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Contents

1 Shankar 12.5.3.

1.1 Expectation Values of J_x and J_y.

Consider a state $|jm\rangle$, and the operators J_+ and J_-, such that

$$J_\pm |j\ m\rangle = \hbar[(j \mp m)(j \pm m + 1)]^{1/2} |j\ m \pm 1\rangle \equiv C_\pm |j\ m \pm 1\rangle \ , \tag{1}$$

it is useful to note that $J_+ = J_-^\dagger$. In terms of $J_\pm$, the angular momentum projection operators J_x and J_y are

$$J_x = \frac{1}{2}(J_+ + J_-) \quad \text{and} \quad J_y = -\frac{1}{2}(J_+ - J_-) \ . \tag{2}$$

With this information, the expectation values of these operators can be found. Begin with the x projection,

$$\langle J_x \rangle = \langle j\ m|J_x|j\ m\rangle = \frac{1}{2}\langle j\ m|J_+ + J_-|j\ m\rangle = \frac{1}{2}\left[\langle j\ m|J_+|j\ m\rangle + \langle j\ m|J_-|j\ m\rangle\right] \tag{3}$$

$$= \frac{1}{2}[C_+ \langle j\ m|j\ m+1\rangle + C_- \langle j\ m|j\ m-1\rangle] = 0 \ , \tag{4}$$

because the states $|j\ m\rangle$ and $|j\ m \pm 1\rangle$ are orthogonal. Similarly for J_y,

$$\langle J_y \rangle = \langle j\ m|J_y|j\ m\rangle = -\frac{i}{2}\langle j\ m|J_+ - J_-|j\ m\rangle = \frac{1}{2}\left[\langle j\ m|J_+|j\ m\rangle - \langle j\ m|J_-|j\ m\rangle\right] \tag{5}$$

$$= \frac{1}{2}[C_+ \langle j\ m|j\ m+1\rangle - C_- \langle j\ m|j\ m-1\rangle] = 0 \ . \tag{6}$$

Note that $\langle j\ m|(J_\pm)^n|j\ m\rangle = 0$ for all $n \neq 0$, because the resultant inner product is between two orthogonal states.

1.2 Expectation Values of J_x^2 and J_y^2.

Using the relations for $J_\pm$ and $J_{x,y}$,

$$J_x^2 = \frac{1}{4}[J_+^2 + J_+J_- + J_-J_+ + J_-^2] \Rightarrow J_x^2 |j\ m\rangle = \frac{1}{4}[J_+J_- + J_-J_+] |j\ m\rangle \tag{7}$$

$$J_y^2 = -\frac{i}{4}[J_+^2 - J_+J_- - J_-J_+ + J_-^2] \Rightarrow J_x^2 |j\ m\rangle = -\frac{1}{4}[-J_+J_- - J_-J_+] |j\ m\rangle = J_x^2 |j\ m\rangle \ , \tag{8}$$

due to the orthonormality condition stated above. The expectation value for the x projection operator squared is

$$4\langle J_x^2 \rangle = \langle j\ m|J_+J_- + J_-J|j\ m\rangle = \langle j\ m|J_+J_-|j\ m\rangle + \langle j\ m|J_-J_+|j\ m\rangle \tag{9}$$

$$= \langle J_-j\ m|J_-j\ m\rangle + \langle J_+j\ m|J_+j\ m\rangle \ , \tag{10}$$

using the Hermiticity of the $J_\pm$ operators. Evaluating this,

$$4\langle J_x^2 \rangle = (C_-)^2 \langle j\ m-1|j\ m-1\rangle + (C_+)^2 \langle j\ m+1|j\ m+1\rangle = (C_-)^2 + (C_+)^2 \tag{11}$$

$$\langle J_x^2 \rangle = \frac{\hbar^2}{4}[(j+m)(j-m+1) + (j-m)(j+m+1)] \tag{12}$$

$$= \frac{\hbar^2}{4}[j^2 - jm + j + jm - m^2 + m + j^2 + jm + j - mj - m^2 - m] \tag{13}$$

$$= \frac{\hbar^2}{4}[2j^2 + j - 2m^2] = \frac{\hbar^2}{2}[j(j+1) - m^2] = \langle J_y^2 \rangle \ . \tag{14}$$

1.3 Uncertainty Relation.

The variance of these operators is given by

$$(\Delta J_x)^2 = \langle J_x^2 \rangle - \langle J_x \rangle^2 = \frac{\hbar^2}{2}[j(j+1) - m^2] = (\Delta J_y)^2 \ . \tag{15}$$

