1)]y^{-1} + \sum_{r=0}^{\infty} C_{r+2}[(r+2)(2|m|+r+2)]y^r + \sum_{r=0}^{\infty} C_r [2\epsilon - 2|m| - 2 - 2r]y^r$$

$$\sum_{r=0}^{\infty} C_{r+2}[2|m| + 2r - 2\epsilon + 2]y^r = C_1(2|m|+1)y^{-1} + \sum_{r=0}^{\infty} C_r[(r+2)(2|m|+r+2)] \ .$$

The coefficient C_1 must be zero, so that the powers of y on each side of the equation match. Therefore the two sums must be equivalent, implying their summands are equal,

$$C_{r+2}[2|m| + 2r - 2\epsilon + 2] = C_r[(r+2)(2|m| + r + 2)] \ , \tag{51}$$

which results in a two term recursion relation, as expected. Note that all other $C_k = 0$, for odd k, because C_1 is zero, and every C_k for higher odd k is just some factor times C_1. In order to maintain the proper behavior in $R_{Em}(\rho)$ as $y \to \infty$, the series must go like ρ to some finite power to cancel that factor in the numerator of R, so the function remains finite. In the large y limit, the term with the highest power of ρ will dominate, so the power series must terminate at some value of r. This means that at some point, there must be a $C_r = 0$, which would imply all terms with power r or greater also vanish. To find the value of r which accomplishes this, set $r = 0$ in Equation 51,

$$2|m| + 2r - 2\epsilon + 2 = 0 \quad \Rightarrow \quad r = \epsilon - |m| - 1 \ , \tag{52}$$

where r still must be even. So the point at which the power series terminates depends on the dimensionless energy. Thusly, the power series will terminate if the energy is $\epsilon = r + |m| + 1$, but because r must be even, a variable k can be introduced so that

$$E = \hbar\omega(2k + |m| + 1) \qquad k = 0, 1, 2, \ldots \tag{53}$$

A new variable n can be defined as $n = 2k + |m|$, which is also a positive definite integer, which results in the energy spectrum

$$E_n = \hbar\omega(n+1) \ . \tag{54}$$

3.8 Energy Spectrum Degeneracy.

For any given value of n, there are specific allowed values for $|m|$,

$$n - 2k = |m| \quad \text{for } k \leq n/2 \ , \tag{55}$$

where n, k are integers. Therefore for every $n > 0$ there are an integer value $n/2$ possible values for k, which gives $n/2$ possible values for $|m|$. For nonzero values of m there are two values that have the same value $|m|$, so m has n possible values for a given energy state n. This, plus the $m = 0$ possibility mean for any n the degeneracy is $n + 1$. This is the same degeneracy of states as in Cartesian coordinates.

3.9 Normalized eigenfunctions.

In Section 3.4 the coefficients of the assumed eigenfunctions were dropped, with the intent of normalizing them. Consider the $n = 0$ case, the radial eigenfunction with normalization constant is

$$R_{nm}(\rho) = A\rho^{|m|} \exp[-\mu\omega\rho^2/2\hbar] \sum_{r=0}^{r_t} C_r \left(\sqrt{\frac{-\mu\omega}{\hbar}}\rho\right)^r \ , \tag{56}$$

with terminal r value r_t, given by $r_t = n - |m|$. Including the rotational part of the eigenfunction $\Phi_m(\phi) = e^{im\phi}/(\sqrt{2\pi})$, the complete, unnormalized eigenfunctions are

$$\psi_{nm} = A\rho^{|m|} \exp[-\mu\omega\rho^2/2\hbar] \sum_{r=0}^{n-|m|} C_r \left(\sqrt{\frac{-\mu\omega}{\hbar}}\rho\right)^r \frac{e^{im\phi}}{\sqrt{2\pi}} \ , \tag{57}$$

for which the normalization condition is the integral over all space (in ρ and ϕ) of the norm squared of the eigenfunction is unity. Consider the $n = 0$ case, which makes $m = 0$ and $r_t = 0$, the normalization condition is

