Dylan J. Temples: Tully Solutions

Particle Physics
Elementary Particle Physics in a Nutshell - C. Tully
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1 Tully 2.1.

Calculate $\mathrm{d}\hat{\mathbf{x}}/\mathrm{d}t$ separately for the nonrelativistic Hamiltonian $\hat{E} = \hat{\mathbf{p}}^2/2m$ and for the Dirac Hamiltonian $\hat{E} = \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m$.

Heisenberg's equation of motion tells us

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathcal{O}} = i\left[\hat{H}, \hat{\mathcal{O}}\right] + \frac{\partial \hat{\mathcal{O}}}{\partial t} \ , \tag{1}$$

in natural units. If the operator is not an explicit function of time, the second term on the right-hand side of the above equation vanishes, and calculating the time rate of change of an operator becomes an excercise of calculating commutators with the Hamiltonian $\hat{H}$.

In the case of the nonrelativistic Hamiltonian, we have:

$$\frac{\mathrm{d}\hat{\mathbf{x}}}{\mathrm{d}t} = i\left[\hat{H}, \hat{\mathbf{x}}\right] = \frac{i}{2m}\left[\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}, \hat{\mathbf{x}}\right] = -\frac{i}{2m}\left[\hat{\mathbf{x}}, \hat{\mathbf{p}} \cdot \hat{\mathbf{p}}\right] \ . \tag{2}$$

Using identities for commutators, we can express this as

$$\frac{\mathrm{d}\hat{\mathbf{x}}}{\mathrm{d}t} = -\frac{i}{2m}\left\{[\hat{\mathbf{x}}, \hat{\mathbf{p}}]\,\hat{\mathbf{p}} + \hat{\mathbf{p}}\,[\hat{\mathbf{x}}, \hat{\mathbf{p}}]\right\} \ , \tag{3}$$

where the canonical commutation relation tells us $[\hat{\mathbf{x}}, \hat{\mathbf{p}}] = i$, in natural units. Therefore:

$$\frac{\mathrm{d}\hat{\mathbf{x}}}{\mathrm{d}t} = -\frac{i}{2m}\left\{i\hat{\mathbf{p}} + \hat{\mathbf{p}}i\right\} = \frac{\hat{\mathbf{p}}}{m} \ , \tag{4}$$

which we expect from a classical point of view: momentum and velocity ($\mathrm{d}x/\mathrm{d}t$) differ only by a factor of the particle's mass.

Things are slightly more difficult with Dirac's Hamiltonian:

$$\left[\hat{H}, \hat{\mathbf{x}}\right] = i\left[\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m, \hat{\mathbf{x}}\right] = -[\hat{\mathbf{x}}, \boldsymbol{\alpha} \cdot \hat{\mathbf{p}}] + m[\beta, \hat{\mathbf{x}}] = -[\hat{\mathbf{x}}, \boldsymbol{\alpha}]\hat{\mathbf{p}} - \boldsymbol{\alpha}[\hat{\mathbf{x}}, \hat{\mathbf{p}}] + m[\beta, \hat{\mathbf{x}}] \ . \tag{5}$$

If we set[1] $[\boldsymbol{\alpha}, \hat{\mathbf{x}}] = [\beta, \hat{\mathbf{x}}] = 0$, we have

$$\frac{\mathrm{d}\hat{\mathbf{x}}}{\mathrm{d}t} = i(-\boldsymbol{\alpha}[\hat{\mathbf{x}}, \hat{\mathbf{p}}]) = i(-i\boldsymbol{\alpha}) = \boldsymbol{\alpha} \ , \tag{6}$$

which is in agreement with Tully equation 2.19.

[1] Here I will argue that the commutators of the parameters $\boldsymbol{\alpha}$ and β with the position and momentum operators are all zero. Let us work in the position-space basis: $\hat{\mathbf{x}} = \hat{\mathbf{x}}$ and $\hat{\mathbf{p}} = -i\boldsymbol{\nabla}$. The components of $\boldsymbol{\alpha}$ and β are the Pauli matrices $\boldsymbol{\sigma}$ and two-dimensional identities $\mathbb{1}_2$, which are all constants (of varying signs). For the commutators with the momentum operator, it is easy to see that taking a derivative then multiplying by a constant is the same as multiplying by a constant and taking a derivative, and thus the commutators must be zero. Similarly, the position operator must commute with the identity, regardless of sign, and therefore $[\beta, \hat{\mathbf{x}}] = 0$. Additionally, the Pauli matrices only act on spin-state vectors, not position-state vectors, and therefore $[\boldsymbol{\alpha}, \hat{\mathbf{x}}] = 0$.