The uncertainty relation given by Shankar 9.2.9 is

$$(\Delta\Omega)^2 (\Delta\Lambda)^2 \geq |\langle \psi | \Omega\Lambda | \psi \rangle|^2 \ . \tag{16}$$

So all that remains to be calculated to find the uncertainty relation is the norm squared of the expectation value of $J_x J_y$,

$$\langle J_x J_y \rangle = -\frac{i}{4}\langle J_+^2 - J_+ J_- + J_- J_+ - J_-^2 \rangle = -\frac{i}{4}[-\langle j\ m|J_+ J_-|j\ m\rangle + \langle j\ m|J_- J_+|j\ m\rangle] \tag{17}$$

$$= \frac{i}{4}[\langle J_- j\ m|J_- j\ m\rangle - \langle J_+ j\ m|J_+ j\ m\rangle] = \frac{i}{4}(C_-^2 - C_+^2) \tag{18}$$

$$= \frac{i}{4}\hbar^2[j^2 - jm + j + jm - m^2 + m - j^2 - jm - j + mj + m^2 + m] = \frac{i\hbar^2}{4}[2m] \ . \tag{19}$$

Therefore the uncertainty relation is

$$\frac{\hbar^4 m^2}{4} \leq \frac{\hbar^4}{4}[j(j+1) - m^2]^2 \quad \Rightarrow \quad m^2 \leq [j^2 + j - m^2]^2 \ , \tag{20}$$

which is satisfied for any j, m because $|m| \leq j$ and $j > 0$.

1.4 Saturated Uncertainty.

Consider the state $|j\ \pm j\rangle$, for which the uncertainty relation

$$(\pm j)^2 \leq [j^2 + j - (\pm j)^2]^2 \quad \Rightarrow \quad j^2 \leq j^2 \ , \tag{21}$$

is saturated.

2 Shankar 12.5.12.

The angular momentum operators L^2 and L_z commute with the parity operator Π, so they all share a common basis. The parity operator reverses the direction of a vector, $\Pi\mathbf{r} = -\mathbf{r}$, in spherical coordinates this is

$$\Pi x \to -x = -r\sin\theta\cos\phi \tag{22}$$
$$\Pi y \to -y = -r\sin\theta\sin\phi \tag{23}$$
$$\Pi z \to -z = -r\cos\theta\ . \tag{24}$$

However, because $r \geq 0$ by definition, the parity operator cannot have any effect on it. Notice that $\cos(\pi+\alpha) = \cos(\pi-\alpha) = -\cos(\alpha)$ and $-\sin(\pi+\alpha) = \sin(\pi-\alpha) = \sin(\alpha)$. For the parity of y to hold, the signs of the transformed coordinates must not cancel, so one must be transformed to $\pi+\alpha$ and the other $\pi-\alpha$ (where α is θ or ϕ). The cosine of either transform always results in a negative sign, so by inspecting the parity of x, the sine term must be positive so θ is transformed to $\pi-\theta$ and ϕ to $\pi+\phi$. So in spherical coordinates the parity operator is

$$\Pi\{r,\theta,\phi\} \to \{r, \pi-\theta, \pi+\phi\}. \tag{25}$$

Furthermore, it can be shown that the angular momentum lowering operator L_z does not alter parity. The definition of the angular momentum ladder operators in the spherical coordinate basis, as given by Shankar Equation 12.5.57, are

$$L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right)\ . \tag{26}$$

Under parity, the partial differentials are

$$\partial\theta \to \partial(\pi-\theta) = -\partial\theta\ , \qquad \partial\phi \to \partial(\pi+\phi) = \partial\phi\ , \tag{27}$$

and using the transformed behavior of cosine and sine stated above, $\Pi\cot\theta = -\cot\theta$, due to picking up a negative from the cosine but not the sine. Therefore the effect of parity,

$$\Pi L_- = -\hbar e^{-i(\pi+\phi)}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) = -\hbar(-1)e^{-i\phi}(-1)\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right) = L_-\ . \tag{28}$$

has no effect on the L_- operator (and therefore the L_+ operator as well).