$$1 = |A|^2 \int_0^{2\pi} (2\pi)^{-1} \int_0^{\infty} C_0^2 \exp[-\mu\omega\rho^2/\hbar]\rho \, d\rho \, d\phi = C_0^2 \frac{\hbar}{2\mu\omega}|A|^2 , \tag{58}$$

and taking $C_0 = 1$,

$$A = \left[\sqrt{\frac{2\mu\omega}{\hbar}}\right]^{1/2} . \tag{59}$$

The $n = 1$ state still has $m = \pm 1$, still with $r_t = 0$ so that

$$1 = |A|^2 \int_0^{2\pi} (2\pi)^{-1} \int_0^{\infty} C_0^2 \rho^2 \exp[-\mu\omega\rho^2/\hbar]\rho \, d\rho \, d\phi = C_0^2 \frac{h^2}{2\mu^2\omega^2}|A|^2 , \tag{60}$$

note both $\pm m$ have the same normalization constant, which with $C_0 = 1$ is

$$A = \sqrt{2}\frac{\mu\omega}{\hbar} . \tag{61}$$

Therefore the normalized eigenfunctions for the $n = 0, 1$ states are

$$\psi_{0,0}(\rho, \phi) = \left[\sqrt{\frac{\mu\omega}{\hbar\pi}}\right]^{1/2} \exp[-\mu\omega\rho^2/2\hbar] \tag{62}$$

$$\psi_{1,\pm 1}(\rho, \phi) = \frac{\mu\omega}{\sqrt{\pi}\hbar}\rho \exp[-\mu\omega\rho^2/2\hbar]e^{\pm i\phi} . \tag{63}$$

3.10 Relation with Cartesian Coordinates.

The $n = 0$ state must be equal in both Cartesian and polar coordinates because that state has no degeneracy. Noting the form of the eigenfunctions, Equation 57, the only place where performing the parity operator has an effect is the power series and the $\rho^{|m|}$ term, but only for even values of $r + |m| = n$ will the eigenstate pick up a negative sign. Therefore the parity eigenvalues are $(-1)^n$. Furthermore the two solutions for $n = 1$ in polar coordinates are simply linear combinations of their counterparts in Cartesian coordinates. Using the Euler identities the $n = 1$ eigenstate in polar coordinates is

$$\psi_{1,\pm 1}(\rho, \phi) = \alpha \exp[-\mu\omega\rho^2/2\hbar]\rho(\cos\phi \pm i \sin\phi) = \alpha \exp[-\mu\omega(x^2 + y^2)/2\hbar](x \pm iy) , \tag{64}$$

and the solutions for the $n = 1$ state in Cartesian coordinates are

$$x \exp\left[-\mu\omega(x^2 + y^2)/(2\hbar)\right] \quad \text{and} \quad y \exp[-\mu\omega(x^2 + y^2)/(2\hbar)] . \tag{65}$$

So as claimed, the $n = 1$ eigenstate in polar coordinates is simply a linear combinations of the $n = 1$ solutions in Cartesian coordinates.

4 Shankar 12.3.8.

Consider a particle of mass μ and charge q in a vector potential given by

$$\mathbf{A} = \frac{B}{2}(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}) \ . \tag{66}$$

4.1 Magnetic Field.

This vector potential gives rise to a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. This cross-product is given by the determinant

$$\frac{B}{2}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \frac{B}{2}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \frac{B}{2}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ -y & x & 0 \end{vmatrix} , \tag{67}$$

with $\frac{\partial}{\partial z} = 0$ because $\mathbf{A}$ has no z dependence. Clearly the only terms that contribute are left in the $\hat{\mathbf{k}}$ direction:

$$\mathbf{B} = \frac{B}{2}\left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right)\hat{\mathbf{k}} = B\hat{\mathbf{k}} \ . \tag{68}$$