2 Tully 2.4.

Show that $\mathrm{d}\hat{\mathbf{L}}/dt = \boldsymbol{\alpha} \times \hat{\mathbf{p}}$ and that $\frac{1}{2}\mathrm{d}\boldsymbol{\Sigma}/\mathrm{d}t = -\boldsymbol{\alpha} \times \hat{\mathbf{p}}$ for a free Dirac particle.

From the definition of the angular momentum, $\hat{\mathbf{L}} = \hat{\mathbf{x}} \times \hat{\mathbf{p}}$, and equation 1, we see

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{L}} = i\left[\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m, \hat{\mathbf{x}} \times \hat{\mathbf{p}}\right] = i\left[\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}, \hat{\mathbf{x}} \times \hat{\mathbf{p}}\right] + im\left[\beta, \hat{\mathbf{x}} \times \hat{\mathbf{p}}\right] \ . \tag{7}$$

Let us investigate the first term:

$$[\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}, \hat{\mathbf{x}} \times \hat{\mathbf{p}}] = \boldsymbol{\alpha}[\hat{\mathbf{p}}, \hat{\mathbf{x}}] \times \hat{\mathbf{p}} + \boldsymbol{\alpha} \cdot \hat{x}[\hat{\mathbf{p}}, \hat{\mathbf{p}}] + [\boldsymbol{\alpha}, \hat{\mathbf{x}}]\hat{\mathbf{p}} \times \hat{\mathbf{p}} + \hat{\mathbf{x}} \times [\boldsymbol{\alpha}, \hat{\mathbf{p}}]\hat{\mathbf{p}} \ , \tag{8}$$

and we can note the middle two terms vanish trivially, and the final term vanishes by the argument presented in footnote 1, yielding

$$[\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}, \hat{\mathbf{x}} \times \hat{\mathbf{p}}] = (-i)\boldsymbol{\alpha} \times \hat{\mathbf{p}} \ . \tag{9}$$

By the same argument used above, the second term in equation 7 vanishes because it is the sum of terms proportional to commutators of β with position and momentum. Therefore, we can insert these results into Heisenberg's equation to obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{L}} = i(-i)\boldsymbol{\alpha} \times \hat{\mathbf{p}} = \boldsymbol{\alpha} \times \hat{\mathbf{p}} \ , \tag{10}$$

our desired result.

The total angular momentum,

$$\hat{\mathbf{J}} = \hat{\mathbf{L}} + \frac{1}{2}\boldsymbol{\Sigma} \ , \tag{11}$$

is a constant of motion, and as such we require:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{J}} = \frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{L}} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Sigma} = 0 \ , \tag{12}$$

yielding the condition

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Sigma} = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{L}} = -(\boldsymbol{\alpha} \times \hat{\mathbf{p}}) \ , \tag{13}$$

using the previous result. If this is unsatisfactory to you, it can be calculated directly using the Heisenberg equation of motion and the explicit forms of $\boldsymbol{\Sigma}$ and $\boldsymbol{\alpha}$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Sigma} = \frac{1}{2i}\left[\boldsymbol{\Sigma}, \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m\right] = \frac{1}{2i}\left\{\left[\Sigma_k, \alpha_i p_j \delta_{ij}\right] + m\left[\Sigma_k, \beta\right]\right\} \ , \tag{14}$$

where

$$\Sigma_k = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \ , \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \ , \quad \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \ , \tag{15}$$

we should also note the (anti) commutator relations for the Pauli matrices:

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \ , \quad \{\sigma_i, \sigma_j\} = 0 \ , \quad \sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k \ . \tag{16}$$

First, we compute:

$$[\Sigma_k, \beta] = \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix} - \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix} = 0 \ , \tag{17}$$

so the second term vanishes. The first term is proportional to

$$[\Sigma_k, \alpha_i p_j \delta_{ij}] = p_j \delta_{ij} \left[\Sigma_k, \alpha_i\right] \ , \tag{18}$$

because the Pauli matrices do not effect the momentum operator. Inserting the explicit representations, we have

$$[\Sigma_k, \alpha_i p_j \delta_{ij}] = p_j \delta_{ij} \begin{pmatrix} 0 & \sigma_k \sigma_i - \sigma_i \sigma_k \\ \sigma_k \sigma_i - \sigma_i \sigma_k & 0 \end{pmatrix} \ , \tag{19}$$

from here, we have two routes: using the commutator, or using the anti-commutator. I will do both, starting with the latter:

$$[\Sigma_k, \alpha_i p_j \delta_{ij}] = 2 p_j \delta_{ij} \begin{pmatrix} 0 & \sigma_k \sigma_i \\ \sigma_k \sigma_i & 0 \end{pmatrix} = 2 p_j \delta_{ij} \begin{pmatrix} 0 & i\epsilon_{kij}\sigma_j \\ i\epsilon_{kij}\sigma_j & 0 \end{pmatrix} = 2 i p_j \delta_{ij} \epsilon_{kij} \alpha_j \ , \tag{20}$$

so

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{\Sigma} = \frac{1}{2i} 2 i p_j \delta_{ij} \epsilon_{kij} \alpha_j = p_i \epsilon_{kij} \alpha_j = -\boldsymbol{\alpha} \times \hat{\mathbf{p}} \ , \tag{21}$$

as expected. We now return to equation 19 and use the commutator:

$$[\Sigma_k, \alpha_i p_j \delta_{ij}] = p_j \delta_{ij} \begin{pmatrix} 0 & [\sigma_k, \sigma_i] \\ [\sigma_k, \sigma_i] & 0 \end{pmatrix} = p_j \delta_{ij} \begin{pmatrix} 0 & 2i\epsilon_{kij}\sigma_j \\ 2i\epsilon_{kij}\sigma_j & 0 \end{pmatrix} = 2 i p_j \delta_{ij} \epsilon_{kij} \alpha_j \ , \tag{22}$$

and then we obtain the same result. (Compare this final expression to equation 21 to see they are equivalent.)

3 Tully 2.5.

Show that $\mathrm{d}\boldsymbol{\alpha}/\mathrm{d}t = -2(\boldsymbol{\Sigma} \times \hat{\mathbf{p}}) - i2m\boldsymbol{\alpha}\beta$.

Again, Heisenberg's equation of motion tells us

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\alpha} = i[\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m, \boldsymbol{\alpha}] = i[\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}, \boldsymbol{\alpha}] + im[\beta, \boldsymbol{\alpha}] = -i[\boldsymbol{\alpha}, \boldsymbol{\alpha} \cdot \hat{\mathbf{p}}] + im[\beta, \boldsymbol{\alpha}] \tag{23}$$

using our previous result. First, we investigate the second term:

$$[\beta, \boldsymbol{\alpha}] = \beta\boldsymbol{\alpha} - \boldsymbol{\alpha}\beta \ , \tag{24}$$

but $\boldsymbol{\alpha}$ and β anticommute (Tully equation 2.8), so $\boldsymbol{\alpha}\beta = -\beta\boldsymbol{\alpha}$, and thus:

$$[\beta, \boldsymbol{\alpha}] = -\boldsymbol{\alpha}\beta - \boldsymbol{\alpha}\beta = -2\boldsymbol{\alpha}\beta \ , \tag{25}$$

so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\alpha} = -i[\boldsymbol{\alpha}, \boldsymbol{\alpha} \cdot \hat{\mathbf{p}}] - 2im\boldsymbol{\alpha}\beta \ . \tag{26}$$

The first term (of the right-most expression) in equation 23 is, component-wise, proportional to

$$[\alpha_i, \alpha_j p_k \delta_{jk}] = p_k \delta_{jk} [\alpha_i, \alpha_j] = p_k \delta_{jk} \begin{pmatrix} [\sigma_i, \sigma_j] & 0 \\ 0 & [\sigma_i, \sigma_j] \end{pmatrix} = p_k \delta_{jk} \begin{pmatrix} 2i\epsilon_{ijk}\sigma_k & 0 \\ 0 & 2i\epsilon_{ijk}\sigma_k \end{pmatrix} \tag{27}$$

$$= 2ip_k \delta_{jk} \epsilon_{ijk} \Sigma_k = 2ip_j \epsilon_{ijk} \Sigma_k \ , \tag{28}$$

therefore:

$$[\boldsymbol{\alpha}, \boldsymbol{\alpha} \cdot \hat{\mathbf{p}}] = 2i\hat{\mathbf{p}} \times \boldsymbol{\Sigma} = -2i\boldsymbol{\Sigma} \times \hat{\mathbf{p}} = \frac{2}{i}\boldsymbol{\Sigma} \times \hat{\mathbf{p}} \ , \tag{29}$$

and inserting this result into equation 26 yields the result:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\alpha} = -i\frac{2}{i}\boldsymbol{\Sigma} \times \hat{\mathbf{p}} - 2im\boldsymbol{\alpha}\beta = -2(\boldsymbol{\Sigma} \times \hat{\mathbf{p}}) - i2m\boldsymbol{\alpha}\beta \ . \tag{30}$$

4 Tully 2.8.

Show that $v^{(2)}(p) = (i\beta\alpha^2)\left[u^{(2)}(p)\right]^*$.