To determine the effect that the parity operator has on the spherical harmonic functions, only one case of the spherical harmonics functional form must be verified. It has been shown that the L_- has no change under parity (they commute), so by finding the effect of parity on Y_ℓ^ℓ will yield the effect parity has on all states,

$$L_-\Pi Y_\ell^\ell = \Pi L_- Y_\ell^\ell = \Pi Y_\ell^{\ell-1}\ , \tag{29}$$

which can be repeated $2\ell+1$ times to see the effect of parity on any spherical harmonic. Therefore showing the behavior of ΠY_ℓ^ℓ will prove the result for general Y_ℓ^m. The effect of parity on this state (given by Shankar Equation 12.5.32) is

$$\Pi Y_\ell^\ell = \Pi\left[(-1)^\ell A_{\ell\ell}(\sin\theta)^\ell e^{i\ell\phi}\right] = (-1)^\ell A_{\ell\ell}[\sin(\pi-\theta)]^\ell e^{i\ell(\pi+\phi)}\ , \tag{30}$$

where $A_{\ell\ell}$ is a positive constant that has no dependence on θ or ϕ. By separating the exponential as done before another factor of $(-1)^\ell$ can be pulled out

$$\Pi Y_\ell^\ell = (-1)^\ell(-1)^\ell A_{\ell\ell}[\sin\theta]^\ell e^{i\ell\phi} = (-1)^\ell Y_\ell^\ell\ . \tag{31}$$

This proves the general result that $\Pi\,|\ell m\rangle = (-1)^\ell\,|\ell m\rangle$.

3 Shankar 12.5.13.

Consider a particle in a state described by

$$\psi = N(x + y + 2z)e^{-\alpha r} \ , \tag{32}$$

where N is a normalization factor.

3.1 Spherical Harmonics in Cartesian Coordinates.

The $\ell = 1$ spherical harmonics,

$$Y_1^0 = (3/4\pi)^{1/2}\cos\theta \qquad Y_1^{\pm 1} = \mp(3/8\pi)^{1/2}\sin\theta e^{\pm i\phi} \ , \tag{33}$$

can be expressed in Cartesian coordinates (and the radial magnitude r), by using the Euler identity for the exponential,

$$Y_1^{\pm 1} = \mp(3/8\pi)^{1/2}\sin\theta(\cos\phi \pm i\sin\phi) = \mp(3/8\pi)^{1/2}\left(\frac{x}{r} \pm i\frac{y}{r}\right) \ , \tag{34}$$

using the standard transformation for spherical to Cartesian coordinates. The $m = 0$ case is even simpler,

$$Y_1^0 = (3/4\pi)^{1/2}\frac{z}{r} \ . \tag{35}$$

3.2 Probabilities of $\ell = 1$ States.

Using these forms, the state described by Equation 32, can be written in terms of the $\ell = 1$ spherical harmonics. The expression for z in terms of spherical harmonics is trivial to get from Equation 35, while the expression for $x + y$ must be found. The spherical harmonics for $m = \pm$ have the from $x \pm iy$, so

$$(x + y) = a(x + iy) + b(x - iy) = (a + b)x + (ai - bi)y \ . \tag{36}$$

This relationship implies that $a = (1 - i)/2$ and $b = (1 + i)/2$. Solving the spherical harmonics for $x \pm iy$, using the above results, and plugging in to Equation 32, gives

$$N(x + y + 2z)e^{-\alpha r} = Ne^{-\alpha r}\left\{\left[\frac{1-i}{2}\right]\left(-\sqrt{\frac{8\pi}{3}}rY_1^1\right) + \left[\frac{1+i}{2}\right]\left(\sqrt{\frac{8\pi}{3}}rY_1^{-1}\right) + 2\sqrt{\frac{4\pi}{3}}rY_1^0\right\} \ . \tag{37}$$

Define constants C_{-1}, C_0, and C_1 to be the constant coefficients of the corresponding spherical harmonic, i.e.

$$N(x + y + 2z)e^{-\alpha r} = Nre^{-\alpha r}\left\{C_1 Y_1^1 + C_{-1}Y_1^{-1} + C_0 Y_1^0\right\} \ . \tag{38}$$