4.2 Classical Circular Motion.

If the particle is treated classically, the force it feels (in CGS units) in this field is

$$\mathbf{F} = \frac{q}{c}(\mathbf{v} \times \mathbf{B}) = \frac{q}{c}vB\sin\theta\hat{\mathbf{n}} , \tag{69}$$

where c is the speed of light and θ is the angle between the directions of particle's velocity and the magnetic field, and $\hat{\mathbf{n}}$ is the unit vector normal to the plane created by the magnetic field and velocity vectors. In the case the particle's velocity is perpendicular to the magnetic field, the force felt by the particle will always be perpendicular to the velocity (and magnetic field) and the particle will move in a circle. Assuming the particle moves in the $\hat{\mathbf{j}}$ direction, it will feel a centripetal force given by

$$\mathbf{F} = \frac{\mu v^2}{r}\hat{\mathbf{n}} = \frac{q}{c}(\mathbf{v} \times \mathbf{B}) = \frac{q}{c}vB\hat{\mathbf{i}} , \tag{70}$$

but velocity is the angular velocity of the circular orbit ω_0 times the radius of the orbit r, so

$$\mu\omega_0^2 r = \frac{q}{c}\omega_0 rB \quad \Rightarrow \quad \omega_0 = \frac{qB}{c\mu} \ . \tag{71}$$

Therefore the particle moves in a circular path with frequency $\omega_0 = qB/\mu c$.

4.3 Quantum Hamiltonian.

Consider the Hamiltonian for the corresponding quantum problem:

$$\hat{H} = \frac{[P_x + qYB/2c]^2}{2\mu} + \frac{[P_y - qXB/2c]^2}{2\mu} , \tag{72}$$

and the operators

$$Q = \frac{cP_x + \frac{1}{2}qYB}{qB} \qquad P = P_y - \frac{qXB}{2c} \ . \tag{73}$$

These operators are canonical if they satisfy the constraints, corresponding to a quantum system, set by Hamilton's canonical equations (Shankar Equation 2.7.18). In this case the constraint is that the commutator of the position and momentum is $[Q, P] = i\hbar$. The commutator for these operators is

$$[Q, P] = \left[\frac{cP_x + \frac{1}{2}qYB}{qB}, P_y - \frac{qXB}{2c} \right] , \tag{74}$$

which condensing constants (which do not effect commutator relations - it also should be clear which constants have been condensed to what) becomes

$$[Q, P] = \left[\alpha P_x + \frac{1}{2}Y, P_y - \beta X \right] = [\alpha P_x, P_y] + \left[\frac{1}{2}Y, P_y \right] + [\alpha P_x, -\beta X] + \left[\frac{1}{2}Y, -\beta X \right] . \tag{75}$$

Noting that the independent components of velocity, and the independent components of position commute with each other respectively, this is

$$[Q, P] = \left[\frac{1}{2}Y, P_y \right] + [\alpha P_x, -\beta X] = \frac{1}{2}[Y, P_y] - \alpha\beta[P_x, X] = \frac{1}{2}[Y, P_y] + \alpha\beta[X, P_x] , \tag{76}$$

and the commutator of a position operator and its conjugate momentum operator is $i\hbar$, so the overall commutator is

$$[Q, P] = \frac{1}{2}(i\hbar) + \alpha\beta(i\hbar) = \frac{1}{2}(i\hbar) + \frac{c}{qB}\frac{qB}{2c}(i\hbar) = (i\hbar) , \tag{77}$$

so the operators P and Q are canonical.

The Hamiltonian can be rewritten in terms of these operators:

$$\hat{H} = \frac{[P_x + qYB/2c]^2}{2\mu} + \frac{[P_y - qXB/2c]^2}{2\mu} = \frac{1}{2\mu}\left[\frac{qB}{c}\frac{(cP_x + qYB/2)}{qB} \right]^2 + \frac{[P_y - qXB/2c]^2}{2\mu} \tag{78}$$

$$= \frac{1}{2}\frac{q^2B^2}{\mu c^2}Q^2 + \frac{P^2}{2\mu} = \frac{P^2}{2\mu} + \frac{1}{2}\mu\omega_0^2 Q^2 , \tag{79}$$

using the definition of ω_0 from the previous section. It is clear this Hamiltonian is that of an equivalent one dimensional harmonic oscillator, and therefore the allowed energy levels are $E = (n + \frac{1}{2})\hbar\omega_0$, where n is an integer.