First, we will investigate the quantity:

$$i\beta\alpha^2 = i\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} , \tag{31}$$

and it is clear the action of this flips a four-component column vector and negates the middle two components. Using the explicit representation of $u^{(2)}(p)$ (Tully 2.56) this is

$$(i\beta\alpha^2)\left[u^{(2)}(p)\right]^* = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ (p_1 + ip_2)/(E+m) \\ -p_3/(E+m) \end{pmatrix} , \tag{32}$$

having taken the complex conjugate. Carrying out the multiplication yields

$$(i\beta\alpha^2)\left[u^{(2)}(p)\right]^* = \begin{pmatrix} -p_3/(E+m) \\ -(p_1 + ip_2)/(E+m) \\ -1 \\ 0 \end{pmatrix} = v^{(2)}(p) , \tag{33}$$

from comparison with Tully 2.60.

5 Tully 2.11.

Show that the matrix representation of the Lorentz group proposed by Dirac, $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$, satisfies the commutation relations (Tully 2.94)

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\left(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}\right) \ . \tag{34}$$

Sol.

6 Tully 2.17.

Show that in the Weyl representation $\bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi = \xi^\dagger i\bar{\sigma}^\mu\partial_\mu\xi + \eta^\dagger i\sigma^\mu\partial_\mu\eta - m(\xi^\dagger\eta + \eta^\dagger\xi)$ where ξ and η are left- and right- handed chirality two-component spinors.

Using $\bar{\Psi} = \Psi^\dagger\gamma^0$, the Dirac Lagrangian becomes

$$\bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi = \Psi^\dagger(i\gamma^0\gamma^\mu\partial_\mu - m\gamma^0)\Psi \ , \tag{35}$$

where

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \ , \quad \text{with} \quad \begin{cases} \sigma^\mu = (\mathbb{1}_2, \boldsymbol{\sigma}) \\ \bar{\sigma}^\mu = (\mathbb{1}_2, -\boldsymbol{\sigma}) \end{cases} \ . \tag{36}$$

We can then identify $\beta = \gamma^0$, and

$$\gamma^0\gamma^\mu = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} = \begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix} \ , \tag{37}$$

so the spatial components give us

$$\gamma^0\gamma^i = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} = \alpha^i \ , \tag{38}$$

in the Weyl representation. If we define $\alpha^0 = \mathbb{1}_4$, then we have $\alpha^\mu = \gamma^0\gamma^\mu$, so the Dirac Lagrangian becomes

$$\bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi = \Psi^\dagger i\alpha^\mu\partial_\mu\Psi - \Psi^\dagger\beta m\Psi \ , \tag{39}$$

where

$$\Psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \ , \quad \text{so } \Psi^\dagger = \begin{pmatrix} \xi^\dagger & \eta^\dagger \end{pmatrix} \ . \tag{40}$$

We can investigate the first term:

$$\Psi^\dagger\alpha^\mu\partial_\mu\Psi = \begin{pmatrix} \xi^\dagger & \eta^\dagger \end{pmatrix}\alpha^\mu\partial_\mu\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi^\dagger & \eta^\dagger \end{pmatrix}\begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix}\begin{pmatrix} \partial_\mu\xi \\ \partial_\mu\eta \end{pmatrix} = \begin{pmatrix} \xi^\dagger & \eta^\dagger \end{pmatrix}\begin{pmatrix} \bar{\sigma}^\mu\partial_\mu\xi \\ \sigma^\mu\partial_\mu\eta \end{pmatrix} \tag{41}$$

$$= \xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi + \eta^\dagger\sigma^\mu\partial_\mu\eta \ , \tag{42}$$

and the second:

$$\Psi^\dagger\beta\Psi = \begin{pmatrix} \xi^\dagger & \eta^\dagger \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi^\dagger & \eta^\dagger \end{pmatrix}\begin{pmatrix} \eta \\ \xi \end{pmatrix} = \xi^\dagger\eta + \eta^\dagger\xi \ . \tag{43}$$

Using these, we have the result:

$$\bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi = \xi^\dagger i\bar{\sigma}^\mu\partial_\mu\xi + \eta^\dagger i\sigma^\mu\partial_\mu\eta - m(\xi^\dagger\eta + \eta^\dagger\xi) \ . \tag{44}$$

7 Tully 2.18.

Given an explicit form for the charge-conjugation operator C in the Weyl representation, compute $C[u^{(1)}(p)]^*$ and $C[u^{(2)}(p)]^*$ explicitly using the matrix representation for C and the column-vector Weyl solutions for $u^{(1)}(p)$ and $u^{(2)}(p)$.