Now that the state is written as a linear combination of eigenstates of the L_z operator, the probability of measuring a specific eigenvalue of L_z can be read off. The probability of measuring a specific m (note the eigenvalues of $L - z$ are $m\hbar$), is just the squared norm of the coefficient of the corresponding eigenstate,

$$P(\ell = 1, m) = |C_m|^2 \ . \tag{39}$$

This fact, along with the definition of the C_m above, say that the probability of each state is as follows

$$P(m = 1 \to \ell_z = +\hbar) = 1/6 \tag{40}$$

$$P(m = 0 \to \ell_z = 0) = 2/3 \tag{41}$$

$$P(m = -1 \to \ell_z = -\hbar) = 1/6 \ . \tag{42}$$

4 Shankar 12.6.8.

Consider the potential forming a spherical box of radius r_0,

$$V = \begin{cases} 0 & r < r_0 \\ \infty & r \geq r_0 \end{cases} , \tag{43}$$

which gives rise to a rotationally invariant problem. Therefore the wavefunction for a particle of mass μ in this potential can be represented as

$$\psi_{E\ell m}(r,\theta,\phi) = \frac{U_{E\ell}(r)}{r} Y_\ell^m(\theta,\phi) , \tag{44}$$

where $U_{E\ell}$ is a function describing the radial behavior of the particle, which obeys the equation

$$\left\{ \frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2} \left[E - V(r) - \frac{\ell(\ell+1)\hbar^2}{2\mu r} \right] \right\} U_{E\ell} = 0 . \tag{45}$$

In the $\ell = 0$ sector, for the potential described in this problem, the differential equation above reduces to

$$\left\{ \frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2} E \right\} U_{E\ell} = 0 \quad \Rightarrow \quad U''_{E\ell} = -k^2 U_{E\ell} , \tag{46}$$

with $k = \sqrt{2\mu E}/\hbar$. Inside the spherical box, the particle behaves as a free particle. The solutions of the above differential equation for $V = 0$ are spherical Bessel functions (note the spherical Neumann functions are ignored because their behavior at the origin and infinity do not obey physical constraints). In the $\ell = 0$ sector, the solution is the zeroth spherical Bessel function,

$$j_o(kr) = \frac{\sin kr}{kr} \tag{47}$$

is the solution to the radial equation inside the well. Outside the solution must be zero because the potential is infinite, therefore the boundary condition on this system is $\psi_{E\ell m}(r_0, \theta, \phi) = 0$, so that

$$U_{E\ell}(r_0) = 0 \quad \Rightarrow \quad \sin kr_0 = 0 \quad \Rightarrow \quad kr_0 = n\pi , \tag{48}$$

where n is an integer. Using the definition of k and solving the final expression above for E yields the energy spectrum of a particle of mass μ in a spherical box of radius r_0,

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2\mu r_0^2} \tag{49}$$

5 Shankar 12.6.9

Consider a particle of mass μ in a potential given by

$$V = \begin{cases} -V_0 & r < r_0 \\ 0 & r \geq r_0 \end{cases} , \tag{50}$$

which has bound states for $-V_0 < E < 0$. As in the previous problem, in the $\ell = 0$ sector, the radial part of the wavefunction obeys

$$\left\{ \frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2} \left[E - V(r) \right] \right\} U_{E\ell} = 0 \ . \tag{51}$$

Due to the piecewise nature of the potential, there will be two solutions: one inside the well U_i and one outside U_o. The differential equation becomes

$$U_i'' = -\frac{2\mu}{\hbar^2}(E + V_0)U_i = -k^2 U_i \tag{52}$$

$$U_o'' = -\frac{2\mu}{\hbar^2}(E)U_o = (i\kappa)^2 U_o \ , \tag{53}$$

where k is the wavenumber inside the well and $i\kappa$ is the complex wavenumber outside the well. This equation has solutions of the form

$$U_i = A\cos(kr) + B\sin(kr) \tag{54}$$

$$U_o = C' e^{i(i\kappa r)} + D' e^{-i(i\kappa r)} = Ce^{\kappa r} + De^{-\kappa r} \ . \tag{55}$$

For the solution to be normalizable, it must be that $U_o(\infty)$ is finite, so $C = 0$. Additionally, the radial coordinate can never be negative so it must be that $U_i(0) = 0$ so $A = 0$. Furthermore, the wavefunction and its first derivative must be continuous at the boundary of the potential well,

$$\psi_{in}(r_0) = \psi_{out}(r_0) \ \rightarrow \ U_i(r_0) = U_o(r_0) \ \rightarrow \ B\sin(kr_0) = De^{-\kappa r_0} \tag{56}$$

$$\psi_{in}'(r_0) = \psi_{out}'(r_0) \ \rightarrow \ \left(\frac{U_i}{r} \right)' \Bigg|_{r_o} = \left(\frac{U_o}{r} \right)' \Bigg|_{r_o} \ . \tag{57}$$