4.4 Isotropic Harmonic Oscillator.

The form of the Hamiltonian in original variables can be expanded to

$$\hat{H} = \frac{1}{2\mu}\left[P_x^2 + \left(\frac{1}{2c}qYB \right)^2 + \frac{1}{c}qYBP_x \right] + \frac{1}{2\mu}\left[P_y^2 + \left(\frac{1}{2c}qXB \right)^2 - \frac{1}{c}qXBP_y \right] \tag{80}$$

$$= \frac{P_x^2}{2\mu} + \frac{P_y^2}{2\mu} + \frac{1}{2}\frac{1}{\mu}\left(\frac{qB}{c}/2 \right)^2 Y^2 + \frac{1}{2}\frac{1}{\mu}\left(\frac{qB}{c}/2 \right)^2 X^2 + \frac{1}{2}\frac{qB}{\mu c}(YP_x - XP_y) \tag{81}$$

$$= \frac{P_x^2}{2\mu} + \frac{P_y^2}{2\mu} + \frac{1}{2}\mu\left(\frac{qB}{\mu c}/2 \right)^2 Y^2 + \frac{1}{2}\mu\left(\frac{qB}{\mu c}/2 \right)^2 X^2 - \frac{1}{2}\frac{qB}{\mu c}(XP_y - YP_x) \tag{82}$$

$$= \frac{P_x^2}{2\mu} + \frac{P_y^2}{2\mu} + \frac{1}{2}\mu\left(\frac{\omega_0}{2} \right)^2 Y^2 + \frac{1}{2}\mu\left(\frac{\omega_0}{2} \right)^2 X^2 - \frac{1}{2}\omega_0(XP_y - YP_x) . \tag{83}$$

If the Hamiltonian for two-dimensional, isotropic harmonic oscillator is defined as

$$\hat{H}_{2d}(\omega', M) = \frac{P_x^2}{2M} + \frac{P_y^2}{2M} + \frac{1}{2}M(\omega')^2X^2 + \frac{1}{2}M(\omega')^2Y^2 \ , \tag{84}$$

then Equation 83 can be written as

$$\hat{H} = \hat{H}_{2d}\left(\frac{\omega_0}{2}, \mu\right) - \frac{\omega_0}{2}\hat{L}_z \ . \tag{85}$$

The basis that diagonalizes $\hat{H}_{2d}$ will diagonalize $\hat{H}$ because the isotropic two dimensional harmonic oscillator Hamiltonian commutes with the angular momentum projection operator. Due to this, and the fact that $\hat{L}_z$ commutes with itself,

$$[\hat{H}, \hat{L}_z] = [\hat{H}_{2d}, \hat{L}_z] - \frac{\omega_0}{2}[\hat{L}_z, \hat{L}_z] = 0 \ , \tag{86}$$

so the operators $\hat{H}$ and $\hat{L}_z$ share a common eigenbasis. Consider a state $|n, m\rangle$ in the eigenbasis, which gives the allowed energies

$$E\,|n,m\rangle = \hat{H}\,|n,m\rangle = \hat{H}_{2d}\,|n,m\rangle - \frac{\omega_0}{2}\hat{L}_z\,|n,m\rangle = \frac{\hbar\omega_0}{2}(2k + |m| + 1)\,|n,m\rangle - \frac{\omega}{2}(\hbar m)\,|n,m\rangle \tag{87}$$

using the energy of the two dimensional isotropic oscillator in terms of angular momentum, and integer $k = (n - |m|)/2 > 0$, from Section 3.7. This implies $E = \hbar\omega_0[k + \frac{1}{2}|m| - \frac{1}{2}m + \frac{1}{2}]$. From the definition of the energy of the isotropic two dimensional harmonic oscillator this is still equal to $E = \hbar\omega_0(n + \frac{1}{2})$, so it agrees with the conclusion of Section 4.3.