In the Weyl representation, the charge conjugation operator is

$$i\beta\alpha^2 = i\begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}\begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma_2 \end{pmatrix} = i\begin{pmatrix} 0 & \sigma^2 \\ -\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} , \tag{45}$$

which simply flips the component order of the 4-component spinor and negates the central two components. We begin with the Dirac equation, with $\psi = (\chi, \phi)$, where χ, ϕ are 2-component spinors:

$$E\begin{pmatrix} \chi \\ \phi \end{pmatrix} = (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)\begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\sigma}\cdot\mathbf{p} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\mathbf{p} \end{pmatrix}\begin{pmatrix} \chi \\ \phi \end{pmatrix} + m\begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}\begin{pmatrix} \chi \\ \phi \end{pmatrix} , \tag{46}$$

yielding the following coupled equations:

$$E\chi = -(\boldsymbol{\sigma}\cdot\mathbf{p})\chi + m\phi \tag{47}$$

$$E\phi = (\boldsymbol{\sigma}\cdot\mathbf{p})\phi + m\chi , \tag{48}$$

slight rearrangement yields

$$\chi = \frac{1}{m}\left(E\mathbb{1}_2 - \boldsymbol{\sigma}\cdot\mathbf{p}\right)\phi \tag{49}$$

$$\phi = \frac{1}{m}\left(E\mathbb{1}_2 + \boldsymbol{\sigma}\cdot\mathbf{p}\right)\chi . \tag{50}$$

The helicity operator is

$$\boldsymbol{\sigma}\cdot\mathbf{p} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_1 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_3 = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} , \tag{51}$$

so

$$\chi = \frac{1}{m}\begin{pmatrix} E - p_3 & -p_1 + ip_2 \\ -p_1 - ip_2 & E + p_3 \end{pmatrix}\phi \tag{52}$$

$$\phi = \frac{1}{m}\begin{pmatrix} E + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & E - p_3 \end{pmatrix}\chi . \tag{53}$$

We can now write the particle spinors as

$$u^{(s)}(p) = \begin{pmatrix} \chi^{(s)} \\ \phi^{(s)} \end{pmatrix} = \begin{pmatrix} \chi^{(s)} \\ \frac{1}{m}\begin{pmatrix} E + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & E - p_3 \end{pmatrix}\chi^{(s)} \end{pmatrix} , \tag{54}$$

where the condition on $\chi^{(s)}$ is

$$\chi^{(s)\dagger}\chi^{(r)} = \delta_{rs} , \tag{55}$$

so we make the choice:

$$\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \quad \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \tag{56}$$

so we have the spinors (and their conjugates)

$$u^{(1)}(p) = \begin{pmatrix} 1 \\ 0 \\ (E+p_3)/m \\ (p_1+ip_2)/m \end{pmatrix} \quad \Rightarrow \quad [u^{(1)}(p)]^* = \begin{pmatrix} 1 \\ 0 \\ (E+p_3)/m \\ (p_1-ip_2)/m \end{pmatrix} \tag{57}$$

$$u^{(2)}(p) = \begin{pmatrix} 0 \\ 1 \\ (p_1-ip_2)/m \\ (E-p_3)/m \end{pmatrix} \quad \Rightarrow \quad [u^{(2)}(p)]^* = \begin{pmatrix} 0 \\ 1 \\ (p_1+ip_2)/m \\ (E-p_3)/m \end{pmatrix} \ . \tag{58}$$

Acting the charge conjugation operator on these spinors yields

$$C[u^{(1)}(p)]^* = i\beta\alpha^2[u^{(1)}(p)]^* = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ (E+p_3)/m \\ (p_1-ip_2)/m \end{pmatrix} = \begin{pmatrix} (p_1-ip_2)/m \\ -(E+p_3)/m \\ 0 \\ 1 \end{pmatrix} \tag{59}$$

$$C[u^{(2)}(p)]^* = i\beta\alpha^2[u^{(2)}(p)]^* = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ (p_1+ip_2)/m \\ (E-p_3)/m \end{pmatrix} = \begin{pmatrix} (E-p_3)/m \\ -(p_1+ip_2)/m \\ -1 \\ 0 \end{pmatrix} \ . \tag{60}$$