The first equation implies that $D = Be^{\kappa r_0} \sin(kr_o)$. Note also that

$$\left(\frac{U}{r} \right)' = U' r^{-1} + U r^{-2} \ \rightarrow \ [U_i' r^{-1} + U_i r^{-2}]_{r_0} = [U_o' r^{-1} + U_o r^{-2}]_{r_0} \ , \tag{58}$$

so multiplying the equation by r^2 and evaluating at $r = r_0$ yields

$$r_0 kB\cos(kr_0) - B\sin(kr_0) = r_0(-\kappa)De^{-\kappa r_0} - De^{-\kappa r_0} \tag{59}$$

$$= r_0(-\kappa)Be^{\kappa r_0}\sin(kr_o)e^{-\kappa r_0} - Be^{\kappa r_0}\sin(kr_o)e^{-\kappa r_0} \ . \tag{60}$$

Note that the exponentials multiply to 1. Dividing the entire equation by $B\sin(kr_0)$ yields

$$r_0 k \frac{\cos(kr_o)}{\sin(kr_0)} - 1 = -\kappa r_0 - 1 \quad \Rightarrow \quad k\cot(kr) = -\kappa \ . \tag{61}$$

Note that κ is defined to be greater than zero so that the term that remains in U_o is an exponential decay. It is also known that $k > 0$ because for a bound state $|E| < V_0$, so the only way to pick up a negative in the above equation is if $\cot(kr_0) < 0$, which first occurs in the second quadrant. Therefore there are no solutions for this transcendental equation if $kr_0 < \pi/2$, which implies it must be that $k > \pi/(2r_0)$. However, from the definition of k, the following condition must be satisfied for a bound state:

$$E = \frac{(\hbar k)^2}{2\mu} - V_0 < 0 \quad \Rightarrow \quad V_0 > \frac{(\hbar k)^2}{2\mu} \ . \tag{62}$$

Since there is a lower bound on k and $V_0 > \alpha k$ (with $\alpha > 0$), then it must be the case that

$$V_0 > \frac{\hbar^2 \pi^2}{2\mu(4r_0^2)} \ . \tag{63}$$

By the combination of restrictions from the quantization condition (Equation 61) and the bound state condition (above), it is true that for a bound state to exist, $V_0 > (\hbar^2\pi^2)/(8\mu r_0^2)$. Therefore, no bound state can exist if $V_0 < (\hbar^2\pi^2)/(8\mu r_0^2)$.

6 Shankar 12.6.10

Given the following expressions are true,

$$(1) \quad \int_{-1}^{1} P_l(\cos\theta) P_{l'}(\cos\theta) d(\cos\theta) = [2/(2l+1)]\delta_{ll'} \tag{64}$$

$$(2) \quad P_l(x) = \frac{1}{2^l l!} \frac{d^l (x^2-1)^l}{dx^l} \tag{65}$$

$$(3) \quad \int_0^1 (1-x^2)^m dx = \frac{(2m)!!}{(2m+1)!!} \ , \tag{66}$$

Shankar Equation 12.6.41 can be proven as follows. The goal is to find the expression for C_l in the expression

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} C_l j_l(kr) P_l(\cos\theta) \ . \tag{67}$$

By multiplying both sides by a different associated Legendre polynomial $P_{l'}$ and integrating with respect to $\cos\theta$, the above expression can be rewritten

$$\int_{-1}^{1} d(\cos\theta) P_{l'}(\cos\theta) e^{ikr\cos\theta} = \int_{-1}^{1} d(\cos\theta) P_{l'}(\cos\theta) \sum_{l=0}^{\infty} C_l j_l(kr) P_l(\cos\theta) \ , \tag{68}$$

and the integral can be moved inside the sum. So using the first given expression, exploiting the orthonormality of the Legendre polynomials, this becomes

$$\int_{-1}^{1} d(\cos\theta) P_{l'}(\cos\theta) e^{ikr\cos\theta} = \sum_{l=0}^{\infty} C_l j_l(kr) [2/(2l+1)]\delta_{ll'} \ . \tag{69}$$

The delta function kills off every term in the sum except $l = l'$, so

$$\int_{-1}^{1} d(\cos\theta) P_l(\cos\theta) e^{ikr\cos\theta} = C_l j_l(kr) [2/(2l+1)] \ , \tag{70}$$

for simplicity let $x = \cos\theta$. Now both sides can be differentiated l times with respect to kr. Note this does not effect the integral because the integral is over x, so

$$\int_{-1}^{1} dx P_l(x) \left(\frac{d}{d(kr)} \right)^l e^{ikrx} = C_l [2/(2l+1)] \left(\frac{d}{d(kr)} \right)^l j_l(kr) \tag{71}$$

Now consider the limit as $kr \to 0$, so Shankar Equation 12.6.33 can be used, and the complex exponential vanishes,

$$\int_{-1}^{1} dx P_l(x) i^l x^l = C_l \frac{2}{2l+1} \left(\frac{d}{d(kr)} \right)^l \frac{(kr)^l}{(2l+1)!!} \ . \tag{72}$$

Now noting that taking the n^{th} derivative of $(kr)^l$ results in a coefficient of $(l-n+1)$, so taking l derivatives yields $l!$, and the final exponent is 0, so

$$\int_{-1}^{1} dx P_l(x) i^l x^l = C_l \frac{2}{2l+1} \frac{l!}{(2l+1)!!} \ , \tag{73}$$

This can be solved for C_l and rewritten

$$C_l = i^l \frac{2l+1}{2} \frac{(2l+1)!!}{l!} \int_{-1}^{1} dx P_l(x) x^l = i^l (2l+1) \frac{(2l+1)!!}{l!} \int_0^1 dx P_l(x) x^l \ , \tag{74}$$

by noting that the quantity $x^l P_l(x)$ is always and even function. This is true because x^l is odd for odd l and even for even l, and likewise for $P_l(x)$, therefore for any l they will always have the same even/odd-ness and their product will always be even. Using the second given expression, this becomes

$$C_l = i^l (2l+1) \frac{(2l+1)!!}{l!} \int_0^1 dx x^l \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2-1)^l \ , \tag{75}$$

and noting $2^l l! = (2l)!!$ gives the expression

$$C_l = i^l (2l+1) \frac{(2l+1)!!}{(l)!(2l)!!} \int_0^1 dx x^l \frac{d^l}{dx^l} (x^2-1)^l \ . \tag{76}$$

The integral can be evaluated using integration by parts l times, with respect to x, using the derivative part as the term to take the antiderivative of in IBP. Integration by parts once (the first one) looks like

$$\int_0^1 dx x^l \frac{d^l}{dx^l}(x^2-1)^l = x^l \frac{d^{l-1}}{dx^{l-1}} (x^2-1)^l \bigg|_0^1 - \int_0^1 (l x^{l-1}) \frac{d^{l-1}}{dx^{l-1}} (x^2-1)^l \ . \tag{77}$$

Note that each on the m^{th} IBP a factor of $l-m+1$ is picked up so after l IBPs there is a coefficient of $l!$, and the final exponent of the x^{l-m} term is zero. Similarly, the order of the derivative is reduced by one at each step, so after l steps, there is no derivative. At each step of IBP, there is an additional factor of -1 picked up due to the subtraction of the integral from the boundary term, which after l times is $(-1)^l$. Therefore (ignoring the boundary term, which will be revisited later) the integral is

$$\int_0^1 dx x^l \frac{d^l}{dx^l}(x^2-1)^l = (-1)^l l! \int_0^1 (x^2-1)^l = l! \int_0^1 (1-x^2)^l = l! \frac{(2l)!!}{(2l+1)!!} \tag{78}$$

using the third given expression. Plugging this back in to Equation 76 yields the final result

$$C_l = i^l (2l+1) \frac{(2l+1)!!}{(l)!(2l)!!} l! \frac{(2l)!!}{(2l+1)!!} = i^l (2l+1) \ . \tag{79}$$

Now return to the boundary terms picked up through integration by parts. The series of these terms is

$$x^l \frac{d^{l-1}}{dx^{l-1}} (x^2-1)^l \bigg|_0^1 - x^{l-1} \frac{d^{l-2}}{dx^{l-2}} (x^2-1)^l \bigg|_0^1 + x^{l-2} \frac{d^{l-3}}{dx^{l-3}} (x^2-1)^l \bigg|_0^1 \cdots \tag{80}$$

clearly the lower boundary does not contribute due to the x^k terms. These factors also do not contribute to the value of the expression at the $x = 1$ boundary either. Expanding out $(x^2-1)^l$ yields a polynomial with coefficients given by the l^{th} row of Pascal's triangle, with the sign of each term alternating (the highest power of x is always positive). Taking the derivatives make the terms with no x dependence (± 1) drop out. At the $x = 1$ boundary, only the coefficients matter, and they will always sum to zero when using alternating negative signs as shown in the series above. This was additionally proved using Mathematica on a case-by-case basis. Therefore the boundary terms do not contribute, as claimed previously.